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All Kronecker coefficients are reduced Kronecker coefficients

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Abstract. We settle the question of where exactly do the reduced Kronecker coefficients lie on the spectrum between the Littlewood-Richardson and Kronecker coefficients by showing that every Kronecker coefficient of the symmetric group is equal to a reduced Kronecker coefficient by an explicit construction. This implies the equivalence of a question by Stanley from 2000 and a question by Kirillov from 2004 about combinatorial interpretations of these two families of coefficients. Moreover, as a corollary, we deduce that deciding the positivity of reduced Kronecker coefficients is NP-hard, and computing them is #P-hard under parsimonious many-one reductions.

Keywords: Kronecker coefficients, reduced Kronecker coefficients, representation theory, symmetric group, general linear group

1 Introduction

The *Kronecker coefficients* $\mathtt{k}(\lambda, \mu, \nu)$ of the symmetric group S_n are some of the most classical, yet largely mysterious, quantities in Algebraic Combinatorics and Representation Theory. The Kronecker coefficient is the multiplicity of the irreducible S_n representation $\mathbb{S}_\nu$ in the tensor product $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$ of two other irreducible S_n representations. Murnaghan defined them in 1938 as an analogue of the Littlewood-Richardson coefficients $c^{\lambda}_{\mu\nu}$ of the general linear group GL_N, which are the multiplicities of the irreducible Weyl modules V_λ in the tensor products $V_\mu \otimes V_\nu$. Yet, the analogy has not translated far into their properties. The Littlewood-Richardson coefficients have a beautiful positive combinatorial interpretation and their positivity is "easy" to decide, formally it is in P. However, positive combinatorial formulas for the Kronecker coefficients have eluded us so far, see Section 1.2, and their positivity is hard to decide.

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The *reduced Kronecker coefficients* $\overline{\mathtt{k}}(\alpha,\beta,\gamma)$ are defined as the stable limit of the ordinary Kronecker coefficients

$$\overline{\mathtt{k}}(\alpha,\beta,\gamma) := \lim_{n\to\infty} \mathtt{k}(\,(n-|\alpha|,\alpha),\ (n-|\beta|,\beta),\ (n-|\gamma|,\gamma)\,).$$

These coefficients are called *extended Littlewood-Richardson numbers* in [13], since in the special case when $|\alpha| = |\beta| + |\gamma|$ we have $\overline{\mathtt{k}}(\alpha,\beta,\gamma) = c^{\alpha}_{\beta,\gamma}$, the Littlewood-Richardson coefficient. Problem 2.32 in [13] asks for a combinatorial interpretation of $\overline{\mathtt{k}}(\alpha,\beta,\gamma)$. As such they have been considered as an intermediate, an interpolation, between the Littlewood-Richardson and Kronecker coefficients. They have been an object of independent interest, see [16, 17, 4, 27, 13, 3, 2, 5, 15, 24, 10, 21, 18, 19], and considered better behaved than the ordinary Kronecker coefficients.

This is, however, not the case. As we show, every Kronecker coefficient is equal to an explicit reduced Kronecker coefficient of not much larger partitions, and in particular:

Theorem 1. *For all partitions λ, μ, ν of equal sizes, we have*

$$\mathtt{k}(\lambda,\mu,\nu) \ = \ \overline{\mathtt{k}}\big(\,\nu_1^{\ell(\lambda)}+\lambda,\ \nu_1^{\ell(\mu)}+\mu,\ (\nu_1^{\ell(\lambda)+\ell(\mu)},\nu)\,\big).$$

Here $a^b := (\underbrace{a,\ldots,a}_{b \text{ many}})$ and $(\nu_1^b,\nu) := (\underbrace{\nu_1,\ldots,\nu_1}_{b \text{ many}},\nu_1,\nu_2,\nu_3,\ldots)$.

This implies that in a very strong sense, on the spectrum between Littlewood-Richardson and Kronecker coefficients, the reduced Kronecker coefficients are at the same point as the ordinary Kronecker coefficients. In particular, Theorem 1 implies that Problem 2.32 in [13] is equivalent to Problem 10 in [26]: Finding a combinatorial interpretation for the Kronecker coefficient or for the reduced Kronecker coefficient are the same problem. Formally, Conjecture 9.1 and 9.4 in [20] are the same. Our result can be interpreted in a positive or in a negative way. On the one hand, the reduced Kronecker coefficients cannot be easier to understand than the ordinary Kronecker coefficients. On the other hand, understanding the reduced Kronecker coefficients is sufficient to understand all ordinary Kronecker coefficients.

We thus settle the conjecture from [21, §4.4] on the hardness of deciding positivity:

Corollary 1. *Given α,β,γ in unary, deciding if $\overline{\mathtt{k}}(\alpha,\beta,\gamma) > 0$ is* NP*-hard.*

Moreover, by the same immediate argument it is now clear that computing the reduced Kronecker coefficient is strongly #P-hard under parsimonious many-one reductions (the argument in [21] gives only the #P-hardness under Turing reductions).

1.1 Background and definitions

We refer to [12, 25, 23] for basic definitions and properties from algebraic combinatorics and representation theory, and employ the following notation. For a *partition*

$\lambda = (\lambda_1, \lambda_2, \ldots)$ of n, denoted $\lambda \vdash n$, its size is denoted $|\lambda| := \sum_i \lambda_i$ and length $\ell(\lambda) = \max\{i \mid \lambda_i > 0\}$. We write λ' do denote the *transpose* partition, i.e., the partition that arises from reflecting the Young diagram at the main diagonal. Formally, $\lambda'_j := \max\{i \mid \lambda_i \geq j\}$. We add partitions row-wise: $(\lambda + \mu)_i = \lambda_i + \mu_i$. We define $\lambda \diamond \mu := (\lambda' + \mu')'$, adding partitions column-wise as Young diagrams. The Specht modules $\mathbb{S}_\lambda$ for $\lambda \vdash n$ are the irreducible representation of the symmetric group S_n, see [12, 25, 23].

