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The characteristic quasi-polynomials for exceptional well-generated complex reflection groups

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Abstract. Kamiya, Takemura, and Terao initiated the theory of the characteristic quasi-polynomial of an integral arrangement, which is a function counting the elements in the complement of the arrangement modulo positive integers. The characteristic quasi-polynomials of crystallographic root systems exhibit many interesting properties. Recently, the authors extended the concept of the characteristic quasi-polynomials for arrangements over a Dedekind domain, where every residue ring with respect to nonzero ideal is finite. In this article, we investigate the characteristic quasi-polynomials for exceptional well-generated complex reflection groups, using the root systems over the rings of definition introduced by Lehrer and Taylor. We demonstrate that a specific relation between the Coxeter numbers and the LCM-periods of the characteristic quasi-polynomials is generalized in this context.

Résumé. Kamiya, Takemura et Terao ont initié la théorie du quasi-polynôme caractéristique d'un agencement intégral, qui est une fonction comptant les éléments dans le complément de l'agencement modulo les entiers positifs. Les quasi-polynômes caractéristiques des systèmes de racines cristallographiques présentent de nombreuses propriétés intéressantes. Récemment, les auteurs ont étendu le concept des quasi-polynômes caractéristiques aux agencements sur un domaine de Dedekind, où chaque anneau résiduel par rapport à un idéal non nul est fini. Dans cet article, nous examinons les quasi-polynômes caractéristiques pour les groupes de réflexion complexes exceptionnellement bien générés, en utilisant les systèmes de racines sur les anneaux définition introduits par Lehrer et Taylor. Nous démontrons qu'une relation spécifique entre les nombres de Coxeter et les LCM-périodes des quasi-polynômes caractéristiques est généralisée dans ce contexte.

Keywords: hyperplane arrangement, complex reflection group, root system, characteristic quasi-polynomial

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1 Introduction

1.1 Characteristic quasi-polynomials

For a positive integer ℓ, let $\mathcal{A} = \{c_1, \dots, c_n\} \subseteq \mathbb{Z}^\ell$ be a finite subset consisting of nonzero integral column vectors. Define the hyperplane arrangement $\mathcal{A}(\mathbb{R})$ in the vector space $\mathbb{R}^\ell$ by $\mathcal{A}(\mathbb{R}) := \{H_1, \dots, H_n\}$, where

$$H_j := \left\{ \boldsymbol{x} := (x_1, \dots, x_\ell) \in \mathbb{R}^\ell \;\middle|\; \boldsymbol{x}c_j = 0 \right\} \quad (j \in [n] := \{1, \dots, n\}).$$

Let $L(\mathcal{A}(\mathbb{R})) := \{ H_J \mid J \subseteq [n] \}$ be the set of intersections $H_J := \bigcup_{j \in J} H_j$. The set $L(\mathcal{A}(\mathbb{R}))$ equipped with the order defined by $X \leq Y \Leftrightarrow X \supseteq Y$ is called the **intersection lattice**. The **characteristic polynomial** $\chi_{\mathcal{A}(\mathbb{R})}$ is defined by

$$\chi_{\mathcal{A}(\mathbb{R})}(t) := \sum_{Z \in L(\mathcal{A}(\mathbb{R}))} \mu(Z) t^{\dim Z},$$

where μ denotes the **Möbius function** on $L(\mathcal{A}(\mathbb{R}))$, which is defined recursively by

$$\mu(\mathbb{R}^\ell) := 1 \qquad \text{and} \qquad \mu(Z) := -\sum_{Y < Z} \mu(Y) \text{ for } Z \neq \mathbb{R}^\ell.$$

The **complement** of an arrangement is the complement of the union of the members of the arrangement in the ambient space. Each connected component of the complement of $\mathcal{A}(\mathbb{R})$ is called a chamber. Zaslavsky [20] proved that the numbers of chambers and bounded chambers coincide with $|\chi_{\mathcal{A}(\mathbb{R})}(-1)|$ and $|\chi_{\mathcal{A}(\mathbb{R})}(1)|$. Orlik and Solomon [13] proved that $\chi_{\mathcal{A}(\mathbb{R})}(t)$ is equivalent to the Poincaré polynomial of the complement of the complexification of $\mathcal{A}(\mathbb{R})$.

