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Colored Permutation Statistics by Conjugacy Class

Jesse Campion Loth*[1], Michael Levet†[2], Kevin Liu‡[3], Sheila Sundaram§[4], and Mei Yin¶[5]

[1]*Department of Mathematics, Simon Fraser University, Burnaby, BC, Canada*
[2]*Department of Computer Science, College of Charleston, Charleston, SC, USA*
[3]*Department of Mathematics, University of Washington, Seattle, WA, USA*
[4]*University of Minnesota, Minneapolis, MN, USA*
[5]*University of Denver, Denver, CO, USA*

Abstract. We consider the moments of statistics on conjugacy classes of colored permutation groups $\mathfrak{S}_{n,r} = \mathbb{Z}_r \wr \mathfrak{S}_n$. We first show that any fixed moment of a statistic coincides on all conjugacy classes when all cycle lengths are sufficiently long. For permutation statistics that can be realized via a process called symmetric extension, we show that for fixed r, this moment on these conjugacy classes is a polynomial in n. Hamaker and Rhoades (arXiv, 2022) established analogous results for the symmetric group as part of their far-reaching representation-theoretic framework. Independently, Campion Loth, Levet, Liu, Stucky, Sundaram, and Yin (arXiv, 2023) arrived at independence and polynomiality results for the symmetric group using instead an elementary combinatorial framework. Our techniques in this paper build on this latter elementary approach. Finally, we extend the work of Fulman (*J. Comb. Theory Ser. A.*, 1998), to establish a central limit theorem for descents in conjugacy classes of the hyperoctahedral group with sufficiently long cycles.

Keywords: colored permutation, Coxeter group, hyperoctahedral group, moment, permutation constraint, permutation statistic

1 Introduction

For a finite group G, a *statistic* is a map $X : G \to \mathbb{R}$. The *distribution* of X is the function (x_k), where x_k is the number of elements $g \in G$ such that $X(g) = k$ (i.e., $x_k := |X^{-1}(k)|$). When G is the symmetric group $G = \mathfrak{S}_n$, we refer to the statistics as *permutation statistics*. The study of permutation statistics is a classical topic in algebraic combinatorics; Stanley's texts [16, 17] serve as a key reference in this area.

*jesse_campion_loth@sfu.ca
†levetm@cofc.edu. Partially supported by J. A. Grochow's NSF award CISE-2047756.
‡kliu15@uw.edu
§shsund@umn.edu
¶mei.yin@du.edu. Partially supported by the University of Denver's Professional Research Opportunities for Faculty Fund 80369-145601.

In this paper, we build on the elementary methods in [4] to investigate the distribution of *colored* permutation statistics by conjugacy class. In contrast to the vast literature on permutation statistics in $\mathfrak{S}_n$, there has been considerably less work on statistics for arbitrary Coxeter groups or the colored permutation groups $\mathfrak{S}_{n,r}$, i.e., the wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n$. We are in particular not aware of work considering colored permutation statistics on individual conjugacy classes.

When $r = 2$, the colored permutation group $\mathfrak{S}_{n,2}$ coincides with the hyperoctahedral group B_n, which is the type B Coxeter group. A study of statistics over the entire Coxeter group for types B and D was initiated by Reiner, see e.g. [15], and carried further by Adin and Roichman, see e.g. [1], and Brenti and Carnevale [3]. There is also work on colored permutation statistics and their distribution, again over the whole group, by Steingrímsson [18], Fire [7], and Moustakas [13].

Recently, Hamaker and Rhoades [11] established a representation-theoretic framework for permutation statistics on $\mathfrak{S}_n$ by conjugacy class C_λ. They introduced so-called *local* permutation statistics; using representation-theoretic methods, they established that the moments of these statistics depend only on n and the number of short cycles in λ. In particular, these moments are independent of the conjugacy class when the cycles in λ are all sufficiently large.

Independently, and subsequent to the paper [11], Campion Loth, Levet, Liu, Stucky, Sundaram, and Yin [4] established similar independence and polynomiality results for conjugacy classes in $\mathfrak{S}_n$, using only elementary combinatorial techniques. The present paper builds on the framework in [4]. The full version of this paper appears in [5].

