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Extended Schur Functions and Bases Related by Involutions

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Abstract. The extended Schur and shin functions are Schur-like bases of $QSym$ and $NSym$. We define a creation operator and a Jacobi-Trudi rule for certain shin functions and show that a similar Jacobi-Trudi rule does not exist for every shin function. We also define the skew extended Schur functions and relate them to the multiplicative structure of the shin basis. Then, we introduce two new pairs of dual bases that result from applying the ρ and ω involutions to the extended Schur and shin functions. These bases are defined combinatorially via variations on shin-tableaux much like the row-strict extended Schur functions.

Keywords: Schur-like, QSym, NSym, extended Schur Function, Shin function

1 Introduction

There has been considerable interest over the last decade in studying Schur-like bases of $NSym$ and $QSym$. A basis $\{\mathcal{S}_\alpha\}_\alpha$ of $NSym$ is generally considered ***Schur-like basis*** if $\chi(\mathcal{S}_\lambda) = s_\lambda$ for any partition λ where the ***forgetful map*** $\chi : NSym \to Sym$ gives the commutative image of an element in $NSym$. A Schur-like basis $\{\mathcal{S}^*_\alpha\}_\alpha$ of $QSym$ is informally defined as a basis dual to a Schur-like basis $\{\mathcal{S}_\alpha\}_\alpha$ of $NSym$. These bases are usually defined combinatorially in terms of tableaux that resemble or generalize the semistandard Young tableaux. The canonical Schur-like bases of $NSym$ and $QSym$ are respectively the immaculate basis [3], the Young noncommutative Schur basis [8], and the shin basis [6], as well as the dual immaculate basis, the Young quasisymmetric Schur basis, and the extended Schur functions.

The shin and extended Schur functions, which are dual bases, are unique among the Schur-like bases for having arguably the most natural relationship with the Schur functions. In $NSym$, the commutative image of a shin function indexed by a partition is a Schur function, while the commutative image of any other shin function is 0. In $QSym$, the extended Schur functions indexed by partitions are equal to Schur functions [6]. The goal of this extended abstract is to answer questions about basis expansions, skew functions, and multiplicative properties of these two bases and introduce new, related

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bases. In the full paper, we present additional results including a second type of skew function and connections to the antipode map on $NSym$ [7].

1.1 The Shin and Extended Schur Functions

The dual shin functions were introduced by Campbell, Feldman, Light, Shuldiner, and Xu in [6] as the duals to the shin functions and defined independently by Assaf and Searles in [2] as the extended Schur functions, which are the stable limits of polynomials related to Kohnert diagrams. We use the name "extended Schur functions" but otherwise retain the notation and terminology of the dual shin functions.

Definition 1.1. Let α and β be a composition and weak composition of n, respectively. A ***shin-tableau*** of shape α and type β is a labeling of the boxes of the diagram of α by positive integers such that the number of boxes labeled by i is β_i, the sequence of entries in each row is weakly increasing from left to right, and the sequence of entries in each column is strictly increasing from top to bottom.

Note that shin-tableaux are a direct generalization of semistandard Young tableaux to composition shapes.

Example 1.2. The shin-tableaux of shape $(3,4)$ and type $(1,2,1,1,2)$ are

1	2	2	
3	4	5	5

1	2	3	
2	4	5	5

1	2	4	
2	3	5	5

A shin-tableau of n boxes is *standard* if each number 1 through n appears exactly once. The *descent set* is defined as $Des_{ש}(U) = \{i : i+1 \text{ is strictly below } i \text{ in } U\}$ for a standard shin-tableau U. Each entry i in $Des_{ש}(U)$ is called a *descent* of U. The *descent composition* of U is defined $co_{ש}(U) = (i_1, i_2 - i_1, \ldots, i_d - i_{d-1}, n - i_d)$ for $Des_{ש}(U) = \{i_1, \ldots, i_d\}$.

The *shin reading word* of a shin-tableau T, denoted $rw_{ש}(T)$, is the word obtained by reading the rows of T from left to right starting with the bottom row and moving up. To *standardize* a shin-tableau T of size n, we will replace entries in T with the numbers 1 through n to obtain a standard shin-tableau. First, replace the 1's in T with $1, 2, \ldots$ in the order they appear in $rw_{ש}(T)$. Then replace the 2's with the next consecutive numbers, again in the order they appear in $rw_{ש}(T)$, then the 3's, etc.

