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Rational Catalan Numbers for Complex Reflection Groups

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Abstract. Assuming standard conjectures, we show that the canonical symmetrizing trace evaluated at powers of a Coxeter element produces rational Catalan numbers for irreducible spetsial complex reflection groups. This extends a technique used by Galashin, Lam, Trinh, and Williams to uniformly prove the enumeration of their noncrossing Catalan objects for finite Coxeter groups.

Keywords: reflection groups, Hecke algebra, parking

1 Introduction

1.1 Catalan combinatorics

This is an extended abstract of [15]. The prototypical noncrossing Coxeter-Catalan objects are the *noncrossing partitions*. In type A with the usual Coxeter element $c = (1, 2, \ldots, n)$, these correspond to partitions $\{B_1, \ldots, B_k\}$ of the set $\{1, 2, \ldots, n\}$ such that there do not exist $a < b < c < d$ such that $a, c \in B_i$ and $b, d \in B_j$ with $i \neq j$. These type A noncrossing partitions are counted by $\mathrm{Cat}_n := \frac{1}{n+1}\binom{2n}{n}$.

In [8], the authors define rational noncrossing Coxeter-Catalan objects called *maximal* $\mathbf{c}^p$*-Deograms*, counted by the *rational Coxeter-Catalan numbers*

$$\mathrm{Cat}_p(W) := \prod_{i=1}^{n} \frac{p + (pe_i \mod h)}{d_i},$$

where $h = d_n$ is the Coxeter number of W, p is coprime to h, $d_1 \leq \cdots \leq d_n$ are the degrees of a set of algebraically independent homogeneous polynomials which generate the algebra of invariants $\mathrm{Sym}(V^*)^W$, and $e_i = d_i - 1$. These objects are defined for finite Coxeter groups. As part of the type-uniform proof of this enumeration, the authors of [8] use Hecke algebra traces to compute the point count of braid Richardson varieties over a finite field, producing q-deformed rational Catalan numbers:

$$\mathrm{Cat}_p(W; q) := \prod_{i=1}^{n} \frac{[p + (pe_i \mod h)]_q}{[d_i]_q}.$$

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Many of the objects used in their proof can also be defined for the well-generated complex reflection groups, so it is natural to try to compute these traces in the complex case. It turns out that the *well-generated* condition is too weak for certain representation-theoretic techniques to work—the necessary condition is for the group to be *spetsial*. These spetsial complex reflection groups (see Definition 3.2) are well-generated complex reflection groups that behave as if they were the Weyl group for some connected reductive algebraic group.

Analogs of unipotent characters, Harish-Chandra theory, and Lusztig's Fourier transform can be defined combinatorially for these groups, which allows techniques from the representation theory of finite groups of Lie type to be extended to spetsial complex reflection groups.

As the main result of our paper, Theorem 8.1, we show that for irreducible spetsial complex reflection groups the trace of a power of a Coxeter element still produces a rational Catalan number even though there are not braid Richardson varieties in this context. Precisely, we prove the following result:

Theorem 1.1. *Let W be an irreducible spetsial complex reflection group with Coxeter number h, and let c be a ζ_h-regular element of W. Let $\mathbf{c} \in B(W)$ be a lift of c such that $\mathbf{c}^h = \boldsymbol{\pi}$. Then*

$$\tau_q(T_{\mathbf{c}}^{-p}) = q^{-np}(1-q)^n \operatorname{Cat}_p(W;q).$$

1.2 Parking combinatorics

The *noncrossing parking functions* in type A are sets of tuples $\{(B_1, L_1), \ldots, (B_k, L_k)\}$, where $B_i, L_i \subseteq \{1, \ldots, n\}$ and

- $\{B_1, \ldots, B_k\}$ is a noncrossing partition of $\{1, \ldots, n\}$,
- $\{L_1, \ldots, L_k\}$ is a set partition of $\{1, \ldots, n\}$, and
- $|B_i| = |L_i|$ for $i = 1, \ldots, k$.

The number of these noncrossing parking functions is $(n+1)^{n-1}$.

