Séminaire Lotharingien de Combinatoire **91B** (2024)
Article #42, 12 pp.

Proceedings of the 36^{th} Conference on Formal Power Series and Algebraic Combinatorics (Bochum)

Poset polytopes and pipe dreams: toric degenerations and beyond

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Abstract. We demonstrate how pipe dreams can be applied to the theory of poset polytopes to produce toric degenerations of flag varieties. Specifically, we present such constructions for marked chain-order polytopes of Dynkin types A and C. These toric degenerations also give rise to further algebraic and geometric objects such as PBW-monomial bases and Newton–Okounkov bodies. We discuss a construction of the former in the type A case and of the latter in type C.

1 Introduction

Recent decades have seen a wide range of new methods for constructing toric degenerations of flag varieties. These methods commonly proceed by attaching a degeneration to every combinatorial or algebraic object of a certain form. Examples of such objects include adapted decompositions in the Weyl group, certain valuations on the function field and certain birational sequences (see [6] for details concerning these results and a partial history of the subject). These correspondences are of great interest for a number of reasons, however, not many explicit constructions are known for the attached objects. This leads to a shortage of concrete recipes that would work in a general situation.

Until recently, the only explicit constructions known to work in the generality of all type A flag varieties were the Gelfand–Tsetlin (GT) degeneration due to [14] and the Feigin–Fourier–Littelmann-Vinberg (FFLV) degeneration due to [10] (as well as slight variations of these two). An important step was made by Fujita in [12] where it is proved that each marked chain-order polytope (MCOP) of the GT poset provides a toric degeneration of a type A flag variety. Each such MCOP $\mathcal{Q}_O(\lambda)$ is given by a subset O of the GT poset P and an integral dominant $\mathfrak{sl}_n$-weight λ. The GT and FFLV polytopes appear as special cases. General MCOPs were defined by Fang and Fourier in [5] and present a far-reaching generalization of the poset polytopes considered by Stanley in [23].

Now, it must be noted that the main objects of study in [12] are Newton–Okounkov bodies, toric degenerations are obtained somewhat indirectly via a general result of [1]

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relating the two notions. This project was initiated with the goal of finding a more direct approach in terms of explicit initial degenerations similar to the classical construction in [14]. Recall that F, the variety of complete flags in $\mathbb{C}^n$, is realized by the Plücker ideal I in the polynomial ring in Plücker variables $X_{i_1,\dots,i_k}$. Meanwhile, the toric variety of $\mathcal{Q}_O(\lambda)$ is realized by a toric ideal I_O in the polynomial ring in variables X_J labeled by order ideals J in P. To obtain a toric initial degeneration of F we may find an isomorphism between the two polynomial rings which would map I_O to an initial ideal of I. The key challenge is then to define this isomorphism, the solution is provided by pipe dreams: a combinatorial rule for associating a permutation w_M with every subset $M \subset P$.

Theorem 0 (cf. Theorem 1)**.** *Fix $O \subset P$. For every order ideal J one can choose $M_J \subset P$ and $k_J \in \mathbb{N}$ so that the map $\psi : X_J \mapsto X_{w_{M_J}(1),\dots,w_{M_J}(k_J)}$ is an isomorphism and $\psi(I_O)$ is an initial ideal of I. Consequently, the toric variety of $\mathcal{Q}_O(\lambda)$ is a flat degeneration of F.*

One reason for the popularity of toric degenerations is that they are accompanied by a collection of other interesting objects: standard monomial theories, Newton–Okounkov bodies, PBW-monomial bases, etc. All of these can also be obtained from our construction, in particular, we explain how PBW-monomial bases are obtained in type A. Consider the irreducible $\mathfrak{sl}_n$-representation V_λ with highest-weight vector v_λ, let $f_{i,j}$ denote the negative root vectors. A basis in V_λ is formed by the vectors $\prod f_{i,j}^{x_{i,j}} v_\lambda$ with x ranging over the lattice points in $\xi(\mathcal{Q}_O(\lambda))$ for a unimodular transformation ξ (see Theorem 2).

We then discuss an extension of our approach to type C. Every integral dominant $\mathfrak{sp}_{2n}$-weight λ and every subset O of the type C GT poset also define an MCOP $\mathcal{Q}_O(\lambda)$. Using a notion of type C pipe dreams (not to be confused with other known symplectic pipe-dream analogs) we state a type C counterpart of Theorem 0, see Theorem 4. A notable new feature of this case is the intermediate degeneration $\widetilde{F}$ of the symplectic flag variety which happens to be a type A Schubert variety (Theorem 3, compare also [3]). All of our toric degenerations are obtained as further degenerations of $\widetilde{F}$.

We also use type C to showcase another aspect of the theory: Newton–Okounkov bodies. Namely, we show how every $\mathcal{Q}_O(\lambda)$ can be realized as a Newton–Okounkov body of the symplectic flag variety (Theorem 5). To us this result is of particular interest because the paper [12] explains in detail why its methods do not extend to type C.

Proofs and further context for the results in type A can be found in [20] (which discusses an extension to semi-infinite Grassmannians); [19] covers types B and C.