Next, for any positive integer q, we define the **q-reduced arrangement** $\mathcal{A}(\mathbb{Z}/q\mathbb{Z})$ in $(\mathbb{Z}/q\mathbb{Z})^\ell$ by $\mathcal{A}(\mathbb{Z}/q\mathbb{Z}) := \{H_{1,q}, \dots, H_{n,q}\}$, where

$$H_{j,q} := \left\{ [\boldsymbol{x}]_q \in (\mathbb{Z}/q\mathbb{Z})^\ell \;\middle|\; \boldsymbol{x}c_j \equiv 0 \pmod{q} \right\} \quad (j \in [n])$$

and $[\boldsymbol{x}]_q$ denotes the equivalence class of $\boldsymbol{x}$.

Athanasiadis [1, Theorem 2.2] provided a method to compute the characteristic polynomial of an integral arrangement by counting the points of the complement of $\mathcal{A}(\mathbb{Z}/p\mathbb{Z})$ for large enough prime numbers p. Athanasiadis [2, Theorem 2.1] also proved that the characteristic polynomial can be computed by counting the points of the complement of $\mathcal{A}(\mathbb{Z}/q\mathbb{Z})$ for large enough integers q relatively prime a constant which depends only on $\mathcal{A}$.

Kamiya, Takemura, and Terao developed Athanasiadis' method by considering the complement of $\mathcal{A}(\mathbb{Z}/q\mathbb{Z})$ for all positive integers q as follows. For a nonempty subset

$J = \{j_1, \dots, j_k\} \subseteq [n]$, suppose that the matrix $C_J := (c_{j_1} \ \cdots \ c_{j_k})$ has the Smith normal form

$$\begin{pmatrix} d_{J,1} & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ 0 & d_{J,2} & & \vdots & & & \vdots \\ \vdots & & \ddots & 0 & & & \\ 0 & \cdots & 0 & d_{J,r(J)} & & & \\ \vdots & & & & 0 & & \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & & & & \cdots & 0 \end{pmatrix},$$

where $d_{J,i}$ is a positive integer such that $d_{J,i}$ divides $d_{J,i+1}$. Define $\rho_{\mathcal{A}} \in \mathbb{Z}_{>0}$ by

$$\rho_{\mathcal{A}} := \operatorname{lcm}\left\{ d_{J,r(J)} \;\middle|\; \varnothing \neq J \subseteq [n] \right\}.$$

Theorem 1.1 (Kamiya–Takemura–Terao [6])**.** *Let $M(\mathcal{A}(\mathbb{Z}/q\mathbb{Z})) := (\mathbb{Z}/q\mathbb{Z})^{\ell} \setminus \bigcup_{J\subseteq[n]} H_{J,q}$ denote the complement of $\mathcal{A}(\mathbb{Z}/q\mathbb{Z})$. Then the function $|M(\mathcal{A}(\mathbb{Z}/q\mathbb{Z}))|$ is a monic integral quasi-polynomial in $q \in \mathbb{Z}_{>0}$ with a period $\rho_{\mathcal{A}}$. Namely, there exist monic polynomials $f^k_{\mathcal{A}}(t) \in \mathbb{Z}[t]$ $(1 \leq k \leq \rho_{\mathcal{A}})$ such that $f^k_{\mathcal{A}}(q) = |M(\mathcal{A}(\mathbb{Z}/q\mathbb{Z}))|$ if $q \equiv k \pmod{\rho_{\mathcal{A}}}$. Furthermore, the quasi-polynomial has the **GCD-property**, that is, $f^k_{\mathcal{A}}(t) = f^{k'}_{\mathcal{A}}(t)$ when $\gcd(k, \rho_{\mathcal{A}}) = \gcd(k', \rho_{\mathcal{A}})$.*

Definition 1.2. We call the quasi-polynomial

$$\chi^{\text{quasi}}_{\mathcal{A}}(q) := |M(\mathcal{A}(\mathbb{Z}/q\mathbb{Z}))|$$

the **characteristic quasi-polynomial** of $\mathcal{A}$. The period $\rho_{\mathcal{A}}$ is called the **LCM-period**. The polynomial $f^k_{\mathcal{A}}(t)$ is said to be the k**-constituent** of $\chi^{\text{quasi}}_{\mathcal{A}}(q)$.