In [8], the authors uniformly defined rational noncrossing parking objects for finite Coxeter groups. These parking objects correspond to certain walks in the Hasse digram of the weak Bruhat order and are counted by p^n. As a corollary of our main result, we prove the following (see Corollary 8.2):

Corollary 1.2. *For W an irreducible spetsial complex reflection group, let $\mathcal{B}$ be a basis of the spetsial Hecke algebra $\mathcal{H}_q(W)$ (adapted to the Wedderburn decomposition), and let $\mathbf{c}$ be a lift of a ζ_h-regular element such that $\mathbf{c}^h = \boldsymbol{\pi}$. Then*

$$\sum_{b \in \mathcal{B}} \tau_q(b^\vee T_{\mathbf{c}^p} b) = (q-1)^n [p]_q^n.$$

In the real case, this is a key algebraic step in the proof of the enumeration of rational noncrossing parking functions [8]. Finding a combinatorial interpretation of the left-hand-side of this equation, e.g. rational noncrossing parking functions for spetsial complex reflection groups, is an open problem.

2 Complex reflection groups

Let V be an n-dimensional complex vector space. A linear transformation $g \in \mathrm{GL}(V)$ is a *reflection* if the order of g is finite and the subspace $\mathrm{Fix}(g) := \{v \in V : gv = v\}$ has codimension 1. In this case, $\mathrm{Fix}(g)$ will be called the *reflection hyperplane* of g. A *(finite) complex reflection group* is a finite subgroup of $\mathrm{GL}(V)$ that is generated by reflections. It is said to be *well-generated* if it can be generated by n reflections. For a complex reflection group W, we will use $\mathcal{R}$ to denote the set of reflections in W, and $\mathcal{A}$ will denote the corresponding set of reflecting hyperplanes (for Coxeter groups, $|\mathcal{R}| = |\mathcal{A}|$).

A complex reflection group is *irreducible* if V is an irreducible W-module. By the classification of Shephard and Todd, an irreducible complex reflection group is either in the infinite family $G(m, p, n)$, for p a divisor of m, or is one of 34 exceptional groups labeled 4 to 37.

For an orbit of hyperplanes $\mathcal{C} \in \mathcal{A}/W$, we will let $e_{\mathcal{C}}$ denote the order of the pointwise stabilizer $W_H = \{w \in W : wh = h,\ \forall h \in H\}$ for any $H \in \mathcal{C}$ (the order does not depend on the choice of H). For any $H \in \mathcal{C}$, the group W_H is cyclic with order $e_{\mathcal{C}}$, and there is a reflection $s_H \in W$ with reflecting hyperplane H and determinant $\zeta_{e_{\mathcal{C}}} = \exp(2\pi i/e_{\mathcal{C}})$. Such reflections are called *distinguished reflections*.

The *field of definition* k_W of a complex reflection group W is the field generated by the traces of the elements of W on the reflection representation. The field of definition is a subfield of $\mathbb{R}$ when W is a finite Coxeter group and equals $\mathbb{Q}$ when W is a Weyl group.

The *degrees* of W are defined to be the degrees $d_1 \leq \cdots \leq d_n$ of a collection of algebraically independent homogeneous polynomials that generate the *algebra of invariants* $\mathrm{Sym}(V^*)^W$. The *Poincaré polynomial* P_W is defined by

$$P_W := \prod_{j=1}^{n} [d_j]_q,$$

where $[n]_q$ denotes the q-analog $[n]_q := (q^n - 1)/(q - 1)$ for $n \in \mathbb{Z}$.

Let S_+^W be the ideal of $S := \mathrm{Sym}(V^*)$ generated by the positive degree elements of S^W. Then the action of W on the *coinvariant algebra* S/S_+^W is the regular representation, so S/S_+^W contains exactly r copies of any irreducible representation M of W of dimension r. The *exponents* of M are defined to be the degrees $e_1(M) \leq \cdots \leq e_r(M)$ of the homogeneous components of S/S_+^W containing a copy of M. If χ is the irreducible character corresponding to M, we will also denote $e_i(\chi) := e_i(M)$ and $e_i := e_i(V)$.

Definition 2.1. The *fake degree* $\mathrm{Feg}_\chi(q)$ of an irreducible character χ of W is the graded multiplicity of the irreducible representation with character χ in S/S_+^W:

$$\mathrm{Feg}_\chi(q) = \sum_{i=1}^{r} q^{e_i(\chi)}.$$

3 Hecke algebras

Let $V^{\mathrm{reg}} := V \setminus \bigcup_{H\in\mathcal{A}} H$ denote the hyperplane complement. The *pure braid group* for a complex reflection group W is $P(W) := \pi_1(V^{\mathrm{reg}})$. Its *braid group* is $B(W) := \pi_1(V^{\mathrm{reg}}/W)$. The quotient $\rho : V^{\mathrm{reg}} \to V^{\mathrm{reg}}/W$ induces a surjection $\varphi : B(W) \to W$, giving a short exact sequence

$$1 \to P(W) \xrightarrow{\rho_*} B(W) \xrightarrow{\varphi} W \to 1,$$

where W can be interpreted as the group of deck transformations of the covering.

