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Levi-spherical varieties and Demazure characters

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Abstract. We prove a short, root-system uniform, combinatorial classification of Levi-spherical Schubert varieties for any generalized flag variety G/B of finite Lie type. We apply this to the study of multiplicity-free decompositions of Demazure modules and their characters.

Keywords: Demazure characters, multiplicity-free, Schubert varieties, Levi subgroups, spherical varieties, toric varieties

1 Introduction

1.1 History and motivation

In his essay [17] on representation theory and invariant theory, R. Howe discusses the significance of multiplicity-free actions as an organizing principle for the subject. Classical invariant theory focuses on actions of a reductive group G on symmetric algebras, which is to say, coordinate rings of vector spaces. Now one also considers G-actions on varieties X and their coordinate rings $\mathbb{C}[X]$. Such an action is multiplicity-free if $\mathbb{C}[X]$ decomposes, as a G-module, into irreducible G-modules each with multiplicity one. An important example is when X is the *base affine space* of a complex, semisimple algebraic group G [3]; in this case the coordinate ring is a multiplicity-free direct sum of the irreducible representations of G. Lustzig's theory of dual canonical bases [24] provides a basis for it. In the early 2000s, understanding this basis was a motivation for S. Fomin and A. Zelevinsky's theory of Cluster algebras [11].

The notion of multiplicity-free actions is closely connected to that of *spherical varieties*. Let G be a connected, complex, reductive algebraic group; we say that a variety X is a G-variety if X is equipped with an action of G that is a morphism of varieties. A spherical variety is a normal G-variety where a Borel subgroup of G has an open, and therefore dense, orbit. A normal, affine G-variety X is spherical if and only if $\mathbb{C}[X]$ decomposes into irreducible G-modules each with multiplicity one [31]. If X is instead a normal,

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projective G-variety then one can still recover one direction of this implication. That is, if the induced G-action on the homogeneous coordinate ring of X is multiplicity-free, then X is G-spherical [15, Proposition 4.0.1].

Spherical varieties possess numerous nice properties. For instance, projective spherical varieties are Mori Dream Spaces. Moreover, Luna-Vust theory describes all the birational models of a spherical variety in terms of colored fans; these fans generalize the notion of fans used to classify toric varieties (which are themselves spherical varieties). N. Perrin's excellent survey covers additional background on spherical varieties [27].

It is an open problem to classify all spherical actions on products of flag varieties. This is solved in the case of Levi subgroups; we point to the work of P. Littelmann [23], P. Magyar–J. Weyman–A. Zelevinsky [25, 26], J. Stembridge [29, 30], R. Avdeev–A. Petukhov [1, 2]. Connecting back to the representation-theoretic perspective of [17], in [29, 30], J. Stembridge relates this classification problem to the multiplicity-freeness of restrictions of *Weyl modules* [12, Lecture 6]. Indeed, the homogeneous coordinate ring of a single flag variety is a multiplicity-free sum of spaces of global sections on the variety with respect to line bundles associated to each dominant integral weight. By the Borel-Weil-Bott theorem, these spaces are isomorphic to the irreducible representations of G. This is the central object of interest in *Standard Monomial Theory* [22] and is closely related to the coordinate ring of base affine space mentioned above. As remarked above a product of flag varieties is G-spherical if its homogeneous coordinate ring is multiplicity-free as an G-module.

This paper solves a related problem. We classify all *Levi-spherical* Schubert varieties in a single flag variety; that is, Schubert varieties that are spherical for the action of a Levi subgroup. Here, the relevant ring is the homogeneous coordinate ring of a Schubert variety and the attendant representation theory is that of *Demazure modules* [10], which are Borel subgroup representations. Critically for this paper, they are also Levi subgroup representations. Multiplicity-freeness in this setting refers to the decomposition of these modules into irreducible Levi subgroup representations. This study was initiated in [16] and the authors solved the problem for the GL_n case in [14]. In [13] we conjectured an answer for all finite rank Lie types; this paper proves that conjecture.[1]

1.2 Background

Throughout, let G be a complex, connected, reductive algebraic group and let $B \leq G$ be a choice of Borel subgroup along with a maximal torus T contained in B. The *Weyl group* is $W := N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. The orbits of the homogeneous space G/B under the action of B by left translations are the *Schubert cells*

[1]During the completion of this article, we learned that M. Can-P. Saha [4] independently proved the conjecture.

$X_w^\circ, w \in W$. Their Zariski closures

$$X_w := \overline{X_w^\circ}$$

are the *Schubert varieties*. It is relevant below that these varieties are normal [9, 28].