Interestingly enough, each constituent of the characteristic quasi-polynomial has a combinatorial interpretation (See [12, 17] for details). In particular, the following holds.

Theorem 1.3 (Kamiya–Takemura–Terao [6, Theorem 2.5])**.** *The 1-constituent of the characteristic quasi-polynomial of $\mathcal{A}$ is the characteristic polynomial of the hyperplane arrangement $\mathcal{A}(\mathbb{R})$. Namely, $f^1_{\mathcal{A}}(t) = \chi_{\mathcal{A}(\mathbb{R})}(t)$.*

For a decade, it was an open problem whether the LCM-period is minimum or not. Recently Higashitani, Tran, and Yoshinaga gave an affirmative answer for central arrangements.

Theorem 1.4 (Higashitani–Tran–Yoshinaga [4, Theorem 1.2])**.** *The LCM-period $\rho_{\mathcal{A}}$ is the minimum period of the characteristic quasi-polynomial $\chi^{\text{quasi}}_{\mathcal{A}}(q)$.*

Remark 1.5. The characteristic quasi-polynomial and its LCM-period can be considered for non-central arrangements [8]. Higashitani, Tran, and Yoshinaga [4] also studied non-central arrangements such that the LCM-periods are not minimum.

1.2 Crystallographic root systems

Let Φ be an irreducible crystallographic root system and Φ^+ a positive system of Φ. Every positive root is expressed as a linear combination of the simple roots with integral coefficients. Gathering the coefficient column vectors, we obtain the set $\mathcal{A}_\Phi$ consisting of integral column vectors. Kamiya, Takemura, and Terao [5, 7] computed the characteristic quasi-polynomial $\chi_\Phi^{\mathrm{quasi}}(q)$ of $\mathcal{A}_\Phi$ and its LCM-period explicitly by using the classification of root systems. Note that Suter [16] gave essentially the same calculation in terms of the number of lattice points in the fundamental alcoves (the Ehrhart quasi-polynomials).

Kamiya, Takemura, and Terao [7, Theorem 3.1] gave an explicit formula of the generating function $\Gamma_\Phi := \sum_{q=1}^{\infty} \chi_\Phi^{\mathrm{quasi}}(q)t^q$ for an irreducible crystallographic root system Φ in terms of the coefficient of the highest root and the Coxeter number. We obtain the following corollaries.

Corollary 1.6 (Kamiya–Takemura–Terao [7, Corollary 3.2])**.** *Let $n_1, \dots, n_\ell$ be the coefficient of the highest root of Φ with respect to the simple roots. Then $\mathrm{lcm}(n_1, \dots, n_\ell)$ coincides with the LCM-period of $\chi_\Phi^{\mathrm{quasi}}(q)$.*

Corollary 1.7 (Kamiya–Takemura–Terao [7, Corollary 3.4])**.** *Let h be the Coxeter number of Φ. Then $\chi_\Phi^{\mathrm{quasi}}(q) > 0$ if and only if $q \geq h$.*

The characteristic quasi-polynomial of an irreducible crystallographic root system also has duality with respect to the Coxeter number. The duality can be shown from the explicit expressions given by Kamiya, Takemura, and Terao [5], or Suter [16]. Yoshinaga [19] gave a classification-free proof.

Theorem 1.8 (Yoshinaga [19, Corollary 3.8])**.** *Let Φ be an irreducible crystallographic root system of rank ℓ and h its Coxeter number. Then $\chi_\Phi^{\mathrm{quasi}}(q) = (-1)^\ell \chi_\Phi^{\mathrm{quasi}}(h-q)$.*

Note that the duality holds as quasi-polynomials but not the level of the constituents. Yoshinaga [18] studied the condition for the constituents to hold the duality in detail.

Combining Theorem 1.8, Theorem 1.3, and the following theorem, we can deduce that the characteristic polynomial $\chi_\Phi(t)$ of the arrangement $\mathcal{A}_\Phi$ satisfies the duality (Corollary 1.10).