The braid group $B(W)$ has a set of generators $\{\mathbf{s}_{H,\gamma}\}$ called *generators of the monodromy* or *braid reflections* [4], such that $\varphi(\mathbf{s}_{H,\gamma}) = s_H$ is a distinguished reflection. Moreover, the pure braid group $P(W)$ is generated by the $\{\mathbf{s}_{H,\gamma}^{e_\mathcal{C}}\}$ (where $H \in \mathcal{C}$), and so $W \cong B(W)/\langle \mathbf{s}_{H,\gamma}^{e_\mathcal{C}}\rangle$.

The *full twist* $\boldsymbol{\pi} \in P(W)$ is given by $t \mapsto v\exp(2\pi i t)$, where the basepoint $v \in V^{\mathrm{reg}}$ is suppressed. The image $\rho_*(\boldsymbol{\pi}) \in B(W)$ is a central element of $B(W)$ and will also be called the full twist and be denoted by $\boldsymbol{\pi}$.

Define $\mathbf{u} := (u_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W),(0\le j\le e_\mathcal{C}-1)}$, and let $\mathbb{Z}[\mathbf{u}^{\pm 1}]$ be the ring of Laurent polynomials. Let J be the ideal of the group algebra $\mathbb{Z}[\mathbf{u}^{\pm}]B(W)$ generated by the elements

$$(\mathbf{s}_H - u_{\mathcal{C},0})(\mathbf{s}_H - u_{\mathcal{C},1})\cdots(\mathbf{s}_H - u_{\mathcal{C},e_\mathcal{C}-1}),$$

where $\mathcal{C} \in \mathcal{A}/W$, $H \in \mathcal{C}$, and $\mathbf{s}_H$ is a braid reflection (since generators $\mathbf{s}_{H,\gamma}$ and $\mathbf{s}_{H,\gamma'}$ of the monodromy around H are conjugate in $B(W)$ [4], it suffices to use only one such braid reflection for each H in the above relations to generate J). The *generic Hecke algebra* $\mathcal{H}(W)$ is the quotient $\mathbb{Z}[\mathbf{u}^{\pm}]B(W)/J$. For $\mathbf{g} \in B(W)$, we'll denote by $T_\mathbf{g}$ the corresponding element in the Hecke algebra.

The *spetsial Hecke algebra* $\mathcal{H}_q(W)$ is the admissible cyclotomic Hecke algebra induced by the map

$$\theta_q : u_{\mathcal{C},j} \mapsto \begin{cases} q & \text{if } j = 0 \\ \zeta_{e_\mathcal{C}}^j & \text{if } j > 0. \end{cases}$$

This is a generalization of the 1-parameter Iwahori-Hecke algebra of Coxeter groups.

The spetsial Hecke algebra $\mathcal{H}_q(W)$ has splitting field $k_W(y)$, where $y^{|\mu(k_W)|} = q$ for $\mu(k_W)$ the group of roots of unity in k_W. By Tits' deformation theorem, there is a bijection between the irreducible characters of $\mathcal{H}_q(W)$ and those of W. We will denote by χ_q the character of $\mathcal{H}_q(W)$ coresponding to $\chi \in \mathrm{Irr}(W)$.