Main Results. Fix $r \geq 1$, and let $\boldsymbol{\lambda}$ be an r-partition of n. For a statistic X on $\mathfrak{S}_{n,r}$, denote by $\mathbb{E}_{\boldsymbol{\lambda}}[X]$ the expected value of X taken over the conjugacy class of $\mathfrak{S}_{n,r}$ indexed by $\boldsymbol{\lambda}$. Our main results are as follows:

- Theorem 12 in Section 3.2 shows that for any statistic X, its kth moment coincides on all conjugacy classes $C_{\boldsymbol{\lambda}}$ of $\mathfrak{S}_{n,r}$ that do not have "short" cycles. For each statistic X, making this notion of "short" precise is done through *colored permutation constraints* as given in Definition 4.

- Theorem 20 in Section 3.3 concerns sequences of statistics $(X_n)_{n\geq 1}$ on $(\mathfrak{S}_{n,r})_{n\geq 1}$ that can be constructed using *symmetric extensions*, as described in Definition 19. This theorem shows that a single polynomial in n gives $\mathbb{E}_{\boldsymbol{\lambda}_n}[X_n^k]$ on conjugacy classes $C_{\boldsymbol{\lambda}_n}$ of $\mathfrak{S}_{n,r}$ without "short" cycles. Note that this result applies to many statistics, including the inversion statistic on B_n defined in (2.2).

- Finally, Theorem 28 in Section 4 establishes asymptotic normality of the descent statistic on B_n for conjugacy classes with no "short" cycles. Our proof leverages a generating function of Reiner [15, Theorem 4.1] for the joint distribution of descent and major index by cycle type, an analogue of the corresponding generating function for the symmetric group [10]. The arguments then follow Fulman's analogous result for descents on conjugacy classes of $\mathfrak{S}_n$ [8, Theorem 1 and proof of Theorem 2], but the technical details are nontrivial and require care to execute.

Remark 1. One essential insight in our work was in developing the notion of *colored permutation constraints* (see Definition 4). It took considerable effort to arrive at this definition, and we discuss these technical difficulties in the full version [5, Remark 3.3]. The fact that Theorem 12 and Theorem 20 generalize analogous results on the symmetric group [11, 4] so cleanly suggests that Definition 4 might in fact be the right notion of colored permutation constraints.

2 Preliminaries

We recall preliminary notions of colored permutation groups. The *colored permutation group* $\mathfrak{S}_{n,r}$ is the wreath product [12, Chapter 4] $\mathbb{Z}_r \wr \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on n elements and $\mathbb{Z}_r$ is the cyclic group on r elements. A *colored permutation* $(\omega, \tau) \in \mathfrak{S}_{n,r}$ can be expressed as an ordered pair consisting of a permutation $\omega \in \mathfrak{S}_n$ along with a function $\tau : [n] \to \mathbb{Z}_r$, where the representative elements of $\mathbb{Z}_r$ are taken in $\{0, \ldots, r-1\}$. The value $\tau(j)$ is called the *color* of the symbol j, and $\tau(j) + \tau'(j)$ is defined as a sum of elements in $\mathbb{Z}_r$.

The colored permutation group $\mathfrak{S}_{n,r}$ has a canonical embedding as a subgroup of the symmetric group $\mathfrak{S}_{rn}$, which we describe explicitly as follows. Writing $[n]^r$ for the set of rn elements $\{i^j | i \in [n], j \in \{0, 1, \ldots, r-1\}\}$ where the exponent indicates the color of an element in $[n]$, we can also think of the colored permutation (ω, τ) as a bijection $f : [n]^r \to [n]^r$ defined by $f(i^j) = \omega(i)^{\tau(\omega(i))+j}$ for all i, j, where $\tau(\omega(i)) + j$ is taken modulo r. In this sense, the coloring of the symbols τ and the underlying permutation ω are independently specified.

We now turn to discussing the conjugacy class structure of $\mathfrak{S}_{n,r}$. An *r-partition* of $n \in \mathbb{N}$ is an r-tuple of partitions $\boldsymbol{\lambda} = (\lambda^j)_{j=0}^{r-1}$ where each λ^j is a partition of some n_j such that $\sum_{j=0}^{r-1} n_j = n$. When $r = 2$, we also call this a *bi-partition.* For a cycle in a permutation in $\mathfrak{S}_{n,r}$, the length of this cycle is the number of elements in it, and the color of this cycle is the sum of the colors in the cycle, taken modulo r. The *cycle type* of $(\omega, \tau) \in \mathfrak{S}_{n,r}$ is the r-partition $\boldsymbol{\lambda} = (\lambda^j)_{0 \le j \le r-1}$, where each λ^j consists of the cycles of color j. Then $m_i(\lambda^j)$ denotes the number of cycles in λ^j of length i, and $C_{\boldsymbol{\lambda}}$ denotes the elements in $\mathfrak{S}_{n,r}$ with cycle type $\boldsymbol{\lambda}$.