For a composition α, the ***extended Schur function*** is defined as $ש^*_\alpha = \sum_T x^T$ where the sum runs over shin-tableaux T of shape α. Their positive expansions into the monomial and fundamental bases in terms of shin-tableaux are known [2, 6]. For a composition α,

$$ש^*_\alpha = \sum_\beta \mathcal{K}_{\alpha,\beta} M_\beta \qquad \text{and} \qquad ש^*_\alpha = \sum_\beta \mathcal{L}_{\alpha,\beta} F_\beta, \tag{1.1}$$

ש is the hebrew character *shin*.

where $\mathcal{K}_{\alpha,\beta}$ denotes the number of shin-tableaux of shape α and type β, and $\mathcal{L}_{\alpha,\beta}$ denotes the number of standard shin-tableaux of shape α with descent composition β.

Example 1.3. The F-expansion of the extended Schur function $\mathfrak{w}^*_{(2,3)}$:

$$\mathfrak{w}^*_{(2,3)} = F_{(2,3)} + F_{(1,2,2)}$$

1	2	
3	4	5

1	3	
2	4	5

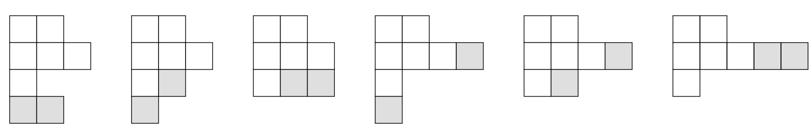
The shin basis of *NSym* was introduced by Campbell, Feldman, Light, Shuldiner, and Xu in [6]. Let α and β be compositions. Then β is said to differ from α by a *shin-horizontal strip* of size r, denoted $\alpha \subset^{\mathfrak{w}}_r \beta$, provided for all i, we have $\beta_i \geq \alpha_i$, $|\beta| = |\alpha| + r$, and for any $i \in \mathbb{N}$ if $\beta_i > \alpha_i$ then for all $j > i$, we have $\beta_j \leq \alpha_i$. The shin functions are defined recursively based on a right Pieri rule using shin-horizontal strips.

Definition 1.4. The ***shin basis*** $\{\mathfrak{w}_\alpha\}_\alpha$ of *NSym* is defined as the unique set of functions $\mathfrak{w}_\alpha$ such that $\mathfrak{w}_\alpha H_r = \sum_{\alpha \subset^{\mathfrak{w}}_r \beta} \mathfrak{w}_\beta$, where the sum runs over all compositions β which differ from α by a shin-horizontal strip of size r.

Intuitively, the compositions β in the summation are given by taking diagrams of α and adding r blocks on the right such that if you add boxes to some row i then no row below i is longer than the original row i. This is referred to as the *overhang rule*.

Example 1.5. The following expression can be visualized with the tableaux below.

$$\mathfrak{w}_{(2,3,1)} H_{(2)} = \mathfrak{w}_{(2,3,1,2)} + \mathfrak{w}_{(2,3,2,1)} + \mathfrak{w}_{(2,4,1,1)} + \mathfrak{w}_{(2,4,2)} + \mathfrak{w}_{(2,5,1)}$$



Repeated application of this right Pieri rule yields the expansion of a complete homogeneous noncommutative symmetric function in terms of the shin functions. This expansion verifies that the extended Schur functions and the shin functions are dual bases. This allows us to expand the ribbon functions into the shin basis dually to the expansion of the extended Schur functions expanded into the fundamental basis [2, 6].

$$H_\beta = \sum_{\alpha \geq_\ell \beta} \mathcal{K}_{\alpha,\beta} \mathfrak{w}_\alpha \qquad \text{and} \qquad R_\beta = \sum_{\beta \leq_\ell \alpha} \mathcal{L}_{\alpha,\beta} \mathfrak{w}_\alpha. \tag{1.2}$$

The extended Schur functions have the special property that $\mathfrak{w}^*_\lambda = s_\lambda$ for a partition λ. Since the forgetful map χ is dual to the inclusion map from *Sym* to *QSym*, $\chi(\mathfrak{w}_\lambda) = s_\lambda$ when λ is a partition and $\chi(\mathfrak{w}_\alpha) = 0$ otherwise. Another interesting feature of the shin functions is their relationship with the other two canonical Schur-like bases, the immaculate functions and the Young noncommutative Schur functions. Given a partition λ, the immaculate function $\mathfrak{S}_\lambda$ equals the Young noncommutative Schur function $\hat{\mathbf{s}}_\lambda$, but the shin function $\mathfrak{w}_\lambda$ differs from the two.

2 A Creation Operator for Certain Shin Functions

The Schur functions and the immaculate functions can both be defined using ***creation operators***. In fact, the immaculate basis was originally defined in terms of noncommutative Bernstein operators [3]. It is using these operators that one can prove various properties of the immaculate basis including the Jacobi-Trudi rule [3], a left Pieri rule [5], a combinatorial interpretation of the inverse Kostka matrix [1], and a partial Littlewood Richardson rule [4]. Here we give similar creation operators for certain shin functions which then allow us to define a ***Jacobi-Trudi rule***. This rule is especially useful because there is currently no other combinatorial way to expand shin functions into the complete homogeneous basis.