We will make the following assumption, called the BMM symmetrizing trace conjecture:

Assumption 3.1 ([2]). *There exists a $\mathbb{Z}[\mathbf{u}^{\pm}]$-linear map $\tau : \mathcal{H}(W) \to \mathbb{Z}[\mathbf{u}^{\pm}]$ such that:*

1. *τ is a symmetrizing trace; that is, τ is the bilinear form $\mathcal{H}(W) \times \mathcal{H}(W) \to \mathbb{Z}[\mathbf{u}^{\pm}]$ given by $(h, h') \mapsto \tau(hh')$ is symmetric and non-degenerate.*
2. *Through the specialization $u_{\mathcal{C},j} \mapsto \zeta^j_{e_\mathcal{C}}$, the form τ becomes the canonical symmetrizing trace on the group algebra: $w \mapsto \delta_{1w}$.*
3. *For all $\mathbf{b} \in B(W)$,*
$$\tau(T_{\mathbf{b}^{-1}})^\vee = \frac{\tau(T_{\mathbf{b}\boldsymbol{\pi}})}{\tau(T_{\boldsymbol{\pi}})},$$
where $\alpha \mapsto \alpha^\vee$ is the automorphism on $\mathbb{Z}[\mathbf{u}^{\pm}]$ consisting of simultaneous inversion of the indeterminates.

If such a symmetrizing trace exists, it is unique. We will call τ the canonical symmetrizing trace on $\mathcal{H}(W)$, and denote by τ_q the specialization to the spetsial Hecke algebra. The BMM symmetrizing trace conjecture has been proven for the infinite family $G(m, p, n)$ and the finite Coxeter groups, but remains open for some of the exceptional groups.

There exist weights $S_\chi(q) \in \mathbb{Z}_{k_W}[y^{\pm}]$ for $\chi \in \mathrm{Irr}(W)$, called *Schur elements*, such that
$$\tau_q = \sum_{\chi \in \mathrm{Irr}(W)} \frac{1}{S_\chi(q)} \chi_q,$$
where $\mathbb{Z}_{k_W}$ is the ring of integers of k_W [2]. The Schur elements have been computed even in the cases for which Assumption 3.1 is still open [12].

Using these Schur elements, we can now define the class of spetisal complex reflection groups, which form a subset of the well-generated complex reflection groups.

Definition 3.2. An complex reflection group is called *spetsial* if all of its irreducible components W satisfy any of the following equivalent conditions (equivalence of the conditions is shown in [12]):

1. $S_{\mathbf{1}}(q) = P_W$, where **1** denotes the trivial representation of W
2. $P_W / S_\chi(q) \in k_W(q)$ for all $\chi \in \mathrm{Irr}(W)$
3. W is one of the following groups:
$$G(m, 1, n), \quad G(m, m, n), \quad G_i \text{ where } i \in \{4, 6, 8, 14, 23, \ldots, 30, 32, \ldots, 37\}.$$

If W is an irreducible spetsial complex reflection group, define the *generic degree* of an irreducible character χ by $\mathrm{Deg}_\chi(q) := P_W / S_\chi(q) \in k_W(q)$.

4 Coxeter elements

The *Coxeter number* of a complex reflection group W is $h := (|\mathcal{R}| + |\mathcal{A}|)/n$. If W is well-generated, then $h = d_n$. More generally, define the *generalized Coxeter number* h_χ associated to a character χ to be the normalized trace of the central element $\sum_{r\in\mathcal{R}}(1-r)$. That is,

$$h_\chi = |\mathcal{R}| - \frac{1}{\chi(1)} \sum_{r\in\mathcal{R}} \chi(r).$$

These generalized Coxeter numbers are integers, and $h_\phi = h$ when ϕ is the character of the reflection representation.

A vector $v \in V$ is *regular* if it is not contained in any reflection hyperplane. An element $c \in W$ is *regular* if it has a regular eigenvector. Moreover, c is *ζ-regular* if this eigenvector may be chosen to have eigenvalue ζ. In this case, the multiplicative order d of ζ is a *regular number* for W.

If W is well-generated, then there exists a ζ-regular element for every h-th root of unity ζ. For ζ a primitive h-th root of unity, the ζ-regular elements of W are called *Coxeter elements*.

Remark 4.1. Some authors define the Coxeter elements to be only the ζ_h-regular elements. In several cases, we will find it useful to restrict our attention to this subset of the Coxeter elements. It follows easily from the definition that each Coxeter element is a power of some ζ_h-regular element.