Example 2. Let $\omega \in \mathfrak{S}_5$ be the permutation specified by $\omega = [45132] = (143)(25)$ in one-line and cycle notation. Let $\tau = (3, 0, 1, 1, 3)$. The colored permutation $(\omega, \tau) \in \mathfrak{S}_{5,4}$ is completely specified by the function $f : [5]^4 \to [5]^4$ satisfying $f(i^0) = \omega(i)^{\tau(\omega(i))}$. Hence in two-line, one-line, and cycle notations we have:

$$(\omega,\tau)=\begin{pmatrix}1^0 & 2^0 & 3^0 & 4^0 & 5^0\\ 4^1 & 5^3 & 1^3 & 3^1 & 2^0\end{pmatrix}=[4^15^31^33^12^0]=(1^34^13^1)(2^05^3).$$

It has a 3-cycle of color 1 and a 2-cycle of color 3. Its cycle type is thus $(\emptyset,(3),\emptyset,(2))$.

The conjugacy classes of $\mathfrak{S}_{n,r}$ are well understood in terms of cycle type.

Proposition 3. *[12, Theorem 4.2.8, Lemmas 4.2.9-4.2.10] The conjugacy classes of $\mathfrak{S}_{n,r}$ are given by $C_{\boldsymbol{\lambda}}$, where $\boldsymbol{\lambda}$ is an r-partition of n.*

In the special case $r=2$, the hyperoctahedral group $\mathfrak{S}_{n,2}=B_n$ can be viewed as the group of signed permutations, i.e., bijections on $[\pm n]=\{\pm 1,\pm 2,\dots,\pm n\}$ where positive and negative elements respectively correspond to colors 0 and 1. In this case, we will denote bipartitions as (λ,μ) and the corresponding conjugacy class as $C_{\lambda,\mu}$.

The type B descent statistic, whose distribution is the subject of Section 4, is then given by the following definition, with the convention that $\omega(0)=0$. See [2, Proposition 8.1.2]:

$$\mathrm{des}_B(\omega)=|\{i\in\{0\}\cup[n-1]\mid\omega(i)>\omega(i+1)\}|. \tag{2.1}$$

Two other B_n-statistics that will be useful for illustrative purposes are inv and negsum, defined by (see [2, Equation 8.1 and page 308])

$$\mathrm{inv}=|\{(i,j)\in[n]\times[n]\mid i<j \text{ and } \omega(i)>\omega(j)\}|,\quad \mathrm{negsum}(\omega)=\sum_{i\in[n],\omega(i)<0}\omega(i). \tag{2.2}$$

Also, the Coxeter length statistic inv_B is given by the formula [2, Proposition 8.1.1]

$$\mathrm{inv}_B(\omega)=\mathrm{inv}(\omega)-\mathrm{negsum}(\omega) \tag{2.3}$$

We will use the des, inv, and negsum statistics as running examples to illustrate our work. Results on inv and negsum naturally lead to statements about inv_B, illustrating the more general fact that our results behave nicely with statistics that are defined as linear combinations of other statistics.

Throughout this paper, we will use $\Pr_{\mathfrak{S}_{n,r}}$ and $\Pr_{\boldsymbol{\lambda}}$ to denote the probabilities in $\mathfrak{S}_{n,r}$ and $C_{\boldsymbol{\lambda}}$ (with respect to the uniform measure). We similarly use $\mathbb{E}_{\mathfrak{S}_{n,r}}$ and $\mathbb{E}_{\boldsymbol{\lambda}}$ for the expected values on the corresponding probability spaces.

3 Moments of colored permutation statistics

In this section, we will discuss the techniques involved in establishing the independence result, Theorem 12, and the polynomiality result, Theorem 20.

3.1 Colored permutation constraints

In this section, we will extend the notion of a permutation constraint from the setting of the symmetric group to the setting of colored permutations. We compare this to [4, Definition 7.1] as well as to the work of Hamaker and Rhoades [11], where permutation constraints are called *partial permutations*. A colored permutation constraint will have two components (K, κ). The first, K, will constrain a permutation ω by specifying a subset of its values. The second component, κ, will assign colors to these values.