Theorem 1.9 (Kamiya–Takemura–Terao [7], Suter [16])**.** *The radical of the LCM period of $\chi_\Phi^{\mathrm{quasi}}(q)$ divides the Coxeter number h.*

Corollary 1.10. *Let Φ be an irreducible crystallographic root system of rank ℓ and h its Coxeter number. Then $\chi_\Phi(q) = (-1)^\ell \chi_\Phi(h-q)$.*

1.3 Characteristic quasi-polynomials over residually finite Dedekind domains

Let $\mathcal{O}$ be a Dedekind domain such that the residue ring $\mathcal{O}/\mathfrak{a}$ is finite for every nonzero ideal $\mathfrak{a}$. Such a ring $\mathcal{O}$ is called a **residually finite Dedekind domain** or a Dedekind domain with the finite norm property. The ring $\mathbb{Z}$ is an example of a residually finite Dedekind domain. More generally, the ring of integers of an algebraic number field is a residually finite Dedekind domain. The authors generalized the notion of characteristic quasi-polynomials for $\mathcal{O}$ as follows.

Let $\mathcal{A} = \{c_1, \dots, c_n\} \subseteq \mathcal{O}^{\ell}$ and $\mathfrak{a} \in I(\mathcal{O})$, where $I(\mathcal{O})$ denotes the set of nonzero ideals of $\mathcal{O}$. Define the **$\mathfrak{a}$-reduced arrangement** $\mathcal{A}(\mathcal{O}/\mathfrak{a})$ by $\mathcal{A}(\mathcal{O}/\mathfrak{a}) := \left\{ H_{j,\mathfrak{a}} \;\middle|\; j \in [n] \right\}$, where

$$H_{j,\mathfrak{a}} := \left\{ [\boldsymbol{x}]_{\mathfrak{a}} \in (\mathcal{O}/\mathfrak{a})^{\ell} \;\middle|\; \boldsymbol{x}c_j \equiv 0 \pmod{\mathfrak{a}} \right\}.$$

Let $M(\mathcal{A}(\mathcal{O}/\mathfrak{a}))$ denote the complement of $\mathcal{A}(\mathcal{O}/\mathfrak{a})$. Namely

$$M(\mathcal{A}(\mathcal{O}/\mathfrak{a})) := (\mathcal{O}/\mathfrak{a})^{\ell} \setminus \bigcup_{j=1}^{n} H_{j,\mathfrak{a}}.$$

Definition 1.11. The function $\chi_{\mathcal{A}}^{\mathrm{quasi}} \colon I(\mathcal{O}) \to \mathbb{Z}$ determined by $\chi_{\mathcal{A}}^{\mathrm{quasi}}(\mathfrak{a}) := |M(\mathcal{A}(\mathcal{O}/\mathfrak{a}))|$ is called the **characteristic quasi-polynomial** of $\mathcal{A}$.

The function $\chi_{\mathcal{A}}^{\mathrm{quasi}}$ is described by using finitely many polynomials periodically as ordinary quasi-polynomials.

Theorem 1.12 ([9, Theorem 3.1])**.** *There exists an ideal $\rho \in I(\mathcal{O})$ such that the following statement holds: For any divisor $\kappa \mid \rho$ there exists a monic polynomial $f_{\mathcal{A}}^{\kappa}(t) \in \mathbb{Z}[t]$ such that*

$$\mathfrak{a} + \rho = \kappa \Longrightarrow \chi_{\mathcal{A}}^{\mathrm{quasi}}(\mathfrak{a}) = f_{\mathcal{A}}^{\kappa}(N(\mathfrak{a})),$$

where $N(\mathfrak{a}) := |\mathcal{O}/\mathfrak{a}|$, the ***absolute norm*** *of $\mathfrak{a}$.*

The ideal ρ above is called a **period**. We can construct a period $\rho_{\mathcal{A}}$ (called the **LCM-period**) for $\chi_{\mathcal{A}}^{\mathrm{quasi}}(\mathfrak{a})$ using the structure theorem for finitely generated modules over Dedekind domains and the authors proved that the LCM-period $\rho_{\mathcal{A}}$ is minimum (See [9, Theorem 5.1] for details). If $\mathcal{O}$ is a Euclidean domain, then we can compute the LCM-period algorithmically by computing the Smith normal forms and elementary divisors.