2 Type A

2.1 Poset polytopes

Choose an integer $n \geq 2$ and consider the set of pairs $P = \{(i,j)\}_{1 \leq i \leq j \leq n}$. We define a partial order $\prec$ on P by setting $(i,j) \preceq (i',j')$ if and only if $i \leq i'$ and $j \leq j'$. The poset $(P,\prec)$ is sometimes referred to as the *Gelfand–Tsetlin* (or *GT*) *poset*. We denote $A=\{(i,i)\}_{i\in[1,n]}\subset P$. Let $\mathcal{J}$ be the set of order ideals (lower sets) in $(P,\prec)$. For $k\in[0,n]$ let $\mathcal{J}_k\subset\mathcal{J}$ consist of J such that $|J\cap A|=k$, i.e. J contains $(1,1),\dots,(k,k)$ but not $(k+1,k+1)$.

We now associate a family of polytopes with this poset. Each polytope is determined by a subset of P and a vector in $\mathbb{Z}^{n-1}_{\geq 0}$. For $k\in[1,n-1]$ we let ω_k denote the kth basis vector in $\mathbb{Z}^{n-1}_{\geq 0}$.

Definition 1. *Consider a subset $O\subset P$ such that $A\subset O$. For $J\in\mathcal{J}$ consider the set*

$$M_O(J)=(J\cap O)\cup\max{}_{\prec}(J)$$

($\max_\prec$ denotes the subset of $\prec$-maximal elements). Let $x_O(J)\in\mathbb{R}^P$ denote the indicator vector $\mathbf{1}_{M_O(J)}$. The marked chain-order polytope (MCOP) $\mathcal{Q}_O(\omega_k)$ is the convex hull of $\{x_O(J)\}_{J\in\mathcal{J}_k}$. For $\lambda=(a_1,\dots,a_{n-1})\in\mathbb{Z}^{n-1}_{\geq 0}$ the MCOP $\mathcal{Q}_O(\lambda)$ is the Minkowski sum

$$a_1\mathcal{Q}_O(\omega_1)+\dots+a_{n-1}\mathcal{Q}_O(\omega_{n-1})\subset\mathbb{R}^P.$$

MCOPs were introduced in [5, 7] in the generality of arbitrary finite posets. The original definition describes the polytope in terms of linear inequalities. The equivalence of the above approach is proved in [11, Subsection 3.5].

The first thing to note is that for $\lambda=(a_1,\dots,a_n)$ and any $x\in\mathcal{Q}_O(\lambda)$ one has $x_{i,i}=a_i+\dots+a_{n-1}$. When $O=P$ one has $M_O(J)=J$. It follows that $\mathcal{Q}_P(\lambda)$ consists of points x with $x_{i,j}\geq x_{i',j'}$ whenever $(i,j)\preceq(i',j')$. Now, identify $\mathbb{Z}^{n-1}$ with the lattice of integral $\mathfrak{sl}_n$-weights by letting ω_k be the kth fundamental weight. Then $\mathcal{Q}_P(\lambda)$ is the GT polytope of [13] corresponding to the integral dominant weight λ (i.e. $\lambda\in\mathbb{Z}^{n-1}_{\geq 0}$). If $O=A$, then $M_O(J)$ is the union of $J\cap A$ and the antichain $\max_\prec J$. One can check that $\mathcal{Q}_A(\lambda)$ consists of points x with all $x_{i,j}\geq 0$ and $\sum_{(i,j)\in K}x_{i,j}\leq a_l+\dots+a_r$ for any chain $K\subset P\backslash A$ starting in $(l,l+1)$ and ending in $(r,r+1)$. This is the FFLV polytope ([9]) of λ. Other MCOPs can be said to interpolate between these two cases.

Note that $|\mathcal{J}_k|=\binom{n}{k}$ and, since $\mathcal{Q}_O(\omega_k)$ is a 0/1-polytope, it has $\binom{n}{k}$ lattice points. More generally, a key property of MCOPs is that the number of lattice points in $\mathcal{Q}_O(\lambda)$ does not depend on O and, moreover, the polytopes with a given λ are pairwise Ehrhart-equivalent ([7, Corollary 2.5]). Now, it is well known that the number of lattice points in the GT or FFLV polytope of λ is $\dim V_\lambda$ where V_λ denotes the irreducible $\mathfrak{sl}_n(\mathbb{C})$-representation with highest weight λ. This immediately provides the following.

Proposition 1. *For any O and λ we have $|\mathcal{Q}_O(\lambda)\cap\mathbb{Z}^P|=\dim V_\lambda$.*