Definition 4. Let $K = \{(i_1, j_1), \ldots, (i_m, j_m)\}$ consist of distinct ordered pairs, where $i_h, j_h \in [n]$. Let $\kappa : \{j_1, \ldots, j_m\} \to \mathbb{Z}_r$. We call the pair (K, κ) a *colored permutation constraint*, and we call m the size of the constraint. For $(\omega, \tau) \in \mathfrak{S}_{n,r}$, we say that ω satisfies K if $\omega(i_h) = j_h$ for all $h \in [m]$, and we say τ satisfies κ if $\tau(x) = \kappa(x)$ for all $x \in \{j_1, \ldots, j_m\}$. Finally we say that $(\omega, \tau) \in \mathfrak{S}_{n,r}$ satisfies (K, κ) if ω satisfies K and τ satisfies κ. We will sometimes denote a constraint as a set of ordered pairs

$$(K, \kappa) = \left\{ \left(i_h^0, j_h^{\kappa(j_h)} \right) \right\}_{h=1}^{m}$$

recording these conditions, and we sometimes omit set braces for brevity.

Recall from Section 2 that we view the hyperoctahedral group $\mathfrak{S}_{n,2} = B_n$ as the group of signed permutations. In this case, a constraint is of the form $(K, \kappa) = \{(i_h, \kappa(j_h) j_h)\}_{h=1}^{m}$, where $\kappa(j_h) = \pm 1$.

Definition 5. Let $\mathcal{C}$ be a set of colored permutation constraints. The *size* of $\mathcal{C}$ is defined as the maximum size over all constraints contained in $\mathcal{C}$, namely,

$$\operatorname{size}(\mathcal{C}) = \max_{(K,\kappa) \in \mathcal{C}} |K|.$$

Recall that a colored permutation statistic is simply a map $X : \mathfrak{S}_{n,r} \to \mathbb{R}$. We now introduce decompositions of colored permutation statistics as weighted sums of indicator functions corresponding to colored permutation constraints.

Definition 6. A colored permutation statistic X is *realizable* over a constraint set of size m if there exists a set of constraints $\mathcal{C}$ of size m and weights $\operatorname{wt}(K, \kappa) \in \mathbb{R}$ such that $X = \sum_{(K,\kappa) \in \mathcal{C}} \operatorname{wt}(K, \kappa) I_{(K,\kappa)}$, where $I_{(K,\kappa)}$ is the indicator function that a permutation satisfies the constraint (K, κ). Note that in general, the decomposition $\sum_{(K,\kappa) \in \mathcal{C}} \operatorname{wt}(K, \kappa) I_{(K,\kappa)}$ is not unique.

Example 7. Many statistics have a natural decomposition in terms of constraints. For the statistics defined on B_n given in Section 2, we have

$$\operatorname{des}_B = \sum_{j \in [n]} I_{(1,-j)} + \sum_{i \in [n-1]} \sum_{\substack{j_1, j_2 \in [\pm n] \\ j_1 < j_2}} I_{(i,j_2),(i+1,j_1)},$$

$$\text{inv} = \sum_{\substack{i_1,i_2\in[n]\\ i_1<i_2}} \sum_{\substack{j_1,j_2\in[\pm n]\\ j_1<j_2}} I_{(i_1,j_2),(i_2,j_1)},$$

$$\text{negsum} = \sum_{i\in[n]} \sum_{j\in[n]} (-j) I_{(i,-j)}.$$

This shows that des_B and inv are realizable over constraint sets of size 2, and negsum is realizable over a constraint set of size 1. Since inv_B is the difference of inv and negsum, we also see that inv_B is realizable over a constraint set of size 2.

Remark 8. We say that (K,κ) is *well-defined* if all of the $i_h \in [n]$ are distinct, and all of the $j_h \in [n]$ are distinct. Observe that if (K,κ) is not well-defined, then $I_{(K,\kappa)}$ is identically 0 on $\mathfrak{S}_{n,r}$, and hence can be omitted from any set realizing a given statistic. Consequently, we are only interested in well-defined constraints.

3.2 Independence of moments

In this section, we outline the steps leading to the proof of our independence result, Theorem 12. Our methods follow the strategy of [4, Section 7]. Proofs appear in [5].

Definition 9. A colored permutation constraint (K,κ) is *acyclic* if K is well-defined and the graph $G(K,\kappa)$, with vertex set $V = [n]$ and directed edge set K, does not contain any cycles. Observe that in this case, $G(K,\kappa)$ consists of a set of paths.