2 Characteristic quasi-polynomials for exceptional well-generated complex reflection groups

Let V be a finite-dimensional complex vector space. A map $r \in \mathrm{GL}(V)$ is called a **reflection** if $\ker(r - \mathrm{id}_V)$ has codimension 1. A finite subgroup $G \subseteq \mathrm{GL}(V)$ is called a

complex reflection group is G is generated by reflections. We say that G is **irreducible** if there are no nontrivial G-invariant subspaces. In this case, the dimension of the ambient space is called the **rank** of G. An irreducible complex reflection group G of rank ℓ is **well-generated** if G is generated by ℓ reflections. Irreducible complex reflection groups are classified by Shephard and Todd [14]. There are an infinite family $G(m, p, \ell)$ and 34 exceptional cases labeled by $G_4, \ldots, G_{37}$. Among the exceptional groups, we have 26 well-generated ones, which are listed in Table 1.

Definition 2.1. Let G be an irreducible reflection group. Define the **field of definition** $K(G)$ by

$$K(G) := \mathbb{Q}(\operatorname{tr}(\sigma) \mid \sigma \in G).$$

Define the **ring of definition** $\mathcal{O}(G)$ as the ring of integers of $K(G)$.

It is shown that G can be representable a vector space U over $K(G)$. Note that since $K(G)/\mathbb{Q}$ is a finite extension, the ring of definition $\mathcal{O}(G)$ is a residually finite Dedekind domain.

Let $(-,-)$ denote the Hermitian inner product of $V = \mathbb{C} \otimes_{K(G)} U$. Let $\boldsymbol{\mu}(\mathcal{O}(G))$ denote the group of roots of unity in $\mathcal{O}(G)$. For every $a \in U \setminus \{0\}$ and $\lambda \in \boldsymbol{\mu}(\mathcal{O}(G))$, we define a reflection $r_{a,\lambda}$ by

$$r_{a,\lambda}(v) := v - (1-\lambda)\frac{(v,a)}{(a,a)}a.$$

Lehrer and Taylor [10] defined a generalization of root systems for algebraic integers and showed that every finite complex reflection group admits a "root system". Namely there exists a pair (Σ, f) satisfying the following.

- Σ is a finite subset of $U \setminus \{0\}$ and Σ spans U.
- $f \colon \Sigma \to \boldsymbol{\mu}(\mathcal{O}(G))$.
- G is generated by the reflections $\left\{ r_{a,f(a)} \;\middle|\; a \in \Sigma \right\}$.
- For all $a \in \Sigma$ and all $\lambda \in K(G)$ we have $\lambda a \in \Sigma \Leftrightarrow \lambda \in \boldsymbol{\mu}(\mathcal{O}(G))$.
- For all $a \in \Sigma$ and $\lambda \in \boldsymbol{\mu}(\mathcal{O}(G))$ we have $f(\lambda a) = f(a) \neq 1$.
- For all $a, b \in \Sigma$ we have $(1 - f(b))(a,b)/(b,b) \in \mathcal{O}(G)$.
- For all $a, b \in \Sigma$ we have $r_{a,f(a)}(b) \in \Sigma$ and $f(r_{a,f(a)}(b)) = f(b)$.

We call (Σ, f) a $\mathcal{O}(G)$**-root system** for G. If $\mathcal{O}(G) = \mathbb{Z}$, then the above definition coincides with the definition of crystallographic root system. Namely a $\mathbb{Z}$-root system is a crystallographic system.

When G is well-generated, there exist roots $a_1, \ldots, a_\ell \in \Sigma$ such that every root in Σ is reperesented by a linear combination of $a_1, \ldots, a_\ell$ over $\mathcal{O}(G)$. Hence we can obtain a finite coefficient column vectors $\mathcal{A}_{(\Sigma,f)} \subseteq \mathcal{O}(G)^\ell$ from the root system (Σ, f) over $\mathcal{O}(G)$. Note that the characteristic quasi-polynomial determined by (Σ, f) does not depend on the choice of roots $a_1, \ldots, a_\ell$.

Lehrer and Taylor listed the Cartan matrices of $\mathcal{O}(G)$-root systems for exceptional irreducible complex reflection groups. We can recover the root system from the corresponding Cartan matrix.