The polytope $\mathcal{Q}_O(\omega_k)$ is normal which means that the associated toric variety is embedded into $\mathbb{P}(\mathbb{C}^{\mathcal{J}_k})$. It is cut out by the kernel of the homomorphism from $\mathbb{C}[X_J]_{J\in\mathcal{J}_k}$ to $\mathbb{C}[P]=\mathbb{C}[z_{i,j}]_{(i,j)\in P}$ mapping X_J to $z^{x_O(J)}=\prod_{(i,j)\in P}z_{i,j}^{x_O(J)_{i,j}}$. For general $\lambda=(a_1,\dots,a_n)$

the definition of $\mathcal{Q}_O(\lambda)$ implies that its normal fan and its toric variety (up to isomorphism) depend only on the set of i for which $a_i > 0$. Hence, for regular λ (all $a_i > 0$) the toric variety coincides with that of $\mathcal{Q}_O(\omega_1) + \cdots + \mathcal{Q}_O(\omega_{n-1})$. The toric variety of a Minkowski sum has a standard multiprojective embedding. Consider the product

$$\mathbb{P}_{\mathcal{J}} = \mathbb{P}(\mathbb{C}^{\mathcal{J}_1}) \times \cdots \times \mathbb{P}(\mathbb{C}^{\mathcal{J}_{n-1}})$$

and its multihomogeneous coordinate ring $\mathbb{C}[\mathcal{J}] = \mathbb{C}[X_J]_{J \in \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_{n-1}}$. Let I_O denote the kernel of the homomorphism $X_J \mapsto z^{x_O(J)}$ from $\mathbb{C}[\mathcal{J}]$ to $\mathbb{C}[P]$.

Proposition 2. *For regular λ the toric variety of $\mathcal{Q}_O(\lambda)$ is isomorphic to the zero set of I_O in $\mathbb{P}_{\mathcal{J}}$.*

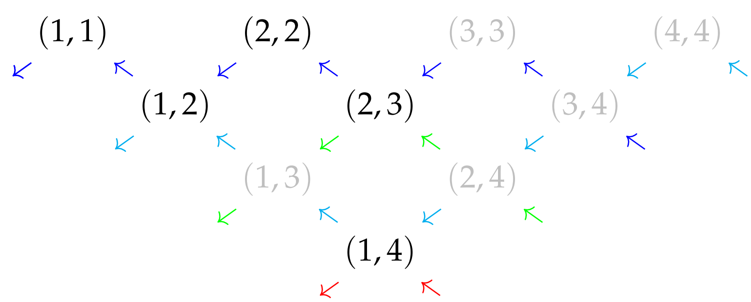
2.2 Pipe dreams

Consider the permutation group $\mathcal{S}_n$ and for $(i, j) \in P$ let $s_{i,j}$ denote the transposition $(i, j) \in \mathcal{S}_n$. In particular, $s_{i,i}$ is always the identity.

Definition 2. *For any subset $M \subset P$ let $w_M \in \mathcal{S}_n$ denote the product of all $s_{i,j}$ with $(i, j) \in M$ ordered first by i increasing from left to right and then by j increasing from left to right.*

Note that w_M is determined by $M \backslash A$ but it is convenient for us to consider subsets of P rather than $P \backslash A$. The term *pipe dream* is due to [17] and refers to a certain diagrammatic interpretation of this correspondence between subsets of P and permutations. The poset P can be visualized as a triangle as shown in (2.1) for $n = 4$. In these terms the pipe dream corresponding to M consists of n polygonal curves or *pipes* described as follows. The ith pipe enters the element (i, n) from the bottom-right, continues in this direction until it reaches an element of $M \cup A$, after which it turns left and continues going to the bottom-left until it reaches an element of M, after which it turns right and again continues to the top-right until it reaches an element of $M \cup A$, etc. The last element passed by the pipe will then be $(1, w_M(i))$.

The pipe dream of the set $M = \{(1, 1), (2, 2), (1, 2), (2, 3), (1, 4)\}$ is shown below, here each pipe is shown in its own colour. Indeed, $s_{1,1}s_{1,2}s_{1,4}s_{2,2}s_{2,3} = (4, 3, 1, 2)$.



(2.1)

We will use a "twisted" version of the correspondence depending on $O \subset P$. For $M \subset P$ we denote $w_M^O = w_O^{-1} w_M$. Diagrammatically, $w_M^O(i) = j$ if the ith pipe of the pipe dream of M ends in the same element as the jth pipe of the pipe dream of O.

2.3 Toric degenerations

For a polynomial ring $\mathbb{C}[x_a]_{a\in A}$ a *monomial order* $<$ on $\mathbb{C}[x_a]_{a\in A}$ is a partial order on the set of monomials that is multiplicative ($M_1 < M_2$ if and only if $M_1 x_a < M_2 x_a$) and weak (incomparability is an equivalence relation). For such an order and a polynomial $p \in \mathbb{C}[x_a]_{a\in A}$ its *initial part* $\operatorname{in}_< p$ is equal to the sum of those monomials occurring in p which are maximal with respect to $<$, taken with the same coefficients as in p. For a subspace $U \subset \mathbb{C}[x_a]_{a\in A}$ its *initial subspace* $\operatorname{in}_< U$ is the linear span of all $\operatorname{in}_< p$ with $p \in U$. The initial subspace of an ideal is an ideal (the *initial ideal*), the initial subspace of a subalgebra is a subalgebra (the *initial subalgebra*). One key property of initial ideals and initial subalgebras is that they define flat degenerations, we explain this phenomenon in the context of flag varieties.