Definition 2.1. For a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ with $k \geq 1$ and a positive integer m, define the action of the linear operator $\beth_m$ on the complete homogeneous basis by

$$\beth_m(1) = H_m \qquad \text{and} \qquad \beth_m(H_\alpha) = H_{m,\alpha_1,\alpha_2,\dots} - H_{\alpha_1,m,\alpha_2,\dots}.$$

Theorem 2.2. *If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ with $k \geq 1$ and $0 < m < \alpha_1$, then $\beth_m(\text{ש}_\alpha) = \text{ש}_{m,\alpha}$.*

This theorem follows from showing inductively that the functions given by $\beth_m(\text{ש}_\alpha)$ satisfy the right Pieri rule defining the shin functions, and then showing $\beth_m(\text{ש}_\alpha)$ equals $\text{ש}_{(m,\alpha)}$ by recursive calculation. These operators allow us to construct shin functions indexed by strictly increasing compositions from the ground up.

Corollary 2.3. *Let $\beta = (\beta_1, \dots, \beta_k)$ where $\beta_i < \beta_{i+1}$. Then, $\beth_{\beta_1} \cdots \beth_{\beta_k}(1) = \text{ש}_\beta$.*

Example 2.4. These creation operators can be used to build up $\text{ש}_{(1,3,4)}$ as follows:

$$\beth_1\beth_3\beth_4(1) = \beth_1\beth_3(H_4) = \beth_1(H_{(3,4)} - H_{(4,3)}) = H_{(1,3,4)} - H_{(1,4,3)} - H_{(3,1,4)} + H_{(4,1,3)}.$$

Using these operators, we define the following Jacobi-Trudi rule to express these same shin functions as matrix determinants. Let $S_k^{\geq}(-1)$ be the set of permutations $\sigma \in S_k$ such that $\sigma(i) \geq i - 1$ for all $i \in [k]$.

Theorem 2.5. *Let $\beta = (\beta_1, \dots, \beta_k)$ be a composition such that $\beta_i < \beta_{i+1}$ for all i. Then,*

$$\text{ש}_\beta = \sum_{\sigma \in S_k^{\geq}(-1)} (-1)^\sigma H_{\beta_{\sigma(1)}} \cdots H_{\beta_{\sigma(k)}}.$$

Equivalently, ש_β can be expressed as the matrix determinant

$\beth$ is the hebrew character ***beth***.

$$\mathfrak{w}_\beta = det \begin{vmatrix} H_{\beta_1} & H_{\beta_2} & H_{\beta_3} & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ H_{\beta_1} & H_{\beta_2} & H_{\beta_3} & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ 0 & H_{\beta_2} & H_{\beta_3} & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ 0 & 0 & H_{\beta_3} & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ 0 & 0 & 0 & \cdots & 0 & H_{\beta_{k-1}} & H_{\beta_k} \end{vmatrix}$$

using the noncommutative determinant obtained by expanding along the first row.

We can show by counterexample that there is not a matrix rule of this form for every shin function, not even those indexed by partitions [7]. It remains open to find a combinatorial or algebraic way of understanding the expansion of the shin basis into the complete homogeneous basis for the general case.

3 Skew Extended Schur Functions

To define skew extended Schur functions, we first use an algebraic approach, and then connect it to tableaux combinatorics. For $F \in QSym$, the operator $F^\perp$ acts on elements $H \in NSym$ based on the relation $\langle H, FG \rangle = \langle F^\perp H, G \rangle$. For dual bases $\{A_\alpha\}_\alpha$ of $QSym$ and $\{B_\alpha\}_\alpha$ of $NSym$ this expands as $F^\perp(H) = \sum_\alpha \langle H, FA_\alpha \rangle B_\alpha$.

Definition 3.1. For compositions α and β with $\beta \subseteq \alpha$, the ***skew extended Schur functions*** are defined as $\mathfrak{w}^*_{\alpha/\beta} = \mathfrak{w}^\perp_\beta(\mathfrak{w}^*_\alpha)$.

By the equation for $F^\perp$ above, we expand $\mathfrak{w}^*_{\alpha/\beta}$ into various bases as follows.