Example 2.2. Consider the group G_4. The rank of G_4 is two and its ring of definition is $\mathbb{Z}[\omega]$, where $\omega = \frac{-1+\sqrt{-3}}{2}$. The matrix

$$C_4 = \begin{pmatrix} 1-\omega & 1 \\ -\omega & 1-\omega \end{pmatrix}$$

is a Cartan matrix for G_4. Let $\{a_1, a_2\}$ a basis for $\mathbb{C}^2$ and r_1, r_2 the correspondence reflections. The Cartan matrix C_4 tells us $r_j(a_i) = a_i - c_{ij}a_j$, where c_{ij} denotes the (i, j) entry of C_4. Thus we have

$$\begin{aligned} r_1(a_1) &= a_1 - (1-\omega)a_1 = \omega a_1, & r_2(a_1) &= a_1 - 1 \cdot a_2 = a_1 - a_2, \\ r_1(a_2) &= a_2 - (-\omega)a_1 = \omega a_1 + a_2, & r_2(a_2) &= a_2 - (1-\omega)a_2 = \omega a_2. \end{aligned}$$

Hence we obtain the matrix representations of r_1, r_2 with respect to the basis $\{a_1, a_2\}$ as follows.

$$r_1 = \begin{pmatrix} \omega & \omega \\ 0 & 1 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} 1 & 0 \\ -1 & \omega \end{pmatrix}.$$

Therefore $G_4 = \langle r_1, r_2 \rangle \subseteq \mathrm{GL}_2(\mathbb{C})$ and Σ can be recovered as $\Sigma = \{\, r(a_i) \mid r \in G_4,\ i = 1, 2 \,\}$ with $a_1 = \binom{1}{0}$ and $a_2 = \binom{0}{1}$. As a result, Σ consists of the following 24 vectors.

$$\lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \lambda \begin{pmatrix} \omega \\ 1 \end{pmatrix}, \quad \lambda \in \boldsymbol{\mu}(\mathbb{Z}[\omega]) = \{\pm 1, \pm\omega, \pm\omega^2\}.$$

Setting $f(a_1) = f(a_2) = \omega$, we obtain the $\mathbb{Z}[\omega]$-root system (Σ, f) and

$$\mathcal{A}_{(\Sigma,f)} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} \omega \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{Z}[\omega]^2.$$

Since $\mathbb{Z}[\omega]$ is a Euclidean domain, we can compute the LCM-period by finding the elementary divisors. The LCM-period is the unit ideal $\langle 1 \rangle$ and hence the characteristic quasi-polynomial has only one constituent (the characteristic polynomial)

$$f^{\langle 1 \rangle}(t) = t^2 - 4t + 3 = (t-1)(t-3).$$

Example 2.3. Consider G_{33} and the Cartan matrix

$$C_{33} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 2 & -\omega & 0 \\ 0 & -1 & -\omega^2 & 2 & -\omega^2 \\ 0 & 0 & 0 & -\omega & 2 \end{pmatrix}.$$

The ring of definition is $\mathbb{Z}[\omega]$ and the LCM-period is $\langle 2\sqrt{-3}\rangle$. The characteristic quasi-polynomial consists of the following constituents:

$$\begin{aligned} f^{\langle 1\rangle}(t) &= t^5 - 45t^4 + 750t^3 - 5590t^2 + 17169t - 12285. \\ &= (t-1)(t-7)(t-9)(t-13)(t-15). \\ f^{\langle 2\rangle}(t) &= t^5 - 45t^4 + 750t^3 - 5590t^2 + 17574t - 18360. \\ &= (t-4)(t-15)(t^3 - 26t^2 + 196t - 306). \\ f^{\langle \sqrt{-3}\rangle}(t) &= t^5 - 45t^4 + 750t^3 - 5590t^2 + 18129t - 20925. \\ &= (t-3)(t-9)(t^3 - 33t^2 + 327t - 775). \\ f^{\langle 2\sqrt{-3}\rangle}(t) &= t^5 - 45t^4 + 750t^3 - 5590t^2 + 18534t - 27000. \end{aligned}$$

The authors calculated the LCM-period for the root systems determined by the Cartan matrices in Table 2. Note that C_{20} is modified from the one in [10] so that it recovers the root system correctly. According to [11] and [15], the rings of definition for exceptional irreducible complex reflection groups are Euclidean domains except for $\mathcal{O}(G_{21})$. Although the authors are not sure whether $\mathcal{O}(G_{21})$ is Euclidean or not, since $\mathcal{O}(G_{21})$ is a principal ideal domain (See [3]), there exist the Smith normal forms. Fortunately the authors could find the Smith normal forms and hence the LCM-period for G_{21}. We summarize the results in Table 1. From this computational result, we have the following theorem, which is a generalization of Theorem 1.9.