Proposition 4.2. *Suppose W is an irreducible spetsial complex reflection group with Coxeter number h, and let c be a ζ_h-regular element of W. Then there exists a lift $\mathbf{c} \in B(W)$ of c such that $\mathbf{c}^h = \boldsymbol{\pi}$, and*

$$\tau_q(T_{\mathbf{c}}^{-p}) = \frac{1}{P_W} \sum_{\chi\in\mathrm{Irr}(W)} q^{(h_\chi - nh)p/h}\, \mathrm{Feg}_\chi(e^{2\pi i p/h})\, \mathrm{Deg}_\chi(q).$$

Proof. It is shown in [1] that the lift $\mathbf{c}$ exists. The trace formula then follows from [5, 18]. □

5 Exterior powers of Galois twists

Let W be a well-generated irreducible complex reflection group with Coxeter number h and reflection representation V of dimension n. The *Galois twist* V^{σ_p} of V is the irreducible representation of W obtained by applying σ_p to the matrices representing the elements $w \in W$ as linear operators on V, where $\sigma_p \in \mathrm{Gal}(\mathbb{Q}(\zeta_h)/\mathbb{Q})$ is defined by $\sigma_p : \zeta_h \mapsto \zeta_h^p$ for some p coprime to h.

Lemma 5.1. *The generalized Coxeter number of $\Lambda^k V^{\sigma_p}$ is kh.*

Proof. This is a straightforward computation. □

Lemma 5.2. *The fake degree of* $\Lambda^k V^{\sigma_p}$ *is*

$$\sum_{i_1<\cdots<i_k} q^{e_{i_1}(V^{\sigma_p})+\cdots+e_{i_k}(V^{\sigma_p})},$$

and the sets $\{e_1(V^{\sigma_p}),\ldots,e_n(V^{\sigma_p})\}$ *and* $\{pe_1 \mod h,\ldots,pe_n \mod h\}$ *coincide.*

Proof. The first part is shown in [17]. For the second part, the result can be checked by computer [16] for the exceptional groups. For the groups $G(m,1,n)$ and $G(m,m,n)$, we use Malle's [11] formulas for the fake degrees of irreducible characters in terms of corresponding *m-symbols*. □

Theorem 5.3. *Suppose* W *is an irreducible well-generated complex reflection group with Coxeter number* h*. Then for* p *relatively prime to* h *and* χ *an irreducible character of* W*,*

$$[S_{\mathbf{1}}/S_\chi](e^{2\pi i p/h}) = \begin{cases} (-1)^k & \text{if } \chi=\chi_{k,p}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\chi_{k,p}$ *is the character corresponding to* $\Lambda^k V^{\sigma_p}$*.*

Proof. For the symmetric groups (in fact, all real groups), the proof is sketched in [8]. For the exceptional irreducible well-generated groups, this is checked by computer [16]. The proof for the groups $G(m,1,n)$ and $G(m,m,n)$ follows the proof of the "untwisted case" in [14], and we will now outline it.

The group $G(m,1,n)$ is generated by the n elements $\{t,s_1,s_2,\ldots,s_{n-1}\}$, where t is given by the matrix $\mathrm{Diag}(\zeta_m,1,\ldots,1)$ and s_i is given by the permutation matrix corresponding to the transposition $(i,i+1)$. The irreducible representations of $G(m,1,n)$ can be parametrized by m-partitions of n, that is, tuples $\boldsymbol{\lambda}=(\lambda^{(0)},\lambda^{(1)},\ldots,\lambda^{(m-1)})$ of partitions such that $|\lambda^{(0)}|+\cdots+|\lambda^{(m-1)}|=n$. The correspondence is then

$$\boldsymbol{\lambda} \leftrightarrow \chi_{\boldsymbol{\lambda}} = \mathrm{Ind}_{G(m,1,|\lambda^{(0)}|)\times\cdots\times G(m,1,|\lambda^{(m-1)}|)}^{G(m,1,n)} \left((\chi_0\otimes\gamma_0)\boxtimes\cdots\boxtimes(\chi_{m-1}\otimes\gamma_{m-1})\right),$$

where

- γ_k is the linear character of $G(m,1,|\lambda^{(k)}|)$ defined by $t\mapsto\zeta_m^k$ and $s_i\mapsto 1$ for $i=1,\ldots,|\lambda^{(k)}|-1$.
- χ_k is the character of the symmetric group $S_{|\lambda^{(k)}|}$ corresponding to $\lambda^{(k)}$ considered as a character of $G(m,1,|\lambda^{(k)}|)$ via the surjection $G(m,1,|\lambda^{(k)}|)\to S_{|\lambda^{(k)}|}$.

There are explicit combinatorial models for these representations [13] which can be used to show that the m-partition of n corresponding to $\Lambda^k V^{\sigma_p}$ is $(n-k, \varnothing, \ldots, \varnothing, 1^k, \varnothing, \ldots, \varnothing)$, where the 1^k is in the pth slot (mod m; indexing starts with 0).