For $n \geq 2$ let F be the variety of complete flags in $\mathbb{C}^n$. The Plücker embedding realizes F as a subvariety in
$$\mathbb{P} = \mathbb{P}(\wedge^1 \mathbb{C}^n) \times \cdots \times \mathbb{P}(\wedge^{n-1} \mathbb{C}^n).$$
The multihomogeneous coordinate ring of $\mathbb{P}$ is $S = \mathbb{C}[X_{i_1,\ldots,i_k}]_{k\in[1,n-1],1\leq i_1<\cdots<i_k\leq n}$ and F is cut out in $\mathbb{P}$ by the *Plücker ideal* $I \subset S$ which can be defined as follows. Consider the $n \times n$ matrix Z with $Z_{i,j} = z_{i,j}$ if $i \leq j$ and $Z_{i,j} = 0$ otherwise. Denote by $D_{i_1,\ldots,i_k}$ the minor of Z spanned by rows $1,\ldots,k$ and columns $i_1,\ldots,i_k$. Then I is the kernel of the homomorphism $\varphi : X_{i_1,\ldots,i_k} \mapsto D_{i_1,\ldots,i_k}$ from S to $\mathbb{C}[P]$. One can also equip S with a $\mathbb{Z}^{n-1}$-grading grad with $\operatorname{grad} X_{i_1,\ldots,i_k} = \omega_k$ and characterize F as $\operatorname{MultiProj} S/I$ with respect to the induced $\mathbb{Z}^{n-1}$-grading. The following fact is essentially classical, for the context of partial monomial orders see [16] (where an algebraic wording is given).

Proposition 3. *For a monomial order $<$ on S the scheme $\operatorname{MultiProj} S/\operatorname{in}_< I$ (i.e. the zero set of $\operatorname{in}_< I$ in $\mathbb{P}$ if the scheme is reduced) is a flat degeneration of F: there exists a flat family $\mathcal{F} \to \mathbb{A}^1$ with fiber over 0 isomorphic to $\operatorname{MultiProj} S/\operatorname{in}_< I$ and all other fibers isomorphic to F.*

Now fix $O \subset P$ containing A. For $J \in \mathcal{J}$ denote $w^O_{M_O(J)} = w^J$. The key ingredient of our first main result is a homomorphism $\psi : \mathbb{C}[\mathcal{J}] \to S$. To define ψ for $J \in \mathcal{J}_k$ we set
$$\psi(X_J) = X_{w^J(1),\ldots,w^J(k)}$$
where we use the convention $X_{i_1,\ldots,i_k} = (-1)^\sigma X_{i_{\sigma(1)},\ldots,i_{\sigma(k)}}$ for $\sigma \in \mathcal{S}_k$. The map ψ encodes a correspondence $J \mapsto (w^J(1),\ldots,w^J(k))$ between order ideals and tuples. When $O = P$ the tuples obtained in this way are precisely the increasing tuples. When $O = A$ one obtains the *PBW tuples* defined in [8]. In general, every subset of $[1,n]$ is represented by exactly one of the obtained tuples, this means that ψ is an isomorphism.

Theorem 1. *The map ψ is an isomorphism and there exists a monomial order $<$ on S such that $\psi(I_O) = \mathrm{in}_< I$. In particular, for regular λ the toric variety of $\mathcal{Q}_O(\lambda)$ is a flat degeneration of F.*

Sketch of proof. It can be checked that $w^J(i) \geq i$ for $J \in \mathcal{J}_k$ and $i \in [1,k]$. Furthermore, there exists a unimodular transformation $\xi \in SL(\mathbb{Z}^P)$ such that for any $J \in \mathcal{J}_k$ one has

$$\xi(x_O(J)) = \mathbf{1}_{\{(1,w^J(1)),\dots,(k,w^J(k))\}}.$$

This means that I_O is the kernel of the homomorphism $X_J \mapsto z_{1,w^J(1)} \dots z_{k,w^J(k)}$.

Using pipe dreams one can define a lexicographic monomial order $\ll$ on $\mathbb{C}[P]$ so that

$$\mathrm{in}_\ll D_{w^J(1),\dots,w^J(k)} = z_{1,w^J(1)} \dots z_{k,w^J(k)}$$

for $J \in \mathcal{J}_k$. Since ξ is bijective, the right-hand sides are distinct monomials for distinct J, hence the sets $\{w^J(1), \dots, w^J(k)\}$ are also pairwise distinct. This provides the isomorphism claim. Note that $\psi(I_O)$ is the kernel of $X_{i_1,\dots,i_k} \mapsto \mathrm{in}_\ll D_{i_1,\dots,i_k}$. Moreover, the $\mathrm{in}_\ll D_{i_1,\dots,i_k}$ generate the initial subalgebra $\mathrm{in}_\ll \varphi(S)$ (i.e. the determinants form a *sagbi basis*). By general properties of initial degenerations, $\ll$ can now be pulled back to a monomial order $<$ on S with the desired property. □

In fact, we could define ψ using the "untwisted" permutation $w_{M(J)}$ and Theorem 1 would still hold since I is invariant under $\mathcal{S}_n$. However, we consider w^J the natural choice because of the property $w^J(i) \geq i$, $i \in [1,k]$ which is also crucial in the next subsection.