As a non-example, the size one constraint induced by $I_{(i,-i)}$ from Example 7 is *not* acyclic.

Lemma 10. *(Compare to $\mathfrak{S}_n$, cf. [4, Lemma 7.15]) Consider the group of all r-colored permutations $\mathfrak{S}_{n,r}$. Let $C_{\boldsymbol{\lambda}}$ be a conjugacy class of $\mathfrak{S}_{n,r}$. Let (K,κ) be a well-defined colored permutation constraint of size $m \leq n$, and suppose that each partition in $\boldsymbol{\lambda}$ has all parts of size at least $m+1$. If K is acyclic, then*

$$\Pr_{\boldsymbol{\lambda}}[(\omega,\tau) \text{ satisfies } (K,\kappa)] = \tfrac{1}{(n-1)(n-2)\cdots(n-m)} \cdot \tfrac{1}{r^m}.$$

If K is not acyclic, then $\Pr_{\boldsymbol{\lambda}}[(\omega,\tau)$ *satisfies* $(K,\kappa)] = 0$.

One essential observation in proving Lemma 10 is that the permutation and the coloring can be treated independently.

Lemma 10 can be used to analyze the first moment of a statistic $\mathbb{E}_{\boldsymbol{\lambda}}[X]$ by expressing X in terms of constraints. We need one final lemma to accommodate arbitrary moments $\mathbb{E}_{\boldsymbol{\lambda}}[X^k]$ in the main result of this section, Theorem 12.

Lemma 11. *Let $X_1, X_2 : \mathfrak{S}_{n,r} \to \mathbb{R}$ be realizable over constraint sets of size m_1, m_2 respectively. Then X_1X_2 is realizable over a constraint set of size $m_1 + m_2$. In particular, for any integer $k \geq 1$, we have that X_1^k is realizable over a constraint set of size km_1.*

This leads to the main theorem of this section.

Theorem 12. *Suppose $X : \mathfrak{S}_{n,r} \to \mathbb{R}$ is realizable over a constraint set of size m. For any $k \geq 1$, the kth moment $\mathbb{E}_{\boldsymbol{\lambda}}[X^k]$ coincides on all conjugacy classes $C_{\boldsymbol{\lambda}}$ with no cycles of length $1, 2, \ldots, mk$.*

Note that the above theorem makes precise the notion of "short" cycles. In particular, if we are considering the kth moment of a statistic X realizable over a constraint set of size m, then the "short" cycles are the ones of length at most mk.

Remark 13. Note that a colored permutation (ω, τ) is itself a colored permutation constraint of size n. Hence, we can express any statistic X using size n constraints. Additionally, one can show that if X is realizable over a constraint set of size m, then it is also realizable over a constraint set of size m' for $m \leq m' \leq n$. For the full strength of our results, we are primarily interested in minimizing m, and we call this minimum possible value the *size* of X.

Remark 14. The arguments leading to the proof of Theorem 12 have practical applications for computing moments of statistics on those conjugacy classes. For example, consider negsum on B_n, which can be expressed as negsum $= \sum_{i\in[n]} \sum_{j\in[n]} (-j) I_{(i,-j)}$. Note that here all constraints are acyclic except for $(i, -i)$. One can then show that for any bi-partition (λ, μ) of n where all the parts have size at least 2,

$$\begin{aligned}\mathbb{E}_{\lambda,\mu}[\text{negsum}] &= -\sum_{i\in[n]} i \cdot \mathbb{E}_{\lambda,\mu}[I_{(i,-i)}] - \sum_{i\in[n]} \sum_{j\in[n]\setminus i} j \cdot \mathbb{E}_{\lambda,\mu}[I_{(i,-j)}] \\ &= -\frac{1}{(n-1)\cdot 2} \cdot \sum_{i\in[n]} \sum_{j\in[n]\setminus i} j = -\frac{1}{2}\binom{n+1}{2}.\end{aligned}$$

More generally, one can use negsum $= \sum_{i\in[n]} \sum_{j\in[n]} (-j) I_{(i,-j)}$ to express negsum^k using constraints of size at most k. On conjugacy classes where all parts have size at least $k+1$, a similar approach as the one above can be used to calculate $\mathbb{E}_{\lambda,\mu}[\text{negsum}^k]$.