The *Kronecker coefficient* $\mathtt{k}(\lambda, \mu, \nu)$ is the structure constant[1] defined as

$$\mathbb{S}_\nu \otimes \mathbb{S}_\mu = \bigoplus_\lambda \mathbb{S}_\lambda^{\oplus \mathtt{k}(\lambda,\mu,\nu)}$$

via Specht modules, giving that $\mathtt{k}(\lambda, \mu, \nu)$ is a nonnegative integer. Yet the problem of finding a combinatorial interpretation of $\mathtt{k}(\lambda, \mu, \nu)$ is wide open [26, 9, 22]. The Kronecker coefficients were defined by Murnaghan [16] in 1938 as the analogues of the *Littlewood-Richardson coefficients* $c^\lambda_{\mu\nu}$, which are the structure constants in the ring of irreducible GL_N representations, the Weyl modules V_λ, given as $V_\mu \otimes V_\nu = \bigoplus_\lambda V_\lambda^{\oplus c^\lambda_{\mu\nu}}$. Some simple properties, see [12, 23] include the transposition invariance $\mathtt{k}(\lambda, \mu, \nu) = \mathtt{k}(\lambda', \mu', \nu)$ and permutation of the terms. We define $\mathtt{k}'(\lambda, \mu, \nu) := \mathtt{k}(\lambda', \mu', \nu') = \mathtt{k}(\lambda', \mu, \nu) = \mathtt{k}(\lambda, \mu', \nu) = \mathtt{k}(\lambda, \mu, \nu')$. It is known that $\mathtt{k}(\lambda, \mu, \nu) = 0$ if $\ell(\lambda) > \ell(\mu) \cdot \ell(\nu)$ [6], which also follows by combining $\mathtt{k}(\lambda, \mu, \nu) = \mathtt{k}(\lambda, \mu', \nu')$ with Lemma 3. We define the *stable range* as the set of triples (λ, μ, ν) that satisfy $\mathtt{k}(\lambda, \mu, \nu) = \mathtt{k}\big(\lambda + (i),\ \mu + (i),\ \nu + (i)\big)$ for all $i \geq 0$. The *reduced Kronecker coefficient* is defined as this limit value:

$$\overline{\mathtt{k}}(\alpha, \beta, \gamma) := \lim_{n \to \infty} \mathtt{k}\big(\,(n - |\alpha|, \alpha),\ (n - |\beta|, \beta),\ (n - |\gamma|, \gamma)\,\big)$$

for arbitrary partitions α, β, γ (in particular, we do *not* require $|\alpha| = |\beta| = |\gamma|$). When $|\alpha| = |\beta| + |\gamma|$, then $\overline{\mathtt{k}}(\alpha, \beta, \gamma) = c^\alpha_{\beta,\gamma}$. For a full list of definitions and properties we refer to the full version of this paper [11].

1.2 Related work

The Littlewood-Richardson (LR) coefficients can be computed by the Littlewood-Richardson rule, stated in 1934 and proven formally about 40 years later. It says that $c^\lambda_{\mu\nu}$ is equal to the number of LR tableaux of shape λ/μ and content ν. The apparent analogy in definitions motivates the community to search for such interpretations for the Kronecker coefficients. Interest in efficient ways to compute $\mathtt{k}(\lambda, \mu, \nu)$ and $\overline{\mathtt{k}}(\alpha, \beta\gamma)$ dates back at least to Murnaghan [16]. Specific interest in nonnegative combinatorial interpretations of $\mathtt{k}(\lambda, \mu, \nu)$ can be found in [Lascoux 1979, Garsia-Remmel 1985] and

[1] In the combinatorics literature these coefficients have usually been denoted by g, e.g. $g(\lambda, \mu, \nu)$, but here we use $\mathtt{k}$ to avoid overlap with the notation used for the representation theory of GL_N.

was formulated clearly again by Stanley as Problem 10 in his list "Open Problems in Algebraic Combinatorics" [26]. See also [22] for a detailed discussion on this topic.

Despite its natural and fundamental nature and the variety of efforts, this question has seen relatively little progress. The state of the art is combinatorial interpretations for specific classes of partitions (ν being a hook, or μ, ν being two-row partitions, etc). It was shown by Murnaghan [17] that the reduced Kronecker coefficients generalize the Littlewood-Richardson coefficients as

$$\overline{\mathtt{k}}(\alpha, \beta, \gamma) = c^{\alpha}_{\beta\gamma} \quad \text{for} \quad |\alpha| = |\beta| + |\gamma|,$$

which motivates Kirillov's naming of $\overline{\mathtt{k}}$ as "extended Littlewood-Richardson numbers". This relationship and other properties have motivated an independent interest in the reduced Kronecker coefficients as intermediates between Littlewood-Richardson and ordinary Kronecker coefficients. Some special cases of combinatorial interpretations can be derived from the existing ones for the ordinary Kronecker coefficients. In [5] a combinatorial interpretation was given when μ, ν are rectangles and λ is one row. A combinatorial interpretation of $\overline{\mathtt{k}}(\alpha, \beta, \gamma)$ in the subcase where $\ell(\alpha) = 1$ was obtained in [1]. Methods to compute them have been discussed in [16, 17] and have been developed in a series of papers, see [3, 2, 18, 19]. As observed in [2] the reduced Kronecker coefficients are also the structure constants for the ring of so called character polynomials. The reduced Kronecker coefficients are a special case of a more general stability phenomenon that if $\mathtt{k}(i\alpha, i\beta, i\gamma) = 1$ for all i, then $\mathtt{k}(\lambda + N\alpha, \mu + N\beta, \nu + N\gamma)$ stabilizes as $N \to \infty$ as seen in [24, 27].