2.4 PBW-monomial bases

The map ξ considered in the proof sketch of Theorem 1 maps $\mathcal{Q}_O(\omega_k)$ to $\Pi_O(\omega_k)$: the convex hull of all $\mathbf{1}_{\{(1,w^J(1)),\dots,(k,w^J(k))\}}$ with $J \in \mathcal{J}_k$. For $\lambda = (a_1, \dots, a_n)$ let $\Pi_O(\lambda)$ denote the image $\xi(\mathcal{Q}_O(\lambda))$. It equals the Minkowski sum $a_1\Pi_O(\omega_1) + \dots + a_{n-1}\Pi_O(\omega_{n-1})$.

Next, let us recall some standard Lie-theoretic notation. We have identified $\mathbb{Z}^{n-1}$ with the lattice of integral $\mathfrak{sl}_n$-weights, let $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{Z}^{n-1}$ denote the simple roots, i.e. $\alpha_i = 2\omega_i - \omega_{i-1} - \omega_{i+1}$ where $\omega_0 = \omega_n = 0$. The positive roots are then $\alpha_{i,j} = \alpha_i + \dots + \alpha_{j-1}$ with $1 \leq i < j \leq n$. Let $f_{i,j} \in \mathfrak{sl}_n(\mathbb{C})$ denote the negative root vector of weight $-\alpha_{i,j}$. For $x \in \mathbb{Z}^P_{\geq 0}$ we write f^x to denote the PBW monomial $\prod_{(i,j)\in P\setminus A} f_{i,j}^{x_{i,j}}$ in $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$ ordered first by i increasing from left to right and then by j increasing from left to right. Finally, let v_λ denote a chosen highest-weight vector in V_λ. Another of our main results is as follows.

Theorem 2. *The vectors $f^x v_\lambda$ with $x \in \Pi_O(\lambda) \cap \mathbb{Z}^P$ form a basis in V_λ.*

When $O = A$ the transformation ξ is almost the identity: one has $\xi(x)_{i,j} = x_{i,j}$ for all $i < j$ so that $f^{\xi(x)} = f^x$. Since $O_A(\lambda)$ is the FFLV polytope, one sees that the obtained basis is the FFLV basis of [9]. For $O = P$ the corresponding basis is also known, see, for instance, [22, 21]. Since in this case the tuples $(w^J(1), \ldots, w^J(k))$ are increasing, the definition of $\Pi_P(\lambda)$ is particularly simple. The observation that such a polytope $\Pi_P(\lambda)$ is unimodularly equivalent to the GT polytope $\mathcal{Q}_O(\lambda)$ is due to [18].

3 Type C

3.1 Type C poset polytopes

For $n \geq 2$ consider the totally ordered set $(N, \lessdot) = \{1 \lessdot \cdots \lessdot n \lessdot -n \lessdot \cdots \lessdot -1\}$.

Definition 3. *The type C GT poset $(P, \prec)$ consist of pairs of integers (i, j) such that $i \in [1, n]$ and $j \in [i, n] \cup [-n, -i]$. The order relation is given by $(i_1, j_1) \preceq (i_2, j_2)$ if and only if $i_1 \leq i_2$ and $j_1 \mathrel{\underline{\lessdot}} j_2$.*

$(P, <)$ has length $2n$, below is its Hasse diagram for $n = 2$.

$$
\begin{array}{ccccccc}
(1,1) & & & & (2,2) & & \\
 & \searrow & & \nearrow & & \searrow & \\
 & & (1,2) & & & & (2,-2) \\
 & & & \searrow & & \nearrow & \\
 & & & & (1,-2) & & \\
 & & & & & \searrow & \\
 & & & & & & (1,-1)
\end{array}
\tag{3.1}
$$

We use notation similar to type A. Let $A \subset P$ be the set of all (i, i). Let $\mathcal{J}$ denote the set of order ideals in $(P, \prec)$. For $k \in [1, n]$ let $\mathcal{J}_k$ consist of J such that $|J \cap A| = k$. We also consider the lattice $\mathbb{Z}^n$ with ω_k denoting the kth basis vector. The definition of MCOPs is almost identical.

Definition 4. *Consider a subset $O \subset P$ such that $A \subset O$. For $J \in \mathcal{J}$ consider the set*

$$M_O(J) = (J \cap O) \cup \max_{\prec}(J).$$

Let $x_O(J) \in \mathbb{R}^P$ denote the indicator vector $\mathbf{1}_{M_O(J)}$. The MCOP $\mathcal{Q}_O(\omega_k)$ is the convex hull of $\{x_O(J)\}_{J \in \mathcal{J}_k}$. For $\lambda = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$ the MCOP $\mathcal{Q}_O(\lambda)$ is the Minkowski sum

$$a_1 \mathcal{Q}_O(\omega_1) + \cdots + a_n \mathcal{Q}_O(\omega_n) \subset \mathbb{R}^P.$$

We identify $\mathbb{Z}^n$ with the lattice of integral $\mathfrak{sp}_{2n}$-weights with ω_k being the kth fundamental weight. Then for an integral dominant weight $\lambda \in \mathbb{Z}^n_{\geq 0}$ one sees that $\mathcal{Q}_P(\lambda)$ is the type C Gelfand-Tsetlin polytope defined in [2] while $\mathcal{Q}_A(\lambda)$ is the type C FFLV polytope defined in [10]. Both of these polytopes are known to parametrize bases in V_λ, the irreducible $\mathfrak{sp}_{2n}(\mathbb{C})$-representation with highest weight λ. This again provides