A *parabolic subgroup* of G is a closed subgroup $B \subset P \subsetneq G$ such that G/P is a projective variety. Each such P admits a *Levi decomposition*

$$P = L \ltimes R_u(P)$$

where L is a reductive subgroup called a *Levi subgroup* of P and $R_u(P)$ is the unipotent radical. One parabolic subgroup is $P_w := \mathrm{stab}_G(X_w)$. Any of the parabolic subgroups $B \subseteq Q \subseteq P_w$ act on X_w.

Let L_Q be a Levi subgroup of Q. A variety X is *H-spherical* for the action of a complex reductive algebraic group H if it is normal and contains an open, and therefore dense, orbit of a Borel subgroup of H. Our reference for spherical varieties is [27]; toric varieties are examples of spherical varieties.

Definition 1.1 ([16, Definition 1.8])**.** Let $B \subseteq Q \subseteq P_w$ be a parabolic subgroup of G. $X_w \subseteq G/B$ is *L_Q-spherical* if X_w has a dense, open orbit of a Borel subgroup of L_Q under left-translations.

1.3 The main result

We give a root-system uniform combinatorial criterion to decide if X_w is L_Q-spherical. Let $\Phi := \Phi(\mathfrak{g}, T)$ be the root system of weights for the adjoint action of T on the Lie algebra $\mathfrak{g}$ of G. It has a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots. Let $\Delta \subset \Phi^+$ be the base of simple roots. The parabolic subgroups $Q = P_I \supset B$ are in bijection with subsets I of Δ; let $L_I := L_Q$. The set of *left descents* of w is

$$\mathcal{D}_L(w) = \{\beta \in \Delta : \ell(s_\beta w) < \ell(w)\},$$

where $\ell(w) = \dim X_w$ is the *Coxeter length* of w. Under the bijection, $P_w = P_{\mathcal{D}_L(w)}$, and $B \subset Q \subseteq P_w = P_{\mathcal{D}_L(w)}$ satisfy $Q = P_I$ for some $I \subseteq \mathcal{D}_L(w)$.

For $I \subset \Delta$, a *parabolic subgroup* $W_I \subseteq W$ is the subgroup generated by $S_I := \{s_\beta : \beta \in I\}$. A *standard Coxeter element* $c \in W_I$ is any product of the elements of S_I listed in some order. Let $w_0(I)$ be the longest element of W_I. The following definition was given in type A in [14, Definition 1.1] and in general type in [13, Section 4]:

Definition 1.2. Let $w \in W$ and $I \subseteq \mathcal{D}_L(w)$ be fixed. Then w is I-spherical if $w_0(I)w$ is a standard Coxeter element for W_J where $J \subseteq \Delta$.

We first note that if $I \subseteq \mathcal{D}_L(w)$, then the left inversion set $\mathcal{I}(w)$, defined in Section 3, contains all the positive roots in the root subsystem generated by I, and thus $w = w_0(I)d$ is a length-additive expression for some $d \in W$, by Proposition 3.1.3 in [5].

Theorem 1.3. *Fix $w \in W$ and $I \subseteq \mathcal{D}_L(w)$. X_w is L_I-spherical if and only if w is I-spherical.*

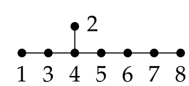
Theorem 1.3 resolves the main conjecture of the authors' earlier work [13, Conjecture 4.1] and completes the main goal set forth in [16]. In [14], Theorem 1.3 was established in the case $G = GL_n$ using essentially algebraic combinatorial methods concerning *Demazure characters* (or in their type A embodiment, the *key polynomials*). In contrast, the geometric arguments of this paper are quite different, significantly shorter, but require more background of the reader in algebraic groups. Theorem 1.3 is a generalization of work of P. Karuppuchamy [21] that characterizes Schubert varieties that are toric, which in our setup means spherical for the action of $L_\emptyset = T$. Using work of R. S. Avdeev–A. V. Petukhov [1], Theorem 1.3 may also be seen as a generalization of some results of P. Magyar–J. Weyman–A. Zelevinsky [25] and J. Stembridge [29, 30] on spherical actions on G/B; see [16, Theorem 2.4]. Previously, there was not even a finite algorithm to decide L_I-sphericality of X_w in general.

1.4 Organization

Examples of the main result are given in Section 2. Sections 3 and 4 prove Theorem 1.3. Section 5 applies our main result to the study of Demazure modules [10].

2 Examples of Theorem 1.3

We begin with a few examples illustrating Theorem 1.3.