In [11], Malle gives a combinatorial construction of unipotent characters and generic degrees for the groups $G(m,1,n)$ and $G(m,m,n)$ which enjoy many of the same properties as the corresponding objects for Weyl groups. Using a simplified formula for the Schur elements [7], we show that evaluating Deg_χ at the root of unity for χ an exterior power of a Galois twist produces $(-1)^k$. We then use a case-by-case argument to show that the evaluation of Deg_χ is zero otherwise.

Using Clifford theory, one can describe a relationship between the irreducible characters of $G(m,1,n)$ and those of $G(m,m,n)$: For an m-partition $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(m-1)})$ of n, denote by $\omega(\boldsymbol{\lambda})$ the cyclic permutation $(\lambda^{(m-1)}, \lambda^{(0)}, \ldots, \lambda^{(m-2)})$. Let $\langle\omega\rangle$ denote the cyclic group of order m, and let $s(\boldsymbol{\lambda})$ be the size of the subgroup of $\langle\omega\rangle$ fixing $\boldsymbol{\lambda}$. Then there is a correspondence:

$$\{\chi_{\boldsymbol{\lambda}}, \chi_{\omega(\boldsymbol{\lambda})}, \ldots, \chi_{\omega^{m/s(\boldsymbol{\lambda})-1}(\boldsymbol{\lambda})}\} \in \mathrm{Irr}(G(m,1,n)) \leftrightarrow \left\{\psi, \psi^t, \ldots, \psi^{t^{s(\boldsymbol{\lambda})-1}}\right\} \in \mathrm{Irr}(G(m,m,n))$$

$$(\chi_{\omega^j(\boldsymbol{\lambda})})_{G(m,m,n)} = \psi + \psi^t + \cdots + \psi^{t^{s(\boldsymbol{\lambda})-1}}$$

$$\chi_{\boldsymbol{\lambda}} + \chi_{\omega(\boldsymbol{\lambda})} + \cdots + \chi_{\omega^{m/s(\boldsymbol{\lambda})-1}(\boldsymbol{\lambda})} = \mathrm{Ind}_{G(m,m,n)}^{G(m,1,n)} \psi^{t^j}.$$

Again, Malle has combinatorial constructions for the unipotent characters and generic degrees [11, 10] which we use to evaluate Deg_χ at the root of unity for χ an exterior power of a Galois twist. We again use a case-by-case argument to show that the evaluation of Deg_χ is zero otherwise. □

6 Families of unipotent characters

Let W be an irreducible spetsial complex reflection group. Define the *Rouquier ring* $\mathcal{R}_W(y)$ to be the $\mathbb{Z}_{k_W}$-subalgebra of $k_W(y)$ given by

$$\mathcal{R}_W(y) := \mathbb{Z}_{k_W}[y, y^{-1}, (y^n - 1)^{-1}_{n \geq 1}].$$

To each $\chi \in \mathrm{Irr}(W)$ we can associate a central primitive idempotent e_χ in $k_W(y)\mathcal{H}_q(W)$ given by

$$e_\chi := \frac{1}{S_\chi(q)} \sum_{b \in \mathcal{B}} \chi_q(b) b^\vee,$$

where $\mathcal{B}$ is a basis of $\mathcal{H}_q(W)$ adapted to the Wedderburn decomposition, and the $b^\vee$ form the dual basis with respect to τ_q [6].

There exists a unique partition RB(W) of $\mathrm{Irr}(W)$ such that

- for each $B \in \mathrm{RB}(W)$, the element $e_B := \sum_{\chi \in B} e_\chi$ is a central primitive idempotent in $\mathcal{R}_W(y)\mathcal{H}_q(W)$,
- $1 = \sum_{B \in \mathrm{RB}(W)} e_B$ and for every central idempotent e of $\mathcal{R}_W(y)\mathcal{H}_q(W)$ there exists a subset $\mathrm{RB}(W, e)$ of $\mathrm{RB}(W)$ such that $e = \sum_{B \in \mathrm{RB}(W,e)} e_B$.