The Kronecker coefficients can be expressed as a small alternating sum of reduced Kronecker coefficients, and reduced Kronecker coefficients are certain sums of ordinary Kronecker coefficients for smaller partitions, see [3]. These relationships showed that reduced Kronecker coefficients are also #P-hard to compute, see [21]. However, these relations did not imply that deciding positivity of reduced Kronecker coefficients is NP-hard.

It is important to note that deciding if $c^{\lambda}_{\mu\nu} > 0$ is in P, since they count integer points in a polytope that has an integral vertex whenever it is nonempty, a consequence of Knutson-Tao's proof of the saturation property: $c^{N\lambda}_{N\mu,N\nu} > 0 \iff c^{\lambda}_{\mu\nu} > 0$. The Kronecker coefficients do not satisfy the saturation property, because $\mathtt{k}(2^2, 2^2, 2^2) = 1$, but $\mathtt{k}(1^2, 1^2, 1^2) = 0$. Until recently it was believed that the reduced Kronecker coefficients have the saturation property: It was conjectured in [13, 14] that if $\overline{\mathtt{k}}(N\alpha, N\beta, N\gamma) > 0$ for some $N > 0$, then $\overline{\mathtt{k}}(\alpha, \beta, \gamma) > 0$. This was disproved in [21] in 2020 and moved the reduced Kronecker coefficients away from the Littlewood-Richardson coefficients on that spectrum.

2 Setting up the proof of Theorem 1

We discovered Theorem 1 using the natural interpretation of $\mathtt{k}(\lambda,\mu,\nu)$ via the general linear group, see §3, and the relationship with 3-dimensional binary contingency arrays. We set the proof up in this section, reducing to a more general Theorem 2, which has a short proof via GL_N and two short, self-contained proofs using basic symmetric function techniques. The complete proofs are available in [11].

Lemma 1. *Let λ,μ,ν be partitions with $\ell(\lambda)\le l$, $\ell(\mu)\le m$. Then*

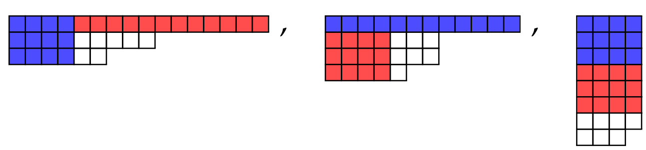
$$\mathtt{k}(\lambda,\mu,\nu) \;=\; \mathtt{k}(\,m^l+\lambda,\; l^m+\mu,\; 1^{lm}+\nu\,).$$

The following Lemma 2 is proved by applying Lemma 1 twice, in different directions.

Lemma 2. *Let λ, μ, ν be partitions of the same size, and let $l\ge\ell(\lambda)$, $m\ge\ell(\mu)$ and $c\ge\nu_1$. Let $d=(m+1)c$, $e=(l+1)c$. Then*

$$\mathtt{k}(\lambda,\mu,\nu) \;=\; \mathtt{k}\big(\,(d)\diamond(c^l+\lambda),\; (e)\diamond(c^m+\mu),\; c^{l+m+1}\diamond\nu\,\big).$$

In lieu of a proof we illustrate this by example with $\lambda=(5,2)$, $\mu=(3,3,1)$ and $\nu=(4,3)$, with $l=2$, $m=3$ and $c=4$. The red boxes are the addition from the first application of Lemma 1 and the blue boxes are the second application.



Theorem 2. *Let λ, μ, ν be partitions of the same size, such that $\lambda_1\ge\ell(\mu)\cdot\nu_1$ and $\mu_1\ge\ell(\lambda)\cdot\nu_1$. Then for every $h\ge 0$ we have*

$$\mathtt{k}(\lambda,\mu,\nu)=\mathtt{k}(\,\lambda+h,\;\mu+h,\;\nu+h\,).$$

Our proofs use an observation on 3-dimensional contingency arrays Q with zeros and ones as entries (Lemma 4), applied differently. We identify subsets $Q\subseteq\mathbb{N}^3$ with their characteristic functions $Q:\mathbb{N}^3\to\{0,1\}$, and we call Q a binary or $\{0,1\}$-contingency array. This means, we interpret Q as a function to $\{0,1\}$, and as the point set of its support. The interpretation will always be clear from the context. The 2-dimensional marginals of Q are defined as $Q_{i**}:=\sum_{j,k}Q_{i,j,k}=|Q\cap(\{i\}\times\mathbb{N}\times\mathbb{N})|$, $Q_{*i*}:=\sum_{j,k}Q_{j,i,k}=|Q\cap(\mathbb{N}\times\{i\}\times\mathbb{N})|$, $Q_{**i}:=\sum_{j,k}Q_{j,k,i}=|Q\cap(\mathbb{N}\times\mathbb{N}\times\{i\})|$. For $\alpha\in\mathbb{N}^{\mathbb{N}}$, $\beta\in\mathbb{N}^{\mathbb{N}}$, $\gamma\in\mathbb{N}^{\mathbb{N}}$, $|\alpha|=|\beta|=|\gamma|<\infty$, we denote by

$$\mathcal{C}(\alpha,\beta,\gamma):=\{Q\subseteq\mathbb{N}^3\mid Q_{i**}=\alpha_i,\; Q_{*i*}=\beta_i,\; Q_{**i}=\gamma_i \text{ for every } i\}.$$

There is a close connection to the Kronecker coefficients via the following (see e.g. §4):