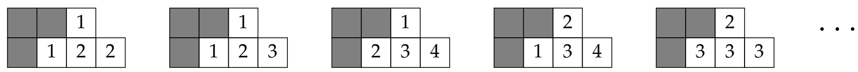
Proposition 3.2. *For compositions* $\beta \subseteq \alpha$, $\mathfrak{w}^*_{\alpha/\beta} = \sum_\gamma \langle \mathfrak{w}_\beta H_\gamma, \mathfrak{w}^*_\alpha \rangle M_\gamma = \sum_\gamma \langle \mathfrak{w}_\beta \mathfrak{w}_\gamma, \mathfrak{w}^*_\alpha \rangle \mathfrak{w}^*_\gamma$. *Furthermore, let* $C^\alpha_{\beta,\gamma} := \langle \mathfrak{w}_\beta \mathfrak{w}_\gamma, \mathfrak{w}^*_\alpha \rangle$. *Then,* $\mathfrak{w}_\beta \mathfrak{w}_\gamma = \sum_\alpha C^\alpha_{\beta,\gamma} \mathfrak{w}_\alpha$.

Using the properties of the forgetful map and the shin basis, we have the following statement about the coefficients that appear in the skew extended Schur functions.

Proposition 3.3. *Let* α, β, γ *be compositions that are not partitions and let* λ, μ, ν *be partitions. Then,* $C^\nu_{\lambda,\beta} = C^\nu_{\alpha,\mu} = C^\nu_{\alpha,\beta} = 0$ *and* $C^\nu_{\lambda,\mu} = c^\nu_{\lambda,\mu}$, *where* $c^\nu_{\lambda,\mu}$ *are the usual Littlewood-Richardson coefficients.*

The skew extended Schur functions can also be expressed in terms of skew shin-tableaux.

Proposition 3.4. *For compositions* α *and* β *such that* $\beta \subseteq \alpha$, $\langle \mathfrak{w}_\beta H_\gamma, \mathfrak{w}^*_\alpha \rangle$ *is equal to the number of skew shin-tableau of shape* α/β *and type* γ. *Moreover,* $\mathfrak{w}^*_{\alpha/\beta} = \sum_T x^T$, *where the sum runs over skew shin-tableau* T *of shape* α/β.



$$\text{ש}^*_{(3,4)/(2,1)} = x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2x_3x_4 + x_1x_2x_3x_4 + x_2x_3^3 + \dots$$

Skew shin-tableaux of shape λ/μ where λ and μ are partitions are simply skew semistandard Young tableaux. By Proposition 3.4, these skew extended Schur functions are equal to the usual skew Schur functions, $\text{ש}^*_{\lambda/\mu} = s_{\lambda/\mu}$.

4 Involutions on $QSym$ and $Nsym$

In $QSym$, we consider three involutions defined on the fundamental basis and their dual maps in $NSym$ [8]. They each are defined as extensions of involutions on compositions. The *complement* of a composition α is defined $\alpha^c = comp(set(\alpha)^c)$, where $set((\alpha_1, \dots, \alpha_k)) = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \dots + \alpha_{k-1}\}$ if α is a composition of n, and $comp(\{s_1, \dots, s_j\}) = (s_1, s_2 - s_1, \dots, s_j - s_{j-1}, n - s_j)$, for $\{s_1, \dots, s_j\} \subseteq [n-1]$. The *reverse* of $(\alpha_1, \dots, \alpha_k)$ is $\alpha^r = (\alpha_k, \dots, \alpha_1)$. The *transpose* of α is defined by $\alpha^t = (\alpha^r)^c = (\alpha^c)^r$.

Definition 4.1. The involutions ψ, ρ and ω on $QSym$ and $NSym$ are defined as

$$\psi(F_\alpha) = F_{\alpha^c} \qquad \rho(F_\alpha) = F_{\alpha^r} \qquad \omega(F_\alpha) = F_{\alpha^t}$$
$$\psi(R_\alpha) = R_{\alpha^c} \qquad \rho(R_\alpha) = R_{\alpha^r} \qquad \omega(R_\alpha) = R_{\alpha^t},$$

extended linearly. All three involutions on $QSym$ and ψ on $NSym$ are automorphisms, while ρ and ω on $NSym$ are anti-automorphisms.

Note that we use the same notation for the corresponding involutions on $QSym$ and $NSym$. These automorphisms commute and $\omega = \rho \circ \psi = \psi \circ \rho$. When ω and ψ are restricted to Sym, they are both equivalent to the classical involution $\omega : Sym \to Sym$ which acts on the Schur functions by $\omega(s_\lambda) = \lambda'$ where λ' is the conjugate of λ. The conjugate of a partition λ is found by flipping the diagram of λ over the diagonal.