We then say that two characters $\chi, \phi \in \mathrm{Irr}(W)$ belong to the same *Rouquier block* of $\mathcal{H}_q(W)$ if they belong to the same element of $\mathrm{RB}(W)$. We then have

Proposition 6.1. *If χ and ψ belong to the same Rouquier block of the spetsial Hecke algebra, then $h_\chi = h_\phi$.*

Proof. The proof that the statistics a_χ and A_χ are constant on Rouquier blocks is described in [6]. It is shown in [2] that $h_\chi = a_\chi + A_\chi$, so the result follows. □

For W be an irreducible Weyl group and q a power of a prime p, let $\mathbf{G}$ be a simple connected reductive group over $\overline{\mathbb{F}}_p$ with connected center which has Weyl group W, and let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius map with respect to some $\mathbb{F}_q$-rational structure which acts trivially on W. We denote by $\mathbf{G}^F$ the corresponding finite group of Lie type and fix a maximally split torus $\mathbf{T}_0$.

A character $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ is called a *unipotent character* if $\langle R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}}), \rho \rangle \neq 0$ for some F-stable maximal torus $\mathbf{T} \subseteq \mathbf{G}$. Here $1_{\mathbf{T}}$ is the trivial character of $\mathbf{T}^F$ and $R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}})$ is the induced Deligne-Lusztig character of $\mathbf{G}^F$. We denote by $\mathrm{Uch}(\mathbf{G}^F)$ the set of all unipotent characters of $\mathbf{G}^F$.

There are two important subsets of the unipotent characters of $\mathbf{G}^F$:

1. For each $\chi \in \mathrm{Irr}(W)$, there is a *unipotent uniform almost character* R_χ which satisfies $R_\chi(1) = \mathrm{Feg}_\chi(q)$
2. For each $\chi \in \mathrm{Irr}(W)$, there is a *unipotent principal series character* ρ_χ which satisfies $\rho_\chi(1) = \mathrm{Deg}_\chi(q)$.

Define a graph on the set of vertices $\mathrm{Uch}(\mathbf{G}^F)$ as follows: two unipotent characters $\rho_1, \rho_2 \in \mathrm{Uch}(\mathbf{G}^F)$ are joined if and only if there is an irreducible character $\chi \in \mathrm{Irr}(W)$ such that $\langle R_\chi, \rho_i \rangle \neq 0$ for $i = 1, 2$. The sets of vertices corresponding to the connected components of the graph are called the *families* in $\mathrm{Uch}(\mathbf{G}^F)$.

These families recover the Rouquier blocks of $\mathrm{Irr}(W)$ via the inclusion $\chi \mapsto \rho_\chi$. For the spetsial groups $G(m, 1, n)$ and $G(m, m, n)$, Malle has defined *families* of his unipotent characters $\mathrm{Uch}(W)$ which recover the Rouquier blocks via an inclusion $\mathrm{Irr}(W) \hookrightarrow \mathrm{Uch}(W)$.

For W an exceptional irreducible spetsial complex reflection group, there is a set $\mathrm{Uch}(\mathbb{G})$ defined in [3] for the corresponding split spets. There is also a principal series $\mathrm{Uch}(\mathbb{G}, 1)$ with bijection $\mathrm{Irr}(W) \to \mathrm{Uch}(\mathbb{G}, 1)$. Moreover, there is a partition of $\mathrm{Uch}(\mathbb{G})$ into *families* which recovers the Rouquier blocks of $\mathcal{H}_q(W)$ when restricted to the principal series.

7 Lusztig's Fourier transform

Lemma 7.1. *Suppose that there exists a pairing* $\{-,-\}_W : \mathrm{Irr}(W) \times \mathrm{Irr}(W) \to \mathbb{C}$ *satisfying*

(T1) For all $\chi \in \mathrm{Irr}(W)$*, we have*

$$\mathrm{Deg}_\chi(q) = \sum_{\phi \in \mathrm{Irr}(W)} \{\chi, \phi\}_W \, \mathrm{Feg}_\phi(q).$$

(T2) For all $\chi, \phi \in \mathrm{Irr}(W)$*, we have* $\{\chi, \phi\}_W = \{\phi, \chi\}_W$*.*

(T3) For all $\chi, \phi \in \mathrm{Irr}(W)$ *with* $\{\chi, \phi\}_W \neq 0$*, we have* $h_\chi = h_\phi$*.*

Then

$$\sum_{\chi \in \mathrm{Irr}(W)} q_1^{f(\chi)} \, \mathrm{Feg}_\chi(q_2) \, \mathrm{Deg}_\chi(q_3) = \sum_{\chi \in \mathrm{Irr}(W)} q_1^{f(\chi)} \, \mathrm{Feg}_\chi(q_3) \, \mathrm{Deg}_\chi(q_2),$$

if f *satisfies* $(h_\chi = h_\phi \implies f(\chi) = f(\phi))$*.*

Proof. This follows from a double-summation argument. □

These conditions can be interpreted as saying that the pairing transforms fake degrees to generic degrees, is symmetric, and is block diagonal on families. The following conjecture has been proven for W a finite Coxeter group and $W = G(m, 1, n)$.