Lemma 3. *For partitions α, β, γ of equal size, we have $\mathtt{k}'(\alpha,\beta,\gamma) \le |\mathcal{C}(\alpha,\beta,\gamma)|$.*

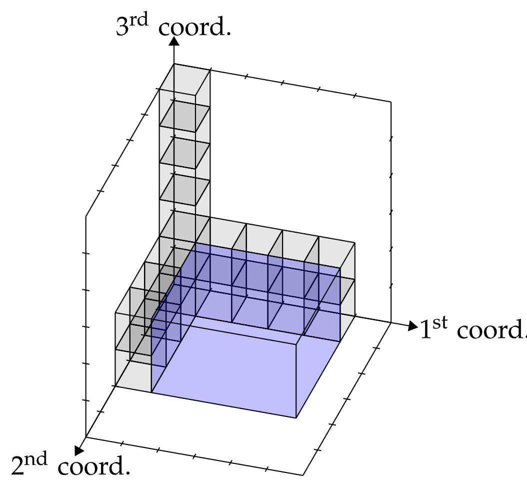
Restrictions on the marginals can result in strong restrictions on the sets Q:

Lemma 4. *Let α,β,γ be compositions with $|\alpha| = |\beta| = |\gamma|$. Let $a \ge \ell(\alpha)$, $b \ge \ell(\beta)$, and let the integers c,h be such that $c+h \ge \ell(\gamma)$ and $\sum_{i>c}\gamma_i \le h$. Furthermore, let $\alpha_1 \ge bc+h$, $\beta_1 \ge ac+h$. Then, for every $Q \in \mathcal{C}(\alpha,\beta,\gamma)$ we have*

$$\{1\}\times[b]\times[c] \subseteq Q, \quad [a]\times\{1\}\times[c] \subseteq Q, \quad \{1\}\times\{1\}\times[c+h] \subseteq Q, \text{ and}$$
$$Q \cap (\mathbb{N}\times\mathbb{N}\times[c+1,c+h]) = \{1\}\times\{1\}\times[c+1,c+h].$$

In particular, if $\mathcal{C}(\alpha,\beta,\gamma)$ is non-empty, then $a = \ell(\alpha)$, $b = \ell(\beta)$, $\gamma_i = 1$ for all $c+1 \le i \le c+h$, and $\alpha_1 = bc+h$, $\beta_1 = ac+h$, $\alpha_2 \le bc$, and $\beta_2 \le ac$.

In other words, if we have 3d point configurations with such marginals, then the walls consist of two rectangles and a long column as depicted in the figure below.



Proof: Assume that there exists a binary contingency array $Q \in \mathcal{C}(\alpha,\beta,\gamma)$. Let $B_\cup := \{1\}\times[b]\times[c+h] \,\cup\, [a]\times\{1\}\times[c+h]$ be the set of points in the planes $x=1$ and $y=1$, and let $B_\cap := \{1\}\times\{1\}\times[c+h]$ be the set of points on the line $x=y=1$. Let $H_i := Q\cap(\mathbb{N}\times\mathbb{N}\times\{i\})\cap B_\cup$ be the entries of Q in $B_\cup$ at the section with the plane $z=i$. In particular, $\sum_{i=1}^{c+h}|H_i| = |Q\cap B_\cup|$. We have $\sum_{i=c+1}^{c+h}|H_i| \le \sum_{i=c+1}^{c+h}\gamma_i \le h$, $|H_i| \le a+b-1$ for all $0 < i \le c$ and $|Q\cap B_\cap| \le c+h$. All these inequalities must be met with equality, because

$$\begin{aligned}
\alpha_1+\beta_1 &= |Q\cap B_\cap| + |Q\cap B_\cup| = |Q\cap B_\cap| + \textstyle\sum_{i=1}^{c+h}|H_i| \\
&= |Q\cap B_\cap| + \textstyle\sum_{i=1}^{c}|H_i| + \sum_{i=c+1}^{c+h}|H_i| \\
&\le (c+h) + (a+b-1)c + h = (a+b)c + 2h \le \alpha_1+\beta_1.
\end{aligned}$$

We thus have the following equalities: $|Q\cap B_\cap| = c+h = |B_\cap|$ and $\forall i \in [c]$ we have $|H_i| = a+b-1 = |(\mathbb{N}\times\mathbb{N}\times\{i\})\cap B_\cup|$. Thus we have $B_\cap \subseteq Q$, and $\{1\}\times[b]\times[c] \subseteq Q$, and $[a]\times\{1\}\times[c] \subseteq Q$, and $Q\cap(\mathbb{N}\times\mathbb{N}\times[c+1,c+h]) = \{1\}\times\{1\}\times[c+1,c+h]$. This gives the desired marginals and the claim follows. □

Proof of Theorem 1. Let $\ell(\lambda) = l$, $\ell(\mu) = m$ and $\nu_1 = c$ and set $d = mc+c$, $e = lc+c$.

Suppose first that $\lambda_1 \le mc$ and $\mu_1 \le lc$. We apply Lemma 2, and obtain

$$\mathtt{k}(\lambda,\mu,\nu) = \mathtt{k}\big(\underbrace{(d)\diamond(c^l+\lambda)}_{=:\hat\lambda}, \underbrace{(e)\diamond(c^m+\mu)}_{=:\hat\mu}, \underbrace{c^{l+m+1}\diamond\nu}_{=:\hat\nu}\big).$$

The top rows of $\hat{\lambda}, \hat{\mu}, \hat{\nu}$ are d, e, c respectively and thus Theorem 2 gives that for all $h \in \mathbb{N}$ we have $\mathtt{k}(\hat{\lambda},\, \hat{\mu},\, \hat{\nu}) = \mathtt{k}(\hat{\lambda}+h,\, \hat{\mu}+h,\, \hat{\nu}+h) =$

$$= \mathtt{k}(\,(d+h)\diamond(c^l+\lambda),\, (e+h)\diamond(c^m+\mu),\, (c+h)\diamond c^{l+m}\diamond\nu\,) = \overline{\mathtt{k}}(c^l+\lambda,\, c^m+\mu,\, c^{l+m}\diamond\nu),$$

where the last identity follows by letting $h \to \infty$. This proves Theorem 1 in the first case.