Example 2.1 (E_8 cf. [16, Example 1.3]). The E_8 Dynkin diagram is 

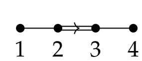
. One associates the simple roots β_i ($1 \leq i \leq 8$) with this labeling and writes $s_i := s_{\beta_i}$. Suppose

$$w = s_2 s_3 s_4 s_2 s_3 s_4 s_5 s_4 s_2 s_3 s_1 s_4 s_5 s_6 s_7 s_6 s_8 s_7 s_6 \in W.$$

Then $\mathcal{D}_L(w) = \{\beta_2, \beta_3, \beta_4, \beta_5, \beta_7, \beta_8\}$. Let $I = \mathcal{D}_L(w)$. Here

$$w_0(I) = s_3 s_2 s_4 s_3 s_2 s_4 s_5 s_4 s_3 s_2 s_4 s_5 \cdot s_7 s_8 s_7 \quad \text{and} \quad w_0(I)w = s_1 s_6 s_7 s_8.$$

Since $w = w_0(I)c$ where $c = s_1 s_6 s_7 s_8$ is a standard Coxeter element, Theorem 1.3 asserts that X_w is L_I-spherical in the complete flag variety for E_8.

Example 2.2 (F_4 cf. [16, Example 1.5]). The F_4 diagram is 

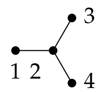
. First suppose

$$w = s_4 s_3 s_4 s_2 s_3 s_4 s_2 s_3 s_2 s_1 s_2 s_3 s_4 \ (I = \mathcal{D}_L(w) = \{\beta_2, \beta_3, \beta_4\}).$$

Then $w_0(I) = s_2s_3s_2s_3s_4s_3s_2s_3s_4$ and $w_0(I)w = s_1s_2s_3s_4$ is standard Coxeter. Hence X_w is L_I-spherical. On the other hand if

$$w' = s_2s_1s_4s_3s_2s_1s_3s_2s_4s_3s_2s_1 \ (I = \mathcal{D}_L(w') = \{\beta_2, \beta_4\}),$$

then $w_0(I) = s_2s_4$ and $w_0(I)w = s_1s_3s_2s_1s_3s_2s_4s_3s_2s_1$ is not standard Coxeter and X_w is not L_I-spherical.

Example 2.3 (D_4)**.** The D_4 diagram is 

. Let

$$w = s_3s_2s_3s_4s_2s_1s_2 \ (I = \mathcal{D}_L(w) = \{\beta_2, \beta_3\}).$$

Thus $w_0(I) = s_2s_3s_2$ and $w_0(I)w = s_4s_2s_1s_2$ is not standard Coxeter. Hence X_w is not L_I-spherical. The interested reader can check w is I-spherical in the (different) sense of [16, Definition 1.2]. Therefore, this w provides a counterexample to [16, Conjecture 1.9] in type D_4. This counterexample was also (implicitly) verified in [13] using a different method, namely Demazure character computations, the topic of Section 5.

3 An equivariant isomorphism

The primary goal of this section is to construct a torus equivariant isomorphism from a specified affine subspace of the open cell of a Schubert variety to the open cell of a distinguished Schubert subvariety. In what follows, we assume standard facts from the theory of algebraic groups. References we draw upon are [18, 6, 22].

Let $w \in W$. Let n_w be a coset representative of w in $N_G(T)$. By definition of $N_G(T)$ being the normalizer of T in G, $t \mapsto n_w t n_w^{-1}$ defines an automorphism $\gamma_w : T \to T$.

Lemma 3.1. *The automorphism γ_w does not depend on our choice of coset representative n_w.*

In light of Lemma 3.1, henceforth for $w \in W$ we will also let w denote a coset representative of w in $N_G(T)$. Let X be a T-variety with action denoted by $\cdot$. For each $w \in W$ we define an action $\cdot_w$ on X by $t \cdot_w x = \gamma_w(t) \cdot x$ for all $x \in X$ and $t \in T$.

Lemma 3.2. *For all $w \in W$, the T-variety X has an open, dense T-orbit for the action $\cdot$ if and only if it has an open, dense T-orbit for the action $\cdot_w$. Indeed, the set of T-orbits in X for these two actions is identical.*

For the remainder, we fix $\cdot$ to be the restriction to T of the action of G on G/B by left multiplication. The *left inversion set* of $w \in W$ is

$$\mathcal{I}(w) := \Phi^+ \cap w(\Phi^-) = \{\alpha \in \Phi^+ | w^{-1}(\alpha) \in \Phi^-\}.$$

Recall two standard facts regarding left inversion sets [19, Chapter 1]. For $w \in W$,

$$|\mathcal{I}(w)| = \ell(w) = \dim_{\mathbb{C}} X_w, \tag{3.1}$$

and

$$\mathcal{I}(w_0(I)) = \Phi^+(I), \tag{3.2}$$

where $\Phi(I) = \Phi(\mathfrak{l}_I, T)$ is the root system for the adjoint action of T on $\mathfrak{l}_I = \mathrm{Lie}(L_I)$.