Proposition 4. *For any O and λ we have $|\mathcal{Q}_O(\lambda) \cap \mathbb{Z}^P| = \dim V_\lambda$.*

We also have multiprojective embeddings for toric varieties. Consider the product

$$\mathbb{P}_{\mathcal{J}} = \mathbb{P}(\mathbb{C}^{\mathcal{J}_1}) \times \cdots \times \mathbb{P}(\mathbb{C}^{\mathcal{J}_n})$$

and its multihomogeneous coordinate ring $\mathbb{C}[\mathcal{J}] = \mathbb{C}[X_J]_{J \in \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_n}$. Let I_O denote the kernel of the homomorphism $\varphi_O : X_J \mapsto z^{x_O(J)}$ from $\mathbb{C}[\mathcal{J}]$ to $\mathbb{C}[P] = \mathbb{C}[z_{i,j}]_{(i,j) \in P}$.

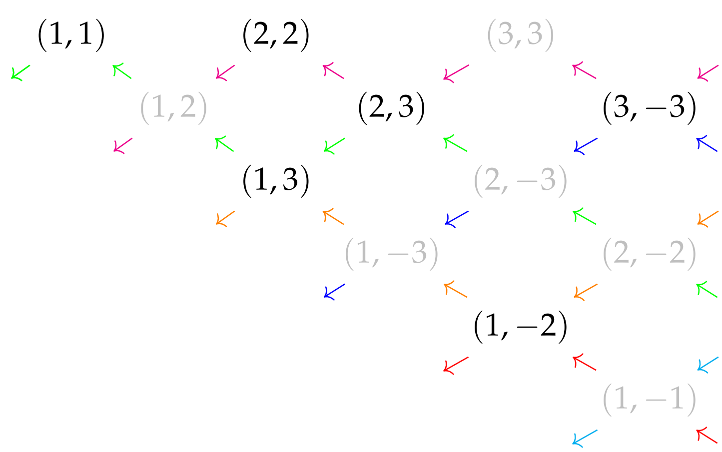
Proposition 5. *For regular λ the toric variety of $\mathcal{Q}_O(\lambda)$ is isomorphic to the zero set of I_O in $\mathbb{P}_{\mathcal{J}}$.*

3.2 Type C pipe dreams

Let S_N denote the group of all permutations of the set N. For $(i,j) \in P$ let $s_{i,j} \in S_N$ denote the transposition which exchanges i and j and fixes all other elements ($s_{i,i} = \mathrm{id}$).

Definition 5. *For $M \subset P$ let $w_M \in S_N$ be the product of all $s_{i,j}$ with $(i,j) \in M$ ordered first by i increasing from left to right and then by j increasing with respect to $\prec$ from left to right.*

In this case the pipe dream consists of $2n$ pipes enumerated by N. In terms of the visualization in (3.1) the ith pipe with $i \in [1,n]$ enters the element $(i,-i)$ from the **bottom**-right and turns at elements of $M \cup A$ while the ith pipe with $i \in [-n,-1]$ enters the element $(i,-i)$ from the **top**-right and then also turns at elements of $M \cup A$.



The pipe dream for the set $M = \{(1,1),(1,3),(1,-2),(2,2),(2,3),(3,-3)\}$ is shown above with each pipe in its own colour. One obtains

$$w_M(1,2,3,-3,-2,-1) = (-2,1,-3,2,3,-1)$$

which agrees with $w_M = s_{1,1}s_{1,3}s_{1,-2}s_{2,2}s_{2,3}s_{3,-3}$.

In fact, pipe dreams for type C_n can be viewed as a special case of pipe dreams for type A_{2n-1}, i.e. for $\mathfrak{sl}_{2n}$. Indeed, one may identify the type C_n GT poset with the "left half" of the type A_{2n-1} GT poset. Then the $2n$ pipes in the type C pipe dream of M will just be end parts of the $2n$ pipes in the type A pipe dream of the same set M.

We again introduce a "twisted" version of the correspondence determined by the choice of $O \subset P$: set $w_M^O = w_O^{-1} w_M$.

3.3 The intermediate Schubert degeneration

We now construct a degeneration $\widetilde{F}$ of the symplectic flag variety which will be used as an intermediate step: toric degenerations will be obtained as further degenerations of $\widetilde{F}$.