Suppose now that $\lambda_1 > mc$, the case $\mu_1 > lc$ is completely analogous. Set $b := m+1$. Then we have $\mathtt{k}(\lambda,\mu,\nu) = \mathtt{k}(\lambda',\mu,\nu') = 0$ since $\ell(\lambda') = \lambda_1 > mc = \ell(\mu)\ell(\nu')$. On the other hand, the reduced Kronecker coefficient is obtained by adding long first rows, $cm+c+h,\, cl+c+h,\, c+h$ respectively, so $\overline{\mathtt{k}}(c^l+\lambda,\, c^m+\mu,\, c^{l+m}\diamond\nu) =$

$$\begin{aligned} &= \mathtt{k}\big((cm+c+h)\diamond(c^l+\lambda),\, (lc+c+h)\diamond(c^m+\mu),\, (c+h)\diamond c^{l+m}\diamond\nu)\big) \\ &= \mathtt{k}'\big(\underbrace{(cm+c+h)\diamond(c^l+\lambda)}_{=:\,\alpha},\, \underbrace{(lc+c+h)\diamond(c^m+\mu)}_{=:\,\beta},\, \underbrace{((l+b)^c+\nu')\diamond(1^h)}_{=:\,\gamma}\big) \end{aligned}$$

for sufficiently large h. Let $\hat{\gamma} = (l+b)^c+\nu'$ be γ without the h many trailing 1s. We observe that $\alpha_2 = \lambda_1 + c$, $\ell(\beta) = b$, and $\ell(\hat{\gamma}) = c$. From $\lambda_1 > mc$ we conclude $\alpha_2 > bc$. Lemma 4 shows that $\mathcal{C}(\alpha,\beta,\gamma) = \emptyset$. Hence $\mathtt{k}'(\alpha,\beta,\gamma) = 0$ by Lemma 3. □

3 Proofs via the general linear group

We refer to [7, §8] for the basic properties of the irreducible representations of the general linear group. The Kronecker coefficients have an interpretation as the structure coefficients arising when decomposing irreducible GL_{ab} representations as $\mathrm{GL}_a \times \mathrm{GL}_b$ representations, which can be seen directly from Schur-Weyl duality:

$$V_\nu(\mathbb{C}^{ab}) \simeq \bigoplus_{\lambda\vdash_a|\nu|,\,\mu\vdash_b|\nu|} \Big(V_\lambda(\mathbb{C}^a)\otimes V_\mu(\mathbb{C}^b)\Big)^{\oplus \mathtt{k}(\lambda,\mu,\nu)}.$$

Another formulation is via the multiplicity of the irreducible $G := \mathrm{GL}_a \times \mathrm{GL}_b \times \mathrm{GL}_c$ representation $V_\alpha(\mathbb{C}^a)\otimes V_\beta(\mathbb{C}^b)\otimes V_\gamma(\mathbb{C}^c)$ in the D-th wedge power of $\mathbb{C}^a\otimes\mathbb{C}^b\otimes\mathbb{C}^c$, see [8]. Formally for partitions $\alpha,\beta,\gamma \vdash D$ we have

$$\mathtt{k}'(\alpha,\beta,\gamma) \;:=\; \mathtt{k}(\alpha,\beta,\gamma') \;=\; \mathrm{mult}_{\alpha,\beta,\gamma}\big(\textstyle\bigwedge^D(\mathbb{C}^a\otimes\mathbb{C}^b\otimes\mathbb{C}^c)\big).$$

A vector v for which $\big(\mathrm{diag}(r_1,\ldots,r_a),\mathrm{diag}(s_1,\ldots,s_b),\mathrm{diag}(t_1,\ldots,t_c)\big)v = r_1^{\lambda_1}\cdots r_a^{\lambda_a}\cdot s_1^{\mu_1}\cdots s_b^{\mu_b}\cdot t_1^{\nu_1}\cdots t_c^{\nu_c}v$ is called a *weight vector* of weight (λ,μ,ν).

For $(A,B,C) \in \mathbb{C}^{a\times a}\times\mathbb{C}^{b\times b}\times\mathbb{C}^{c\times c}$, the Lie algebra action on $\bigwedge^D(\mathbb{C}^a\otimes\mathbb{C}^b\otimes\mathbb{C}^c)$ is defined as $(A,B,C).v := \lim_{\varepsilon\to 0}\varepsilon^{-1}((\varepsilon(A,B,C)+(\mathrm{id}_a,\mathrm{id}_b,\mathrm{id}_c))v - v)$. A raising operator is the Lie algebra action of $(E_{i-1,i},0,0)$, where $E_{i,j}$ is the matrix with a 1 at position (i,j) and zeros everywhere else. The other raising operators are $(0,E_{i-1,i},0)$ and