We say that an algebraic group H is *directly spanned* by its closed subgroups $H_1, \ldots, H_n$, in the given order, if the product morphism

$$H_1 \times \cdots \times H_n \to H$$

is bijective. For $w \in W$, define $U_w := U \cap wU^- w^{-1}$, where U consists of the unipotent elements of B and similarly, U^- is the unipotent part of $B^- := w_0 B w_0$. This is a subgroup of U that is closed and normalized by T. Hence, by [6, §14.4], U_w is directly spanned, in any order, by the *root subgroups* U_α, $\alpha \in \Phi^+$, contained in U_w. Since by [20, Part II, 1.4(5)],

$$wU_\alpha w^{-1} = U_{w(\alpha)}, \tag{3.3}$$

these are the U_α such that $\alpha \in \Phi^+ \cap w(\Phi^-) = \mathcal{I}(w)$. Thus

$$U_w = \prod_{\alpha \in \mathcal{I}(w)} U_\alpha, \tag{3.4}$$

where the products U_α may be taken in any order.

Lemma 3.3. *For a coset $wB \in G/B$, we have*

$$X_w^\circ := BwB = U_w wB = \prod_{\alpha \in \mathcal{I}(w)} U_\alpha \; wB. \tag{3.5}$$

Moreover, X_w° is isomorphic to the affine space $\mathbb{A}^{\ell(w)}$ (as varieties).

We say that $w = uv \in W$ is *length additive* if $\ell(uv) = \ell(u) + \ell(v)$. Under this hypothesis, by [7, Ch. VI, §1, Cor. 2 of Prop. 17] one has

$$\mathcal{I}(uv) = \mathcal{I}(u) \sqcup u(\mathcal{I}(v)).$$

Therefore, in particular, if we assume $w_0(I)d \in W$ is *length additive*, then

$$\mathcal{I}(w_0(I)d) = \mathcal{I}(w_0(I)) \sqcup w_0(I)(\mathcal{I}(d)). \tag{3.6}$$

Define

$$V_d := w_0(I) U_d w_0(I)^{-1} = w_0(I) U_d w_0(I).$$

Lemma 3.4. *V_d is a closed subgroup of $U_{w_0(I)d}$ that is normalized by T.*

Lemma 3.5. *$U_{w_0(I)d}$ is directly spanned by $U_{w_0(I)}$ and V_d:*

$$U_{w_0(I)d} = U_{w_0(I)} V_d = V_d U_{w_0(I)}. \tag{3.7}$$

Define

$$\tilde{O} := V_d w_0(I) dB \subseteq G/B.$$

Lemma 3.6. *$\tilde{O}$ is T-stable for the action $\bullet$.*

The following is the main point of this section:

Proposition 3.7. *If $w_0(I)d \in W$ is length additive then*

$$X^\circ_{w_0(I)d} = U_{w_0(I)d}\, w_0(I)dB.$$

Hence $\tilde{O} \subset X^\circ_{w_0(I)d}$. Moreover, $\tilde{O}$ with the T-action $\bullet$ is T-equivariantly isomorphic to X°_d with the T-action $\bullet_{w_0(I)}$.

4 Proof of the main result

We need a lemma examining the L_I-action on $\tilde{O}$. This lemma is then used in conjunction with Proposition 3.7 to prove our main result.

Let $B_{L_I} = L_I \cap B$ and let U_{L_I} be the unipotent radical of B_{L_I}. Then B_{L_I} is a Borel subgroup in L_I [6, §14.17] with $U_{L_I} = B_{L_I} \cap U$ and $B_{L_I} = T \ltimes U_{L_I}$. Since L_I is the subgroup of G generated by T and $\{U_\alpha \mid \alpha \in \Phi(I)\}$ [22, §3.2.2], it is straightforward to show that

$$U_{L_I} = \prod_{\alpha \in \Phi^+(I)} U_\alpha,$$

where the product is taken in any order [6, §14.4].

Lemma 4.1. *Let $w = w_0(I)d \in W$ be length additive. Let $x \in X^\circ_{w_0(I)d} \setminus \tilde{O}$ and $y, z \in \tilde{O}$.*

(i) $uy \notin \tilde{O}$ for all $u \in U_{L_I}$ with $u \neq e$.