Definition 6. *A tuple $(i_1, \dots, i_k)$ of elements of N is admissible if for every $l \in [1, n]$ the number of elements with $|i_j| \le l$ does not exceed l. Let Θ denote the set of all admissible tuples of the form $(i_1 \lessdot \dots \lessdot i_k)$ and $\Theta_k \subset \Theta$ denote the subset of k-tuples.*

Consider the space $\mathbb{C}^N \simeq \mathbb{C}^{2n}$ with basis $\{e_i\}_{i \in N}$. One has a standard embedding $V_{\omega_k} \subset \wedge^k \mathbb{C}^N$. Let $\{X_{i_1,\dots,i_k}\}_{i_1 \lessdot \dots \lessdot i_k}$ be the basis in $(\wedge^k \mathbb{C}^N)^*$ dual to $\{e_{i_1} \wedge \dots \wedge e_{i_k}\}_{i_1 \lessdot \dots \lessdot i_k}$. It is known ([4]) that the set $\{X_{i_1,\dots,i_k}\}_{(i_1,\dots,i_k) \in \Theta_k}$ projects to a basis in $V^*_{\omega_k}$.This allows us to identify $V^*_{\omega_k}$ with $\mathbb{C}^{\Theta_k}$. The multihomogeneous coordinate ring of

$$\mathbb{P} = \mathbb{P}(V_{\omega_1}) \times \dots \times \mathbb{P}(V_{\omega_n})$$

is then identified with $\mathbb{C}[\Theta] = \mathbb{C}[X_{i_1,\dots,i_k}]_{(i_1,\dots,i_k) \in \Theta}$. The Plücker embedding of the complete symplectic flag variety $F \hookrightarrow \mathbb{P}$ is defined by the symplectic Plücker ideal $I \subset \mathbb{C}[\Theta]$.

Next, consider the variety F_{A} of type A partial flags in $\mathbb{C}^N$ of signature $(1, \dots, n)$. It is embedded into

$$\mathbb{P}_{\mathrm{A}} = \mathbb{P}(\wedge^1 \mathbb{C}^N) \times \dots \times \mathbb{P}(\wedge^n \mathbb{C}^N)$$

where it is cut out by the Plücker ideal I_{A} in the multihomogeneous coordinate ring $S = \mathbb{C}[X_{i_1,\dots,i_k}]_{k \in [1,n], \{i_1 \lessdot \dots \lessdot i_k\} \subset N}$. Consider the surjection $\pi : S \to \mathbb{C}[\Theta]$ mapping all $X_{i_1,\dots,i_k} \notin \mathbb{C}[\Theta]$ to 0 and fixing $\mathbb{C}[\Theta]$. Set $\widetilde{I} = \pi(I_{\mathrm{A}})$.

Theorem 3. *There exists a monomial order $\widetilde{<}$ on $\mathbb{C}[\Theta]$ such that $\mathrm{in}_{\widetilde{<}} I = \widetilde{I}$.*

This means that $\widetilde{F}$, the zero set of $\widetilde{I}$ in $\mathbb{P}$, is a flat degeneration of F. Also, importantly to us, every initial ideal of $\widetilde{I}$ is an initial ideal of I. The advantage of working with $\widetilde{I}$ instead of degenerating I directly is that the former is more simply expressed as a homomorphism kernel, allowing us to use the technique of sagbi degenerations. Consider Z, the $n \times 2n$ matrix with rows indexed by $[1, n]$ and columns indexed by N such that $Z_{i,j} = z_{i,j}$ if $(i, j) \in P$ and $Z_{i,j} = 0$ otherwise. Denote by $D_{i_1,\dots,i_k}$ the minor of Z spanned by rows $1, \dots, k$ and columns $i_1, \dots, i_k$.

Proposition 6. *$\widetilde{I}$ is the kernel of the homomorphism $X_{i_1,\dots,i_k} \mapsto D_{i_1,\dots,i_k}$ from $\mathbb{C}[\Theta]$ to $\mathbb{C}[P]$.*

A noteworthy property of $\widetilde{F}$ is that it is a type A Schubert variety in F_{A}. Indeed, $\pi^{-1}(\widetilde{I})$ cuts out $\widetilde{F}$ in $\mathbb{P}_{\mathrm{A}}$ and $\pi^{-1}(\widetilde{I})$ is generated by I_{A} and all $X_{i_1,\dots,i_k} \notin \mathbb{C}[\Theta]$. Consider the alternative order $-1 \lessdot' 1 \lessdot' \cdots \lessdot' -n \lessdot n$ on N. One sees that $X_{i_1,\dots,i_k} \notin \mathbb{C}[\Theta]$ if and only if $(i_1,\dots,i_k)$ has a reordering $(j_1 \lessdot' \cdots \lessdot' j_k)$ such that $j_l \lessdot' -l$ for some l. Now one sees that $\pi^{-1}(\widetilde{I})$ is indeed the defining ideal of a Schubert variety in F_{A}. Namely, the Schubert variety corresponding to the Borel subgroup in $SL(\mathbb{C}^N)$ given by the ordering $\lessdot'$ and the torus-fixed point $y \in F_{\mathrm{A}}$ with all multihomogeneous coordinates zero except for $y_{-1,\dots,-k}$ with $k \in [1,n]$.

3.4 Toric degenerations

Fix $O \subset P$ containing A. We can now realize the toric variety of $\mathcal{Q}_O(\lambda)$ as a degeneration of F by identifying I_O with an initial ideal of $\widetilde{I}$. This is again done via an isomorphism between $\mathbb{C}[\mathcal{J}]$ and $\mathbb{C}[\Theta]$. For $J \in \mathcal{J}$ denote $w^O_{M_O(J)} = w^J$.