Applying ψ to the extended Schur and shin functions recovers the row strict extended Schur and row strict extended shin functions ($\mathfrak{R}\text{ש}$) of Niese, Sundaram, van Willigenburg, Vega, and Wang in [9]. We define two new pairs of dual bases ($\reflectbox{ש}$ and $\mathfrak{R}\reflectbox{ש}$) in $QSym$ and $NSym$ by applying ρ and ω to the extended Schur and shin functions as

$$\psi(\text{ש}^*_\alpha) = \mathfrak{R}\text{ש}^*_\alpha \qquad \rho(\text{ש}^*_\alpha) = \reflectbox{ש}^*_{\alpha^r} \qquad \omega(\text{ש}^*_\alpha) = \mathfrak{R}\reflectbox{ש}^*_{\alpha^r}$$
$$\psi(\text{ש}_\alpha) = \mathfrak{R}\text{ש}_\alpha \qquad \rho(\text{ש}_\alpha) = \reflectbox{ש}_{\alpha^r} \qquad \omega(\text{ש}_\alpha) = \mathfrak{R}\reflectbox{ש}_{\alpha^r}.$$

We give combinatorial interpretations of these two new pairs of bases in terms of variations on shin-tableaux. While specific definitions are to follow, we first describe intuitively how ψ, ρ, and ω act on the tableaux defining each basis. Recall that shin-tableaux have weakly increasing columns and strictly increasing rows. The ψ map switches whether the strictly changing condition is on rows or columns (while the other has a weakly changing condition). The ρ map switches the row condition from increasing to decreasing, or vice versa. The ω map does both. Through this combinatorial interpretation, each of the four pairs of dual bases is related to any other by one of the three involutions ψ, ρ, or ω as shown in the figure below.

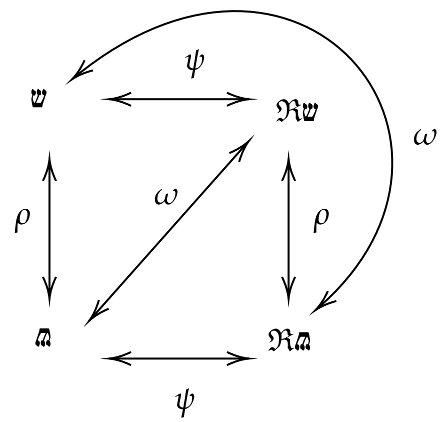


Figure 1: Mappings between shin variants in *NSym*.

Essentially, ψ, ρ, and ω in *QSym* and *NSym* collectively serve as the analogue to the classical ω in *Sym*. In *Sym*, the Schur basis is its own image under ω but in *QSym* and *NSym* our Schur-like basis is instead part of a system of four related bases that are in a sense closed under the three involutions ψ, ρ, and ω. While the combinatorics of these bases are similar, they may have very different applications. For example, the quasisymmetric Schur basis and the Young quasisymmetric Schur basis are related by the involution ρ but the former is much more compatible with Macdonald polynomials while the latter is more useful when working with Schur functions [8].

The table below serves to summarize the tableaux defined over the course of this section. It lists each type of tableaux, the position of $i+1$ relative to i that makes i a descent, the order the boxes appear in the reading word (Left, Right, Top, Bottom), the condition on entries of each row, and the condition on entries in each column.

	Descent	**Reading Word**	**Rows**	**Columns**
Shin	strictly below	L to R, B to T	weakly increasing	strictly incr.
Row-strict	weakly above	L to R, T to B	strictly increasing	weakly incr.
Reverse	strictly below	R to L, B to T	weakly decreasing	strictly incr.
Row-strict rev.	weakly above	R to L, T to B	strictly decreasing	weakly incr.

We now briefly review row-strict extended Schur and row-strict shin functions but reserve more details for the full paper [7]. Let α be a composition and let β be a weak composition, allowing for zero entries. A ***row-strict shin-tableaux*** (RSST) of shape α and type β is a filling of the composition diagram of α with positive integers such that each row strictly increases from left to right, each column weakly increases from top to bottom, and each integer i appears β_i times. A *standard* row-strict shin-tableaux (SRSST) with n boxes is one containing the entries 1 through n each exactly once. For a composition α, define the ***row strict extended Schur function*** as $\mathfrak{R}\mathfrak{w}^*_\alpha = \sum_T x^T$, where the sum runs over all row-strict shin-tableaux T of shape α. The ***row strict shin functions*** are defined as the duals in $NSym$ to the row strict extended Schur functions in $QSym$.

For a standard row-strict shin-tableau U, the *descent set* is defined to be $Des_{\mathfrak{R}\mathfrak{w}}(U) = \{i : i+1 \text{ is weakly above } i \text{ in } U\}$. Each entry i in $Des_{\mathfrak{R}\mathfrak{w}}(U)$ is called a *descent* of U. The *descent composition* of U is defined to be $co_{\mathfrak{R}\mathfrak{w}}(U) = (i_1, i_2 - i_1, \ldots, i_d - i_{d-1}, n - i_d)$ for $Des_{\mathfrak{R}\mathfrak{w}}(U) = \{i_1, \ldots, i_d\}$. Equivalently, the descent composition is found by counting the number of entries in U (in the order they are numbered) between each descent. Note that the set of standard row-strict shin-tableaux is exactly the same as the set of standard shin-tableaux. Using the framework of standard row-strict shin-tableaux, it is shown in [9] that for a composition α, the row-strict extended Schur function expands into the fundamental basis as $\mathfrak{R}\mathfrak{w}^*_\alpha = \sum_U F_{co_{\mathfrak{R}\mathfrak{w}}(U)}$, where the sum runs over all standard row-strict shin-tableaux.