Conjecture 7.2. *For all irreducible spetsial complex reflection groups* W*, there exists a pairing* $\{-,-\}_W$ *satisfying (T1), (T2), and (T3), which we will call the truncated Lusztig Fourier transform.*

For W a Weyl group, the transform is given by the inner product of characters $\langle \rho_\chi, R_\phi \rangle$. For non-Weyl Coxeter groups, the pairing is described in [9]. Malle constructs a Fourier transform in [11] for $G(m, 1, n)$, and it is shown in [10] that it satisfies (T1), (T2), and (T3).

In [10], Lasy conjectures the existence of a Fourier transform for $G(m, m, n)$ and describes its relation to a "pre-Fourier" transform which is a slight modification of the construction in [11]. This conjectured transform will satisfy (T1), (T2), and (T3).

The Fourier transforms for the exceptional groups (and for the families $G(m, 1, n)$ and $G(m, m, n)$) are contained in the data for GAP3, but their properties have not yet appeared in publication. See [3].

8 Rational Catalan numbers

Theorem 8.1. *Let W be an irreducible spetsial complex reflection group with Coxeter number h, and let c be a ζ_h-regular element of W. Let $\mathbf{c} \in B(W)$ be a lift of c such that $\mathbf{c}^h = \boldsymbol{\pi}$. Then*

$$\tau_q(T_{\mathbf{c}}^{-p}) = q^{-np}(1-q)^n \operatorname{Cat}_p(W;q).$$

Proof. Assuming Conjecture 7.2,

$$\begin{aligned}
\tau_q(T_{\mathbf{c}}^{-p}) &= \frac{1}{P_W} \sum_{\chi \in \operatorname{Irr}(W)} q^{(h_\chi - nh)p/h} \operatorname{Feg}_\chi(e^{2\pi i p/h}) \operatorname{Deg}_\chi(q) \\
&= \frac{1}{P_W} \sum_{\chi \in \operatorname{Irr}(W)} q^{(h_\chi - nh)p/h} \operatorname{Feg}_\chi(q) \operatorname{Deg}_\chi(e^{2\pi i p/h}) \\
&= \frac{1}{P_W} \sum_{k=0}^{n} (-1)^k q^{(k-n)p} \sum_{i_1 < \cdots < i_k} q^{e_{i_1}(V^{\sigma_p}) + \cdots + e_{i_k}(V^{\sigma_p})} \\
&= \frac{1}{P_W} q^{-np} \prod_{i=1}^{n} \left(1 - q^{p + e_i(V^{\sigma_p})}\right) = q^{-np}(1-q)^n \prod_{i=1}^{n} \frac{[p + e_i(V^{\sigma_p})]_q}{[d_i]_q} \\
&= q^{-np}(1-q)^n \operatorname{Cat}_p(W;q).
\end{aligned}$$

□

These "trace techniques" also allow us to extend a result from [8] related to the enumeration of rational parking functions.

Corollary 8.2. *For W an irreducible spetsial complex reflection group, let $\mathcal{B}$ be a basis of the spetsial Hecke algebra $\mathcal{H}_q(W)$ (adapted to the Wedderburn decomposition), and let $\mathbf{c}$ be a lift of a ζ_h-regular element such that $\mathbf{c}^h = \boldsymbol{\pi}$. Then*

$$\sum_{b \in \mathcal{B}} \tau_q(b^\vee T_{\mathbf{c}^p} b) = (q-1)^n [p]_q^n.$$

This corollary motivates future work: Are there noncrossing parking objects for spetsial complex reflection groups analogous to those in [8] whose enumeration is related to the sum $\sum_{b \in \mathcal{B}} \tau_q(b^\vee T_{\mathbf{c}^p} b)$?

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