$(0,0,E_{i-1,i})$. Let $e_i := (0,\ldots,0,1,0,\ldots,0)^T$ and let $e_{i,j,k} := e_i \otimes e_j \otimes e_k$. Then, for example, $(E_{i,j},0,0)e_{r,1,1} = e_{i,1,1}$ iff $r = j$ and 0 otherwise. A highest weight vector (HWV) of weight (α,β,γ) is a nonzero weight vector of weight (α,β,γ) that is mapped to zero by all raising operators. The irreducible $\mathrm{GL}_a \times \mathrm{GL}_b \times \mathrm{GL}_c$ representation $V_\alpha \otimes V_\beta \otimes V_\gamma$ contains exactly one HWV (up to scale), and that is of weight (α,β,γ). Hence ([8, Lemma 2.1]),

$$\mathtt{k}'(\alpha,\beta,\gamma) = \dim\left(\mathrm{HWV}_{\alpha,\beta,\gamma}\bigwedge^D(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)\right),$$

where $\mathrm{HWV}_{\alpha,\beta,\gamma}$ denotes the space of HWVs of weight (α,β,γ). Note that each standard basis vector in $\bigwedge^D(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)$ is a weight vector, and hence for each weight vector space of weight w we have a basis given by the set of standard basis vectors of weight w. Let $e_{i,j,k} := e_i \otimes e_j \otimes e_k$, and for a list of points $Q \in (\mathbb{N}^3)^D$ we define $\psi_Q := e_{Q_1} \wedge e_{Q_2} \wedge \cdots \wedge e_{Q_D}$. If Q has marginals (α,β,γ), then ψ_Q has weight (α,β,γ). This immediately implies the result of Lemma 3.

Proof of Theorem 2 via contingeny arrays and highest weight vectors. Let $a := \ell(\lambda)$, $b := \ell(\mu)$, $c := \nu_1$. Let $\gamma := \nu'$, so $\ell(\gamma) = c$. We have $\lambda_1 \geq bc$ and $\mu_1 \geq ac$. Observe that $\mathtt{k}(\lambda,\mu,\nu) = \mathtt{k}'(\lambda,\mu,\gamma)$. Let $\widetilde{\lambda} = \lambda + (h)$, $\widetilde{\mu} = \mu + (h)$, $\widetilde{\gamma} = \gamma \diamond (1^h)$. We define an injective linear map φ as follows.

$$\begin{array}{rcl} \varphi:\ \bigwedge^D(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c) & \to & \bigwedge^{D+h}(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^{c+h}) \\ v & \mapsto & v \wedge e_{1,1,c+1} \wedge e_{1,1,c+2} \wedge \cdots \wedge e_{1,1,c+h} \end{array}$$

Note that φ maps vectors of weight (λ,μ,γ) to vectors of weight $(\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma})$. It remains to show that φ maps HWVs to HWVs, and that every HWV of weight $(\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma})$ has a preimage under φ.

We first prove that φ sends HWVs to HWVs. By construction of φ, we observe that for $1 \leq i < i' \leq a$, we have

$$(E_{i,i'},0,0)\varphi(u) = \varphi((E_{i,i'},0,0)u) = \varphi(0) = 0.$$

Analogously, $(0,E_{j,j'},0)\varphi(u) = 0$ for $1 \leq j < j' \leq b$, and $(0,0,E_{k,k'})\varphi(u) = 0$ for $1 \leq k < k' \leq c$. The remaining raising operators also vanish by construction of φ, because

$$\begin{array}{rl} & (0,0,E_{k,k'})(v \wedge e_{1,1,c+1} \wedge \cdots \wedge e_{1,1,c+h}) \\ = & v \wedge e_{1,1,c+1} \wedge \cdots \wedge \widehat{e_{1,1,c+k}} \wedge e_{1,1,c+k'} \wedge e_{1,1,c+k'} \wedge \cdots \wedge e_{1,1,c+h} = 0 \end{array}$$

because of the repeated factor $e_{1,1,c+k'}$. Here the $\widehat{e_{1,1,c+k}}$ means omission of that factor.

We now show that every weight vector of weight $(\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma})$ has a preimage under φ, which finishes the proof. It is sufficient to show this for basis vectors. Let $u = \psi_P$ be a basis weight vector of weight $(\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma})$, i.e., $Q \subseteq \mathbb{N}^3$ with marginals $(\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma})$. We apply Lemma 4 to see that $\{1\} \times \{1\} \times [c+1,c+h] \subset Q$ and $Q \cap (\mathbb{N} \times \mathbb{N} \times \{i\}) = \{(1,1,i)\}$ for all $c+1 \leq i \leq c+h$. Therefore, ψ_Q has a preimage under φ, namely ψ_P, where P arises from Q by deleting all points with 3rd coordinate $> c$. $\square$

4 Proofs via symmetric functions

Here we use basic definitions and facts from symmetric function theory, see [25, 23] and will skip the definitions of SSYTs, Schur function etc.. The *multi-LR coefficients* $c^{\lambda}_{\alpha^1\cdots\alpha^k}$ are defined as

$$c^{\lambda}_{\alpha^1\cdots\alpha^k} := \langle s_\lambda, s_{\alpha_1} s_{\alpha^2}\cdots s_{\alpha^k}\rangle = \sum_{\beta^1,\beta^2,\ldots} c^{\lambda}_{\alpha^1\beta^1} c^{\beta^1}_{\alpha^2\beta^2}\cdots c^{\beta^{k-1}}_{\alpha^{k-1}\alpha^k} \tag{4.1}$$

from where it is easy to see that they count SSYTs T of shape λ and type $(\alpha^1 \diamond \alpha^2 \diamond \cdots)$, such that the reading word of each skew subtableau corresponding to the entries with values between $1+\sum_{i=1}^{r}\ell(\alpha^i)$ and $\sum_{i=1}^{r+1}\ell(\alpha^i)$ is a lattice permutation for every $r=1,\ldots,k-1$. For example, $\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&4&4&6\\ \hline 2&2&2&4&5&7\\ \cline{1-6} 3&5&5&6&6\\ \cline{1-5}\end{array}$ and $\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&4&4&6\\ \hline 2&2&2&4&6&6\\ \cline{1-6} 3&5&5&5&7\\ \cline{1-5}\end{array}$ are two multi-LR tableaux of shape $\lambda=(7,6,5)$ and types $\alpha^1=(4,3,1)$, $\alpha^2=(3,3)$, $\alpha^3=(3,1)$.