(ii) $tx \notin \tilde{O}$ for all $t \in T$.

(iii) There exists $b \in B_{L_I}$ such that $by = z$ if and only if there exists $t \in T$ such that $ty = z$.

We now have the necessary ingredients to complete the proof of Theorem 1.3.

5 Application to Demazure modules

As an application of these results we give a sufficient condition for a Demazure module to be a multiplicity-free L_I-module; equivalently, a sufficient condition for a Demazure character to be multiplicity-free with respect to the basis of irreducible L_I-characters.

Let $\mathfrak{X}(T)$ denote the lattice of weights of T; our fixed Borel subgroup B determines a subset of dominant integral weights $\mathfrak{X}(T)^+ \subset \mathfrak{X}(T)$. The finite-dimensional irreducible G-representations are indexed by $\lambda \in \mathfrak{X}(T)^+$. Denoting the associated representation by V_λ, there is a class of B-submodules of V_λ, first introduced by Demazure [10], that are indexed by $w \in W$. If v_λ is a nonzero highest weight vector, then the *Demazure module* V_λ^w is the minimal B-submodule of V_λ containing wv_λ.

There is a geometric construction of these Demazure modules. For $\lambda \in \mathfrak{X}(T)^+$, let $\mathfrak{L}_\lambda$ be the associated line bundle on G/B. For $w \in W$, we write $\mathfrak{L}_\lambda|_{X_w}$ for the restriction of $\mathfrak{L}_\lambda$ to the Schubert subvariety $X_w \subseteq G/B$. Then the Demazure module V_λ^w is isomorphic to the dual of the space of global sections of $\mathfrak{L}_\lambda|_{X_w}$, that is

$$V_\lambda^w \cong H^0(X_w, \mathfrak{L}_\lambda|_{X_w})^*.$$

This geometric perspective highlights the fact that V_λ^w is not just a B-module, but is in fact also a L_I-module via the action induced on $H^0(X_w, \mathfrak{L}_\lambda|_{X_w})$ by the left multiplication action of L_I on X_w.

As L_I is a reductive group over characteristic zero, any L_I-module decomposes into a direct sum of irreducible L_I-modules. Let $\mathfrak{X}_{L_I}(T)^+$ be the set of dominant integral weights with respect to the choice of maximal torus and Borel subgroup $T \subseteq B_I \subseteq L_I$. For $\mu \in \mathfrak{X}_{L_I}(T)^+$, let $V_{L_I,\mu}$ be the associated irreducible L_I-module. If M is a L_I-module and

$$M = \bigoplus_{\mu \in \mathfrak{X}_{L_I}(T)^+} V_{L_I,\mu}^{\oplus m_{L_I,\mu}}$$

is the decomposition into irreducible L_I-modules, then we say that M is a *multiplicity-free L_I-module* if $m_{L_I,\mu} \in \{0,1\}$. Similarly, if $\mathrm{char}(M)$ is the formal T-character of M and

$$\mathrm{char}(M) = \sum_{\mu \in \mathfrak{X}_{L_I}(T)^+} m_{L_I,\mu} \mathrm{char}(V_{L_I,\mu}),$$

then we say that $\mathrm{char}(M)$ is *I-multiplicity-free* if $m_{L_I,\mu} \in \{0,1\}$.

The following argument was given for type A in [16, Theorem 4.13(II)]. We prove the general type argument (which is essentially the same) for sake of completeness:

Theorem 5.1. *Let $w \in W$ with $I \subseteq D_L(w)$. Then X_w is L_I-spherical if and only if for all $\lambda \in \mathfrak{X}(T)^+$, the Demazure module V_λ^w is multiplicity-free L_I-module.*

Corollary 5.2. *Let $w \in W$ be I-spherical for $I \subseteq D_L(w)$. For all $\lambda \in \mathfrak{X}(T)^+$, the Demazure module V_λ^w is a multiplicity-free L_I-module.*

Corollary 5.3. *Let $w \in W$ be I-spherical for $I \subseteq D_L(w)$. For all $\lambda \in \mathfrak{X}(T)^+$, the Demazure character* $\mathrm{char}(V_\lambda^w)$ *is I-multiplicity-free.*

These two corollaries appear non-trivial from a combinatorial perspective, even for a *specific choice* of dominant weight λ with fixed $w \in W$. The Demazure character can be recursively computed using Demazure operators. There is also a combinatorial rule for the character in terms of crystal bases (in instantiations such as the *Littelmann path model* or the *alcove walk model*); see, e.g., the textbook [8]. However, an argument based on these methods eludes in general type, although we have an argument in type A [14].

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