3.3 Symmetric colored permutation statistics

We now turn to extending the notion of a *symmetric* permutation statistic from [4] to the colored setting. We begin with some definitions.

Definition 15. The *support* of a colored permutation constraint $(K, \kappa) = \{(i_r^0, j_r^{\kappa(j_r)})\}_{r=1}^m$ is $\text{supp}(K, \kappa) = \{i_1, \ldots, i_m, j_1, \ldots, j_m\}$. We emphasize that $\text{supp}(K, \kappa)$ is a set and not a multiset.

Definition 16. Consider any colored permutation constraint (K,κ) with support given by $a_1 < \cdots < a_s$. For any order-preserving injection $f : \{a_1, \ldots, a_s\} \to [n]$, define $f(K,\kappa)$ to be the constraint

$$f(K,\kappa) = (\ \{(f(i_1), f(j_1)), \ldots, (f(i_m), f(j_m))\}, \{\kappa(f(j_1)) = k_1, \ldots, \kappa(f(j_m)) = k_m\}\ ).$$

Definition 17. A set of colored permutation constraints $\mathcal{C}$ is *symmetric* if for all $(K,\kappa) \in \mathcal{C}$ and any order-preserving injection $f : \operatorname{supp}(K,\kappa) \to [n]$, we have $f(K,\kappa) \in \mathcal{C}$. A statistic X is *symmetric* if it has the form $X = \sum_{(K,\kappa)\in\mathcal{C}} I_{(K,\kappa)}$ for some symmetric $\mathcal{C}$.

Many statistics naturally satisfy this condition.

Example 18. Consider the statistic inv on B_n that can be realized as

$$\mathrm{inv} = \sum_{\substack{i,j\in[n] \\ i<j}} \sum_{\substack{k,\ell\in[\pm n] \\ k<\ell}} I_{\{(i,\ell),(j,k)\}}.$$

We denote the constraint set $\mathcal{C}$. If $k, \ell > 0$, then for any order preserving $f : \{i, j, k, \ell\} \to [n]$, we see that $\{(f(i), f(\ell)), (f(j), f(k))\} \in \mathcal{C}$. Note that the set $\{i, j, k, \ell\}$ need not consist of four distinct elements. If $k < 0$ and $\ell > 0$, we see that for any order-preserving $f : \{i, j, |k|, \ell\} \to [n]$, we have $\{(f(i), f(\ell)), (f(j), -f(|k|))\} \in \mathcal{C}$. The same argument holds for the case when $k, \ell < 0$.

Definition 19. Fix $n_0 \geq 2$. Let $X = \sum_{(K,\kappa)\in\mathcal{C}} I_{(K,\kappa)}$ be a symmetric statistic defined on $\mathfrak{S}_{n_0,r}$. Define the *r-colored symmetric extensions* of X to be the statistics $X_n = \sum_{(K,\kappa)\in\mathcal{C}_n} I_{(K,\kappa)}$ on $\mathfrak{S}_{n,r}$ with $\mathcal{C}_n$ defined as follows:

- If $n \leq n_0$, then $\mathcal{C}_n$ contains all $(K,\kappa) \in \mathcal{C}$ with support contained in $[n]$.
- If $n \geq n_0$, then $\mathcal{C}_n$ is the set of all $f(K,\kappa)$ where $(K,\kappa) \in \mathcal{C}$ and $f : [n_0] \to [n]$ is order-preserving.

Observe that by construction, each X_n is a symmetric statistic. We emphasize here that r is kept constant throughout this construction.

Many statistics can be constructed in this manner. For example, if $\mathcal{C}$ is the set of constraints for inv on B_4, then this results in the inv statistics on all B_n. In general, the moments of these statistics satisfy the following polynomial property.

Theorem 20. *Fix $r \geq 1$. Let (X_n) be the symmetric extensions of a symmetric statistic $X = X_{n_0}$ on $\mathfrak{S}_{n,r}$ induced by a constraint set $\mathcal{C}$ of size m. There exists a polynomial $p_X(n)$ of degree at most mk depending only on X such that $p_X(n) = \mathbb{E}_{\boldsymbol{\lambda}_n}[X_n^k]$ for any r-partition $\boldsymbol{\lambda}_n$ of n where all $\lambda_n^{(j)}$ have parts of size at least $mk + 1$.*

Note that one can show this polynomiality property for other statistics that are not symmetric extensions. The key requirement is that the weights for the various I_K behave in a way that allows us to divide by the denominators that result from applying Lemma 10.