The Kronecker coefficient can be studied via the following two [equivalent] identities

$$s_\lambda[x\cdot y] = \textstyle\sum_{\mu,\nu} \mathtt{k}(\lambda,\mu,\nu) s_\mu(x)s_\nu(y), \quad \sum_{\lambda,\mu,\nu}\mathtt{k}(\lambda,\mu,\nu)s_\lambda(x)s_\mu(y)s_{\nu'}(z) = \prod_{i,j,k}(1+x_iy_jz_k).$$

Extracting coefficients in both gives us the following formulas via multi-LRs:

$$\mathtt{k}(\lambda,\mu,\nu) = \sum_{\sigma\in S_\ell}\mathrm{sgn}(\sigma)\sum_{\alpha^i\vdash\lambda_i-i+\sigma_i} c^{\mu}_{\alpha^1\cdots\alpha^k}c^{\nu}_{\alpha^1\cdots\alpha^k}. \tag{4.2}$$

and via 3d point configurations with given marginals:

$$\textstyle\sum_{\lambda,\mu,\nu}\mathtt{k}(\lambda,\mu,\nu)s_\lambda(x)s_\mu(y)s_{\nu'}(z) = \sum_{\alpha,\beta,\gamma} C(\alpha,\beta,\gamma)x^\alpha y^\beta z^\gamma \tag{4.3}$$

Note that this identity immediately gives the upper bound in Lemma 3 by comparing coefficients at $x^\lambda y^\mu z^{\nu'}$ on both sides. Replacing the Schurs by Weyl determinantal formula and extracting monomials gives

$$\mathtt{k}(\lambda,\mu,\nu) = \sum_{\sigma\in S_a,\,\pi\in S_b,\,\rho\in S_c}\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\pi)\,\mathrm{sgn}(\rho)C(\lambda+\sigma-\mathrm{id},\mu+\pi-\mathrm{id},\nu'+\rho-\mathrm{id}). \tag{4.4}$$

where a permutation σ is interpreted as the vector $(\sigma(1),\ldots,\sigma(a))$ and $\mathrm{id}=(1,2,\ldots)$.

Proof of Theorem 2 via contingency arrays and symmetric functions. From now on we will use formula (4.4) and Lemma 4 to show that the only possible contingency arrays are the ones depicted there. Consider now $\mathtt{k}(\lambda+h,\mu+h,\nu+h)$ as in the problem, and let $\alpha=(\lambda+h),\beta=(\mu+h)$, $\gamma=(\nu+h)'$ so that $\mathtt{k}(\alpha,\beta,\gamma')=\mathtt{k}(\lambda+h,\mu+h,\nu+h)$. Let $\nu_1=c$, $\ell(\lambda)=a$ and $\ell(\mu)=b$, so we have $\alpha_1\ge bc+h,\beta_1\ge ac+h$, $\gamma_i=1$ for $i=c+1,\ldots,c+h$ and

$$\mathtt{k}(\alpha,\beta,\gamma') = \sum_{\sigma\in S_a,\,\pi\in S_b,\,\rho\in S_{c+h}}\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\pi)\,\mathrm{sgn}(\rho)C(\alpha+\sigma-\mathrm{id},\beta+\pi-\mathrm{id},\gamma+\rho-\mathrm{id}). \tag{4.5}$$

In formula (4.5) we then consider $\{0,1\}$-contingency arrays Q with marginals

$$Q_{1**} := \sum_{j,k} Q_{1,j,k} = \lambda_1 + \sigma_1 - 1 \geq bc + h, \quad Q_{*1*} := \sum_{i,k} Q_{i,1,k} = \mu_1 + \pi_1 - 1 \geq ac + h,$$
$$Q_{**k} := \sum_{i,j} Q_{i,j,k} = 1 + \rho_k - k, \text{ for } k = c+1, \dots, c+h.$$

Note that then we have $\sum_{k>c} Q_{**k} = h + \sum_{k=c+1}^{c+h} \rho_k - \sum_{k=c+1}^{c+h} k \leq h$, and the support of the array is in $[1,a] \times [1,b] \times [1,c+h]$, so we can apply Lemma 4 and conclude that $Q_{1,j,k} = 0$ iff $(j,k) \in [2,b] \times [c+1,c+h]$ and $Q_{i,1,k} = 0$ iff $(i,k) \in [2,a] \times [c+1,c+h]$. Thus, we must have $Q_{1**} = bc+h$, $Q_{*1*} = ac+h$ and so $\sigma_1 = \pi_1 = 1$, $\{\rho_{c+1}, \dots, \rho_{c+h}\} = \{c+1, \dots, c+h\}$ and for $k \in [c+1, c+h]$ we must have $Q_{i,j,k} = 0$ unless $i = j = 1$. This also forces us to have $Q_{1,1,k} = 1$ for all these k, and so $\rho_k = k$ for $k = c+1, \dots, c+h$.