Example 4.2. The F-expansion of the row-strict extended Schur function $\mathfrak{R}\mathfrak{w}^*_{(2,3)}$:

$$\mathfrak{R}\mathfrak{w}^*_{(2,3)} = F_{(1,2,1,1)} + F_{(2,2,1)}$$

1	2	
3	4	5

1	3	
2	4	5

We can now relate the extended Schur and row-strict extended Schur functions by using ψ on their F-expansions. This relationship follows from the fact that the set of standard tableaux is the same but the definitions of descent sets are in a sense complementary and the map ψ is using complements. For all compositions α, $\psi(\mathfrak{w}^*_\alpha) = \mathfrak{R}\mathfrak{w}^*_\alpha$, and $\{\mathfrak{R}\mathfrak{w}^*_\alpha\}_\alpha$ is a basis of $QSym$. Additionally, $\psi(\mathfrak{w}_\alpha) = \mathfrak{R}\mathfrak{w}_\alpha$ and $\{\mathfrak{R}\mathfrak{w}_\alpha\}_\alpha$ is a basis of $NSym$.

4.1 Reverse extended Schur and shin functions.

Let α be a composition and β a weak composition. A *reverse shin-tableau* of shape α and type β is a composition diagram α filled with positive integers that weakly decrease along the rows and strictly increase along the columns (from top to bottom) where each positive integer i appears β_i times.

Definition 4.3. For a composition α, the ***reverse extended Schur function*** is defined as $\reflectbox{$\mathfrak{w}$}^*_\alpha = \sum_T x^T$, where the sum runs over all reverse shin-tableaux T of shape α.

A *standard* reverse shin-tableau of shape α is one containing the entries 1 through n each exactly once. For a standard reverse shin-tableau S, the *descent set* is defined as $Des_{\mathfrak{m}}(S) = \{i : i+1 \text{ is strictly below } i \text{ in } S\}$. Each entry i in $Des_{\mathfrak{m}}(S)$ is called a *descent* of S. The *descent composition* of S is defined $co_{\mathfrak{m}}(S) = comp(Des_{\mathfrak{m}}(S))$. Define $flip(S)$ to be the tableau U obtained by flipping S horizontally (in other words, reversing the order of the rows of S) and then replacing each entry i with $n-i$. It is easy to see that the map $flip$ is an involution between the set of standard shin-tableaux and the set of standard reverse shin-tableaux.

$$flip\left(\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \cline{1-2}\end{array}\right) = \begin{array}{|c|c|c|}\cline{2-3} & 4 & 1 \\ \hline 5 & 3 & 2 \\ \hline\end{array}$$

The ***reverse shin-reading word*** of a reverse shin-tableau T, denoted $rw_{\mathfrak{m}}(T)$, is the word obtained by reading the rows of T from right to left starting with the top row and moving down. To *standardize* a reverse shin-tableau T, replace the 1's with $1, 2, \ldots$ in the order they appear in $rw_{\mathfrak{m}}(T)$, then the 2's starting with the next consecutive number, etc.

Proposition 4.4. *For a composition α, $\mathfrak{m}^*_\alpha = \sum_S F_{co_{\mathfrak{m}}(S)}$, where the sum runs over standard reverse shin-tableaux U of shape α.*

Example 4.5. The F-expansion of the reverse extended Schur function $\mathfrak{m}^*_{(3,2)}$:

$$\mathfrak{m}^*_{(3,2)} = F_{(3,2)} + F_{(2,2,1)} \qquad \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 & \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 5 & 3 & \\ \cline{1-2}\end{array}$$

The descent composition of a standard shin-tableau U is the reverse of the descent composition of the standard reverse shin-tableau given by $flip(U)$. Using this fact, we show that the reverse extended Schur functions are the image of the extended Schur functions under ρ.

Theorem 4.6. *For a composition α, $\rho(\mathfrak{w}^*_\alpha) = \mathfrak{m}^*_{\alpha^r}$, and $\{\mathfrak{m}_\alpha\}_\alpha$ is a basis of QSym.*

Now, we define the reverse shin basis by applying ρ to the shin basis.

Definition 4.7. For a composition α, the ***reverse shin function*** is defined as $\mathfrak{m}_\alpha = \rho(\mathfrak{w}_{\alpha^r})$.