Lemma 1. *For every $J \in \mathcal{J}_k$ and $i \in [1,k]$ one has $|w^J(i)| \ge i$. In particular, $(w^J(1),\dots,w^J(k))$ is admissible.*

The lemma lets us define a homomorphism $\psi : \mathbb{C}[\mathcal{J}] \to \mathbb{C}[\Theta]$, for $J \in \mathcal{J}_k$ we set

$$\psi(X_J) = X_{w^J(1),\dots,w^J(k)}.$$

Theorem 4. *The map ψ is an isomorphism and for a certain (explicitly defined) monomial order $<$ on S one has $\psi(I_O) = \mathrm{in}_< \widetilde{I}$. In particular, for regular λ the toric variety of $\mathcal{Q}_O(\lambda)$ is a flat degeneration of $\widetilde{F}$ and, subsequently, of F.*

3.5 Newton–Okounkov bodies

Following [15] we associate a Newton–Okounkov body of F with a line bundle $\mathcal{L}$, a global section τ of $\mathcal{L}$ and a valuation ν on the function field $\mathbb{C}(F)$. We choose an integral dominant $\lambda = (a_1,\dots,a_n)$ and let $\mathcal{L}$ be the $Sp_{2n}(\mathbb{C})$-equivariant line bundle on F associated with the weight λ. In terms of the multiprojective embedding $F \subset \mathbb{P}$ this is the restriction of $\mathcal{O}(a_1,\dots,a_n)$ to F. Consider the $\mathbb{Z}^N$-grading on $\mathbb{C}[\Theta]$ given by $\operatorname{grad} X_{i_1,\dots,i_k} = \omega_k$ and the induced grading on the Plücker algebra $R = C[\Theta]/I$. Then $H^0(F,\mathcal{L})$ is identified with the homogeneous component of grading λ in R. No we choose $\tau \in H^0(F,\mathcal{L})$ as the image of $\prod_k X_{1,\dots,k}^{a_k}$ in R.

To define the valuation ν we first define a valuation on R. Theorems 3 and 4 provide a monomial order $<_0$ on $\mathbb{C}[\Theta]$ such that $\mathrm{in}_{<_0} I = \psi(I_O)$. The proofs of the theorems show that $<_0$ arises from a total monomial order $\ll$ on $\mathbb{C}[P]$ in the following sense. Consider the homomorphism $\rho : \mathbb{C}[\Theta] \to \mathbb{C}[P]$ mapping the variable $\psi(X_J)$ to $z^{x_O(J)}$, note that

$\rho = \varphi_O \psi^{-1}$ and $\ker \rho = \psi(I_O)$. Then for two monomials one has $M_1 <_0 M_2$ if and only if $\rho(M_1) \ll \rho(M_2)$. The monomial order $\ll$ corresponds to a semigroup order on $\mathbb{Z}^P_{\geq 0}$ which we also denote by $\ll$. We have a $(\mathbb{Z}^P_{\geq 0}, \ll)$-filtration on R with component R_x, $x \in \mathbb{Z}^P_{\geq 0}$ spanned by the images of monomials $M \in \mathbb{C}[\Theta]$ such that $\rho(M) \ll z^x$. By general properties of initial degenerations we then have $\operatorname{gr} R \simeq \mathbb{C}[\Theta] / \operatorname{in}_{<_0} I$ (which is the toric ring of $\mathcal{Q}_O(\lambda)$). For nonzero $p \in R$ we now define $\nu(p)$ as the $\ll$-minimal x such that $p \in R_x$: by definition, such a map is a valuation if and only if $\operatorname{gr} R$ is an integral domain. One sees that ν maps the (image in R of) $\psi(X_J)$ to $x_O(J)$ and, consequently,

$$\nu(H^0(F, \mathcal{L}) \backslash \{0\}) = \mathcal{Q}_O(\lambda) \cap \mathbb{Z}^P.$$

Since $\mathbb{C}(F)$ consists of fractions p/q where grad-homogeneous $p, q \in R$ satisfy $\operatorname{grad} p = \operatorname{grad} q$, we can now extend the valuation to $\mathbb{C}(F)$ by $\nu(p/q) = \nu(p) - \nu(q)$.

Definition 7. *The Newton–Okounkov body of F defined by $\mathcal{L}$, τ and ν is the convex hull closure*

$$\Delta = \overline{\operatorname{conv}\left\{ \frac{\nu(\sigma/\tau^m)}{m} \,\middle|\, m \in \mathbb{Z}_{>0}, \sigma \in H^0(F, \mathcal{L}^{\otimes m}) \backslash \{0\} \right\}} \subset \mathbb{R}^P.$$

For every $k \in [1, n]$ we have a unique $J \in \mathcal{J}_k$ such that $w^J(\{1, \ldots, k\}) = \{1, \ldots, k\}$, denote $x_k = x_O(J)$. For $\lambda = (a_1, \ldots, a_n)$ denote $x_\lambda = a_1 x_1 + \cdots + a_n x_n$. We have $\nu(\tau) = x_\lambda$ and it is now straightforward to deduce

Theorem 5. $\Delta = \mathcal{Q}_O(\lambda) - x_\lambda$.

Acknowledgements

We thank Michael Joswig for helpful comments.

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