4 Descents in conjugacy classes of hyperoctahedral groups

In this section we discuss the techniques involved in establishing our central limit theorem for descents in conjugacy classes of B_n that do not have short cycles. The descent statistic on B_n was defined in Eqn. (2.1). Let $(\lambda(\omega), \mu(\omega))$ denote the cycle type of $\omega \in B_n$, and let $m_i(\lambda)$ denote the number of parts of λ equal to i. While Reiner [15] uses a different notion of descents, the generating function [9, Theorem 5.3]

$$\sum_{\omega \in B_n} t^{\mathrm{des}_B(\omega)} \prod_i x_i^{m_i(\lambda(\omega))} y_i^{m_i(\mu(\omega))} \tag{4.1}$$

is unaffected.

Following Fulman [8], our approach involves examining the generating function given in (4.1), which allows us to analyze the generating function for des_B on a conjugacy class. We then relate this with the generating function for descents on all of B_n. In the case where there are no short cycles in $C_{\lambda,\mu}$, we will ultimately conclude that certain moments of des_B agree on $C_{\lambda,\mu}$ and B_n, and this in turn enables us to use the method of moments with a known central limit theorem of Chow and Mansour for des_B on B_n given below.

Proposition 21. *[6, Thm 3.4] Let X_n be des_B defined on B_n. Then X_n has mean $n/2$ and variance $(n+1)/12$, and as $n \to \infty$, the standardized random variable $(X_n - n/2)/\sqrt{(n+1)/12}$ converges to a standard normal distribution.*

We will need the well-known generating function of des_B over all of B_n.

Proposition 22. *[14, Eqn. (13.3)] Let $B_n(t) = \sum_{\omega \in B_n} t^{\mathrm{des}_B(\omega)+1}$. Then*

$$\frac{B_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 1} (2k-1)^n t^k.$$

We now analyze (4.1), which will allow us to derive an expression for the generating function of des_B on a conjugacy class $C_{\lambda,\mu}$. The following expression features prominently in our analysis.

Definition 23. [15] Let $\mu(d)$ be the number-theoretic Möbius function. Define, for nonnegative integers r and m,

$$N(r, 2m) = \frac{1}{2m} \sum_{\substack{d \mid m \\ d \text{ odd}}} \mu(d) \left(r^{m/d} - 1 \right).$$

Reiner [15, Theorem 4.1, Theorem 4.2] shows that $N(2k-1, 2m)$ must be a nonnegative integer for all $k, m \geq 1$.

For a fixed bi-partition (λ, μ) of n, we use the special case of [15, Theorem 4.1] appearing in [9, Theorem 5.3] to derive the following expressions for the generating function $B_{\lambda,\mu}(t) = \sum_{\omega \in C_{\lambda,\mu}} t^{\mathrm{des}_B(\omega)+1}$ of descents over the conjugacy class $C_{\lambda,\mu}$.

Proposition 24. *Let $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \ldots)$ and $\mu = (1^{m_1(\mu)}, 2^{m_2(\mu)}, \ldots)$. Then the following are equal to $B_{\lambda,\mu}(t)/(1-t)^{n+1}$:*

$$t\delta_{((1^n),\varnothing)} + \sum_{k\geq 2} t^k \left(\prod_{i\geq 1} \binom{N(2k-1,2i)}{m_i(\mu)} \prod_{i\geq 2} \binom{N(2k-1,2i)+m_i(\lambda)-1}{m_i(\lambda)}\right) \binom{N(2k-1,2)+m_1(\lambda)}{m_1(\lambda)}$$
$$= t\delta_{((1^n),\varnothing)} + \sum_{k\geq 2} t^k \frac{m_1(\lambda)+k-1}{k-1} \prod_{i\geq 1} \binom{N(2k-1,2i)-1+m_i(\lambda)}{m_i(\lambda)} \binom{N(2k-1,2i)}{m_i(\mu)}.$$

Here $\delta_{((1^n),\varnothing)}$ is the Kronecker delta which is 1 for the conjugacy class $\lambda, \mu = ((1^n), \varnothing)$, and zero otherwise.

By solving for $B_{\lambda,\mu}$ and extracting the coefficient of t^d, we also obtain the following corollary.