Theorem 2.4. *Every exceptional well-generated irreducible complex reflection group G admits an $\mathcal{O}(G)$-root system such that the radical of the LCM-period divides the Coxeter number.*

Remark 2.5. We anticipated phenomenon analogous to Corollary 1.6, Corollary 1.7, Theorem 1.8, and Theorem 1.9. However, only Theorem 1.9 has been observed.

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Table 1: LCM-periods and Coxeter numbers

G	$\mathcal{O}(G)$	LCM-period	h	Coexponents
G_4	$\mathbb{Z}[\omega]$	$\langle 1 \rangle$	6	$1, 3$
G_5	$\mathbb{Z}[\omega]$	$\langle 2\sqrt{-3} \rangle$	12	$1, 7$
G_6	$\mathbb{Z}[i, \omega]$	$\langle 1+i \rangle$	12	$1, 9$
G_8	$\mathbb{Z}[i]$	$\langle 1+i \rangle$	12	$1, 5$
G_9	$\mathbb{Z}[\zeta_8]$	$\langle 6 \rangle$	24	$1, 17$
G_{10}	$\mathbb{Z}[i, \omega]$	$\langle (1+i)\sqrt{-3} \rangle$	24	$1, 13$
G_{14}	$\mathbb{Z}[\omega, \sqrt{-2}]$	$\langle 6 \rangle$	24	$1, 19$
G_{16}	$\mathbb{Z}[\zeta_5]$	$\langle 1-\zeta_5 \rangle$	30	$1, 11$
G_{17}	$\mathbb{Z}[i, \zeta_5]$	$\langle 6\sqrt{5} \rangle$	60	$1, 41$
G_{18}	$\mathbb{Z}[\omega, \zeta_5]$	$\langle 2\sqrt{-3}(1-\zeta_{15}^3) \rangle$	60	$1, 31$
G_{20}	$\mathbb{Z}[\omega, \tau]$	$\langle 2\sqrt{-3} \rangle$	30	$1, 19$
G_{21}	$\mathbb{Z}[i, \omega, \tau]$	$\langle 6\sqrt{5} \rangle$	60	$1, 49$
$G_{23} = \mathrm{H}_3$	$\mathbb{Z}[\tau]$	$\langle 2 \rangle$	10	$1, 5, 9$
G_{24}	$\mathbb{Z}[\lambda]$	$\langle 4 \rangle$	14	$1, 9, 11$
G_{25}	$\mathbb{Z}[\omega]$	$\langle \sqrt{-3} \rangle$	12	$1, 4, 7$
G_{26}	$\mathbb{Z}[\omega]$	$\langle 6 \rangle$	18	$1, 7, 13$
G_{27}	$\mathbb{Z}[\omega, \tau]$	$\langle 4\sqrt{-3} \rangle$	30	$1, 19, 25$
$G_{28} = \mathrm{F}_4$	$\mathbb{Z}$	$\langle 12 \rangle$	12	$1, 5, 7, 11$
G_{29}	$\mathbb{Z}[i]$	$\langle 10(1+i) \rangle$	20	$1, 9, 13, 17$
$G_{30} = \mathrm{H}_4$	$\mathbb{Z}[\tau]$	$\langle 6\sqrt{5} \rangle$	30	$1, 11, 19, 29$
G_{32}	$\mathbb{Z}[\omega]$	$\langle 2\sqrt{-3} \rangle$	30	$1, 7, 13, 19$
G_{33}	$\mathbb{Z}[\omega]$	$\langle 2\sqrt{-3} \rangle$	18	$1, 7, 9, 13, 15$
G_{34}	$\mathbb{Z}[\omega]$	$\langle 84 \rangle$	42	$1, 13, 19, 25, 31, 37$
$G_{35} = \mathrm{E}_6$	$\mathbb{Z}$	$\langle 6 \rangle$	12	$1, 4, 5, 7, 8, 11$
$G_{36} = \mathrm{E}_7$	$\mathbb{Z}$	$\langle 12 \rangle$	18	$1, 5, 7, 9, 11, 13, 17$
$G_{37} = \mathrm{E}_8$	$\mathbb{Z}$	$\langle 60 \rangle$	30	$1, 7, 11, 13, 17, 19, 23, 29$

$$i = \sqrt{-1},\ \omega = \frac{-1+\sqrt{-3}}{2},\ \tau = \frac{1+\sqrt{5}}{2},\ \lambda = \frac{-1+\sqrt{-7}}{2},\ \zeta_k = e^{2\pi i/k}.$$