By the invariance of ρ under duality, we have that the reverse shin functions are the dual basis to the reverse extended Schur functions, that is $\langle \mathfrak{m}_\alpha, \mathfrak{m}^*_\beta \rangle = \delta_{\alpha,\beta}$.

Let $\mathcal{K}^{\mathfrak{m}}_{\alpha,\beta}$ be the number of reverse shin-tableaux of shape α and type β, and let $\mathcal{L}^{\mathfrak{m}}_{\alpha,\beta}$ be the number of reverse shin-tableaux with shape α and descent composition β. Then,

$$\mathfrak{m}^*_\alpha = \sum_\beta \mathcal{K}^{\mathfrak{m}}_{\alpha,\beta} M_\beta = \sum_\beta \mathcal{L}^{\mathfrak{m}}_{\alpha,\beta} F_\beta \quad \text{and} \quad H_\beta = \sum_\alpha \mathcal{K}^{\mathfrak{m}}_{\alpha,\beta} \mathfrak{m}_\alpha \quad \text{and} \quad R_\beta = \sum_\alpha \mathcal{L}^{\mathfrak{m}}_{\alpha,\beta} \mathfrak{m}_\alpha.$$

By applying ρ, we can translate many of the results on the shin functions to the reverse shin functions.

Theorem 4.8. *For compositions α, β and a positive integer m,*

1. $H_m \text{ש}_\alpha = \sum_{\alpha^r \subset_m^{\text{ש}} \beta^r} \text{ש}_\beta.$

2. $H_\beta = \sum_\alpha \mathcal{K}_{\alpha^r,\beta^r} \text{ש}_\alpha$ *and* $R_\beta = \sum_\alpha \mathcal{L}_{\alpha^r,\beta^r} \text{ש}_\alpha.$

3. $\text{ש}^*_{\lambda^r} = s_\lambda$. *Also,* $\chi(\text{ש}_{\lambda^r}) = s_\lambda$ *and* $\chi(\text{ש}_\alpha) = 0$ *when* α^r *is not a partition.*

4. *For a composition γ such that $\gamma_i > \gamma_{i+1}$ for all $1 \leq i \leq \ell(\gamma)$,*

$$\text{ש}_\gamma = \sum_{\sigma \in S_{\ell(\gamma)}} (-1)^\sigma H_{\gamma_{\sigma(1)}} \cdots H_{\gamma_{\sigma(\ell(\gamma))}},$$

where the sum runs over $\sigma \in S_{\ell(\gamma)}$ such that $\sigma(i) \geq i - 1$ for all $i \in [\ell(\gamma)]$.

4.2 Row-strict reverse extended Schur and shin functions.

Let α be a composition and β be a weak composition. A *row-strict reverse shin-tableau* (BST) of shape α and type β is a filling of the diagram of α with positive integers such that the entries in each row are strictly decreasing from left to right and the entries in each column are weakly increasing from top to bottom where each integer i appears β_i times. These are essentially a row-strict version of the reverse shin-tableaux.

Definition 4.9. For a composition α, the ***row-strict reverse extended Schur function*** is defined as $\mathfrak{R}\text{ש}^*_\alpha = \sum_T x^T$, where the sum runs over all row-strict reverse shin-tableaux T of shape α.

A row-strict reverse shin-tableau of shape α is *standard* if it includes the entries 1 through n each exactly once. For a standard row-strict reverse shin-tableau S, the *descent set* is defined to be $Des_{\mathfrak{R}\text{ש}}(S) = \{i : i+1 \text{ is weakly above } i \text{ in } S\}$. Each entry i in $Des_{\mathfrak{R}\text{ש}}(S)$ is called a *descent* of S. Then, we define the *descent composition* of S to be $co_{\mathfrak{R}\text{ש}}(S) = comp(Des_{\mathfrak{R}\text{ש}}(S))$. Equivalently, the descent composition is found by counting the number of entries in S (in the order they are numbered) between each descent. Note that the set of standard row-strict reverse shin-tableaux is exactly the same as the set of standard reverse shin-tableaux.

The *row-strict reverse shin reading word* of a row-strict reverse shin-tableau T, denoted $rw_{\mathfrak{R}\text{ש}}(T)$ is the word obtained by reading the rows of T from right to left starting with the bottom row and moving up. We can *standardize* a standard row-strict reverse shin-tableau as follows. To *standardize* a row-strict reverse shin-tableau T, replace the 1's in T with $1, 2, \ldots$ in the order they appear in $rw_{\mathfrak{R}\text{ש}}(T)$, then the 2's starting with the next consecutive integer, then 3's, etc.