Corollary 25. *The number of permutations $\omega \in B_n$ that are of cycle type (λ, μ) and have $d-1$ descents is*

$$\sum_{k=1}^{d} (-1)^{d-k} \binom{n+1}{d-k} \binom{m_1(\lambda)+k-1}{m_1(\lambda)} \prod_{i\geq 2} \binom{N(2k-1,2i)+m_i(\lambda)-1}{m_i(\lambda)} \prod_{i\geq 1} \binom{N(2k-1,2i)}{m_i(\mu)}.$$

We now give an elegant analogue of a result of Fulman [8, Proof of Theorem 2], which will relate $B_{\lambda,\mu}(t)$ and $B_n(t)$.

Theorem 26. *Let $C_{\lambda,\mu}$ be the conjugacy class of B_n indexed by the bi-partition (λ, μ) of n, let $B_n(t) = \sum_{\omega\in B_n} t^{\mathrm{des}_B(\omega)+1}$, and let $B_{\lambda,\mu}(t) = \sum_{\omega\in C_{\lambda,\mu}} t^{\mathrm{des}_B(\omega)+1}$. Then*

$$\frac{B_{\lambda,\mu}(t)}{|C_{\lambda,\mu}|} = \frac{B_n(t)}{2^n n!} + \frac{1-t}{2n} \frac{B_{n-1}(t)}{2^{n-1}(n-1)!}[m_1(\lambda)^2 - m_1(\mu)^2] + (1-t)^2 g(t),$$

where $g(t)$ is some polynomial in t. Furthermore, when all cycles in $C_{\lambda,\mu}$ have length larger than $2k$,

$$\frac{B_{\lambda,\mu}(t)}{|C_{\lambda,\mu}|} = \frac{B_n(t)}{2^n n!} + (1-t)^{k+1} h(t),$$

where $h(t)$ is some polynomial in t.

The latter case allows us to obtain the following result involving moments of des_B on B_n and $C_{\lambda,\mu}$.

Corollary 27. *Let $C_{\lambda,\mu}$ be the conjugacy class of B_n indexed by the bi-partition (λ, μ) of n. The kth moment of des_B in $C_{\lambda,\mu}$ is equal to the kth moment of des_B in B_n if all cycles in $C_{\lambda,\mu}$ have length greater than $2k$.*

The main result of this section, Theorem 28, now follows by applying Corollary 27, the method of moments, and the asymptotic normality theorem for descents in B_n given in Proposition 21.

Theorem 28. *For every $n \geq 1$, pick a conjugacy class C_{λ_n,μ_n} in B_n indexed by the bi-partition (λ_n, μ_n) of n, where $\lambda_n = (1^{m_1(\lambda_n)}, 2^{m_2(\lambda_n)}, \ldots)$ and $\mu_n = (1^{m_1(\mu_n)}, 2^{m_2(\mu_n)}, \ldots)$. Define X_n to be des_B on C_{λ_n,μ_n}. Suppose that for all i, $m_i(\lambda_n) \to 0$ and $m_i(\mu_n) \to 0$ as $n \to \infty$. For sufficiently large n, X_n has mean $n/2$ and variance $(n+1)/12$. Furthermore, as $n \to \infty$, the random variable $(X_n - n/2)/\sqrt{(n+1)/12}$ converges to a standard normal distribution.*

5 Conclusion

In this paper, we have introduced a notion of constraints and size for any colored permutation statistic $X : \mathfrak{S}_{n,r} \to \mathbb{R}$, and we have used this framework to study the moments of X on conjugacy classes $C_{\boldsymbol{\lambda}}$. In particular, we have established that for a statistic of size m, the kth moment on $C_{\boldsymbol{\lambda}}$ is independent of conjugacy class $C_{\boldsymbol{\lambda}}$ when all parts of the partitions in $\boldsymbol{\lambda}$ have length at least $mk + 1$. For statistics on $\mathfrak{S}_{n,r}$ that can be expressed as symmetric extensions, these moments are polynomials in n. Our results directly generalize those in [4] on $\mathfrak{S}_n$. Given the numerous connections to [11], one natural problem is the following.

Problem 29. Use the representation theory of B_n and $\mathfrak{S}_{n,r}$ to establish analogues of the results in [11].

Finally, we note that $\mathfrak{S}_n$ and B_n are respectively the type A and type B Coxeter groups. The following is a natural problem to consider next.

Problem 30. Establish analogues of the results in this paper for the type D Coxeter groups.

It would also be of interest to establish analogous results for (irreducible) complex reflection groups.

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