Table 2: Cartan matrices (C_{20} is modified)

$$C_4 = \begin{pmatrix} 1-\omega & 1 \\ -\omega & 1-\omega \end{pmatrix}, \quad C_5 = \begin{pmatrix} 1-\omega & 1 \\ -2\omega & 1-\omega \end{pmatrix}, \quad C_6 = \begin{pmatrix} 2 & 1 \\ 1-\omega+i\omega^2 & 1-\omega \end{pmatrix},$$

$$C_8 = \begin{pmatrix} 1-i & 1 \\ -i & 1-i \end{pmatrix}, \quad C_9 = \begin{pmatrix} 2 & 1 \\ (1+\sqrt{2})\zeta_8 & 1+i \end{pmatrix}, \quad C_{10} = \begin{pmatrix} 1-\omega & 1 \\ -i-\omega & 1-i \end{pmatrix},$$

$$C_{14} = \begin{pmatrix} 1-\omega & 1 \\ 1-\omega+i\omega^2\sqrt{2} & 2 \end{pmatrix}, \quad C_{16} = \begin{pmatrix} 1-\zeta_5 & 1 \\ -\zeta_5 & 1-\zeta_5 \end{pmatrix}, \quad C_{17} = \begin{pmatrix} 2 & 1 \\ 1-\zeta_5-i\zeta_5^3 & 1-\zeta_5 \end{pmatrix},$$

$$C_{18} = \begin{pmatrix} 1-\omega & 1 \\ -\omega-\zeta_5 & 1-\zeta_5 \end{pmatrix}, \quad C_{20} = \begin{pmatrix} 1-\omega & \tau-1 \\ \omega(1-\tau) & 1-\omega \end{pmatrix}, \quad C_{21} = \begin{pmatrix} 2 & 1 \\ 1-\omega-i\omega^2\tau & 1-\omega \end{pmatrix},$$

$$C_{23} = \begin{pmatrix} 2 & -\tau & 0 \\ \tau & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad C_{24} = \begin{pmatrix} 2 & -1 & -\lambda \\ -1 & 2 & -1 \\ 1+\lambda & -1 & 2 \end{pmatrix}, \quad C_{25} = \begin{pmatrix} 1-\omega^2 & \omega^2 & 0 \\ -\omega^2 & 1-\omega & -\omega^2 \\ 0 & \omega^2 & 1-\omega \end{pmatrix},$$

$$C_{26} = \begin{pmatrix} 1-\omega & -\omega^2 & 0 \\ \omega^2 & 1-\omega & -1 \\ 0 & -1+\omega & 2 \end{pmatrix}, \quad C_{27} = \begin{pmatrix} 2 & -\tau & -\omega \\ -\tau & 2 & -\omega^2 \\ -\omega^2 & -\omega & 2 \end{pmatrix},$$

$$C_{28} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad C_{29} = \begin{pmatrix} 2 & -1 & i+1 & 0 \\ -1 & 2 & -i & 0 \\ -i+1 & i & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix},$$

$$C_{30} = \begin{pmatrix} 2 & -\tau & 0 & 0 \\ -\tau & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad C_{32} = \begin{pmatrix} 1-\omega & \omega^2 & 0 & 0 \\ -\omega^2 & 1-\omega & -\omega^2 & 0 \\ 0 & \omega^2 & 1-\omega & \omega^2 \\ 0 & 0 & -\omega^2 & 1-\omega \end{pmatrix},$$

$$C_{33} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 2 & -\omega & 0 \\ 0 & -1 & -\omega^2 & 2 & -\omega^2 \\ 0 & 0 & 0 & -\omega & 2 \end{pmatrix}, \quad C_{34} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -\omega & 0 \\ 0 & 0 & -1 & -\omega^2 & 2 & -\omega^2 \\ 0 & 0 & 0 & 0 & -\omega & 2 \end{pmatrix}.$$

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