Proposition 4.10. *For a composition α, $\mathfrak{R}\varpi^*_\alpha = \sum_S F_{co_{\mathfrak{R}\varpi}(S)}$, where the sum runs over standard row-strict reverse shin-tableaux S.*

Example 4.11. The F-expansion of the reverse extended Schur function $\mathfrak{R}\varpi^*_{(3,2)}$:

$$\mathfrak{R}\varpi^*_{(3,2)} = F_{(1,1,2,1)} + F_{(1,2,2)}$$

3	2	1
5	4	

4	2	1
5	3	

The descent compositions of row-strict reverse shin tableaux of shape α^r are complementary to the descent compositions of reverse tableaux of shape α^r, thus $\psi(\varpi_{\alpha^r}) = \mathfrak{R}\varpi^*_{\alpha^r}$. Given that $\varpi^*_{\alpha^r} = \rho(\mathfrak{w}^*_\alpha)$ and $\psi \circ \rho = \omega$, we have the following result.

Theorem 4.12. *For a composition α, $\omega(\mathfrak{w}^*_\alpha) = \mathfrak{R}\varpi^*_{\alpha^r}$ and $\{\mathfrak{R}\varpi^*_\alpha\}_\alpha$ is a basis of QSym.*

The row-strict reverse extended Schur basis is not equivalent to the extended Schur basis, the row-strict extended Schur basis, or the reverse extended Schur basis. Again, it is simple to check that there exist row-strict reverse extended Schur functions that are not elements in the extended Schur, row-strict extended Schur, or reverse extended Schur bases. Like with ψ and ρ, it follows from the dual definitions of ω in *NSym* and *QSym* that ω is invariant under duality. Thus, the row-strict reverse extended Schur functions are dual to the row-strict reverse shin functions when defined as follows.

Definition 4.13. For a composition α, define the *row-strict reverse shin function* $\mathfrak{R}\varpi_\alpha = \omega(\mathfrak{w}_{\alpha^r})$.

Let $\mathcal{K}^{\mathfrak{R}\varpi}_{\alpha,\beta}$ be the number of row-strict reverse shin-tableaux of shape α and type β, and let $\mathcal{L}^{\mathfrak{R}\varpi}_{\alpha,\beta}$ be the number of standard row-strict reverse shin-tableaux with shape α and descent composition β. The expansions of the row-strict reverse extended Schur functions into the monomial and fundamental bases follow those of the extended Schur functions. That is,

$$\mathfrak{R}\varpi^*_\alpha = \sum_\beta \mathcal{K}^{\mathfrak{R}\varpi}_{\alpha,\beta} M_\beta = \sum_\beta \mathcal{L}^{\mathfrak{R}\varpi}_{\alpha,\beta} F_\beta, \quad \text{and} \quad H_\beta = \sum_\alpha \mathcal{K}^{\mathfrak{R}\varpi}_{\alpha,\beta} \mathfrak{R}\varpi_\alpha \quad \text{and} \quad R_\beta = \sum_\alpha \mathcal{L}^{\mathfrak{R}\varpi}_{\alpha,\beta} \mathfrak{R}\varpi_\alpha.$$

Now, we apply ω to the various results on the shin and extended Schur bases and find analogous results on the row-strict reverse shin and row-strict reverse extended Schur bases.

Theorem 4.14. *For compositions α, β and a positive integer m,*

1. $E_m \mathfrak{R}\varpi_\alpha = \sum_{\alpha^r \subset^{\mathfrak{w}}_m \beta^r} \mathfrak{R}\varpi_\beta$.

2. $E_\beta = \sum_\alpha \mathcal{K}_{\alpha^r,\beta^r} \mathfrak{R}\varpi_\alpha$ *and* $R_\beta = \sum_\alpha \mathcal{L}_{\alpha^r,\beta^t} \mathfrak{R}\varpi_\alpha$

3. $\mathfrak{R}\mathfrak{a}^*_{\lambda^r} = s_{\lambda'}$. *Also,* $\chi(\mathfrak{R}\mathfrak{a}_{\lambda^r}) = s_{\lambda'}$ *and* $\chi(\mathfrak{R}\mathfrak{a}_{\alpha}) = 0$ *when* α^r *is not a partition.*

4. *For a composition* γ *such that* $\gamma_i > \gamma_{i+1}$,

$$\mathfrak{R}\mathfrak{a}_{\gamma} = \sum_{\sigma \in S_{\ell(\gamma)}} (-1)^{\sigma} E_{\gamma_{\sigma(1)}} E_{\gamma_{\sigma(2)}} \cdots E_{\gamma_{\sigma(\ell(\gamma))}},$$

where the sum runs over $\sigma \in S_{\ell(\gamma)}$ *such that* $\sigma(i) \geq i-1$ *for all* $i \in [\ell(\gamma)]$.

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