This completely determines $Q_{i,j,k}$ for $k > c$, as well as ρ_k for $k > c$, and $\rho = \bar{\rho}, (c+1), \dots, (c+h)$ for $\bar{\rho} \in S_c$. We can thus write formula (4.5) as

$$\begin{aligned} &\mathtt{k}(\lambda + h, \mu + h, \nu + h) \\ &\quad = \sum_{\sigma \in S_a,\, \pi \in S_b,\, \rho \in S_{c+h}} \operatorname{sgn}(\sigma) \operatorname{sgn}(\pi) \operatorname{sgn}(\rho)\, C(\alpha + \sigma - \mathrm{id}, \beta + \pi - \mathrm{id}, \gamma + \rho - \mathrm{id}) \\ &\quad = \sum_{\sigma \in S_a,\, \pi \in S_b,\, \eta \in S_c} \operatorname{sgn}(\sigma) \operatorname{sgn}(\pi) \operatorname{sgn}(\eta)\, C(\bar{\alpha} + \sigma - \mathrm{id}, \bar{\beta} + \pi - \mathrm{id}, \bar{\gamma} + \eta - \mathrm{id}), \end{aligned}$$

where $\bar{\alpha} = \alpha - (h) = \lambda$, $\bar{\beta} = \beta - (h) = \mu$ and $\bar{\gamma} = (\gamma_1 \dots, \gamma_c) = \nu'$. As the last part coincides with the expression for $\mathtt{k}(\lambda, \mu, \nu)$ in (4.4), we get the desired identity. □

Proof of Theorem 2 via Littlewood-Richardson coefficients.
Let again $\ell(\lambda) = a$, $\ell(\mu) = b$ and $\nu_1 = c$.

We have that $\mathtt{k}(\lambda + h, \mu + h, \nu + h) = \mathtt{k}(\nu' \diamond (1^h), \lambda' \diamond (1^h), \mu + h)$ and we are going to apply formula (4.2) with that triple of partitions. Set $\hat{\mu} = \mu + h$, $\hat{\lambda} = \lambda' \diamond (1^h) = (\lambda + h)'$ and $\hat{\nu} = \nu' \diamond (1^h)(\nu + h)'$. Here $\ell(\nu' \diamond (1^h)) = c + h$, so

$$\mathtt{k}(\lambda + h, \mu + h, \nu + h) = \sum_{\sigma \in S_{c+h}} \operatorname{sgn}(\sigma) \sum_{\alpha^i \vdash \hat{\nu}_i - i + \sigma_i} c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \dots} c^{\hat{\mu}}_{\alpha^1 \alpha^2 \dots}$$

From the iterated definition of the multi-LR coefficients (4.1) we see that in order for the coefficients to be nonzero, we must have $\alpha^i \subset \hat{\mu}$ and $\alpha^i \subset \hat{\lambda}$. Tthen $\ell(\alpha^i) \leq \ell(\mu) = b$ and $\alpha^i_1 \leq \hat{\lambda}_1 = a$. Note that multi-LR coefficients count certain SSYTs of type $(\alpha^1 \diamond \alpha^2 \diamond \dots \diamond \alpha^c \diamond \dots)$ and thus in the shape $\hat{\lambda}$ the first column will have at most $\ell(\alpha^1) + \cdots + \ell(\alpha^c) \leq bc$ many entries from the first c partitions. So there are at least h boxes in the first column which need to be covered by the partitions $\alpha^{c+1}, \dots, \alpha^{c+h}$. We then have

$$h \leq \ell(\alpha^{c+1}) + \cdots + \ell(\alpha^{c+h}) \leq |\alpha^{c+1}| + \cdots + |\alpha^{c+h}| = \sum_{i=c+1}^{c+h} 1 - i + \sigma_i \leq h,$$

as $\sigma_{c+1} + \cdots + \sigma_{c+h} \leq c + 1 + \cdots c + h$. Thus we need to have equalities, and so

$$|\alpha^{c+1}| + \cdots + |\alpha^{c+h}| = h, \ell(\alpha^i) = |\alpha^i|,$$

so α^i are single column partitions, possibly empty. Further, we have $\alpha^i \leq a$, $\alpha^i \subset \hat{\mu}$. As there is a multi-LR of type $(\alpha^1 \diamond \alpha^2 \cdots)$, the first row of that tableaux can only be occupied by the smallest entries of each type. So we must have

$$ac + h = \hat{\mu}_1 \leq \sum_i \alpha_1^i \leq \sum_{i=1}^{c} a + \sum_{i=c+1}^{c+h} \alpha_1^i.$$

Thus $\alpha_1^{c+1} + \cdots + \alpha_1^{c+h} \geq h$. Since $\alpha_1^i \leq 1$ by the above consideration, we must have $\alpha^i = (1)$ for all $i > c$. So $\sigma_i = i$ for $i = c+1, \ldots, c+h$. Then

$$c^{\hat{\lambda}}_{\alpha^1\alpha^2\cdots\alpha^{c+h}} = c^{\lambda'}_{\alpha^1\cdots\alpha^c} \quad \text{and} \quad c^{\hat{\mu}}_{\alpha^1\alpha^2\cdots\alpha^{c+h}} = c^{\mu}_{\alpha^1\cdots\alpha^c}.$$

We thus get that
$$\mathtt{k}(\lambda + h, \mu + h, \nu + h) = \sum_{\sigma \in S_{c+h}} \operatorname{sgn}(\sigma) \sum_{\alpha^i \vdash \hat{\nu}_i - i + \sigma_i} c^{\hat{\lambda}}_{\alpha^1\alpha^2\cdots} c^{\hat{\mu}}_{\alpha^1\alpha^2\cdots}$$
$$= \sum_{\sigma \in S_c} \operatorname{sgn}(\sigma) \sum_{\alpha^i \vdash \nu'_i - i + \sigma_i} c^{\lambda'}_{\alpha^1\alpha^2\cdots} c^{\mu}_{\alpha^1\alpha^2\cdots} = \mathtt{k}(\nu', \lambda', \mu) = \mathtt{k}(\lambda, \mu, \nu). \quad \square$$

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