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Inhomogeneous particle process defined by canonical Grothendieck polynomials

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Abstract. We construct a time, particle, and position inhomogeneous discrete time particle process on the nonnegative integers that generalizes one of those studied in a Dieker and Warren. The particles move according to an inhomogeneous geometric distribution and stay in (weakly) decreasing order, where smaller particles block larger particles. We show that the transition probabilities for our particle process is given by a (refined) canonical Grothendieck function up to a simple overall factor.

Keywords: Grothendieck polynomial, particle process, transition probability

1 Introduction

The *totally asymmetric simple exclusion process* (TASEP) with sites on $\mathbb{Z}$ is a classical one dimensional model that has many interesting features and applications. In this stochastic process, there is at most one particle in each site and the particles move in one direction — say, to the right — according to some specified dynamic. For the continuous time process, particle p jumps one step with rate π_p, subject to an exclusion interaction, where a particle immediately to the right of p blocks it. For discrete time, then particles decide to move by flipping (biased) coins, where success rate π_p depends on particle. However, we need a rule to resolve when two particles move simultaneously that could interact. For the rule particles update right-to-left, this is Case B studied by Dieker and Warren [4]. On the other hand, if the particle keeps moving one step each time it flips the (biased) coin successfully until it fails, then it moves by the geometric distribution. This is [4, Case C] with instead updating the particles from left-to-right.

In a seemingly different area, the (refined) Grothendieck polynomials $G_{\lambda /\!/ \mu}(\mathbf{x}_n; \boldsymbol{\beta})$ originated from the (connective) K-theoretic Schubert calculus of the Grassmannian, and so they are a natural generalization of the Schur polynomials. They have been well-studied since their inception (for the unrefined case $\boldsymbol{\beta} = \beta$) in the work of Lascoux and

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Schützenberger [13], which includes explicit combinatorial descriptions [2, 3, 7]. However, this is related to the aforementioned particle processes as follows. When we additionally make the success probability of Case C TASEP depend on the time t according to $\pi_p x_t$, then the n-step transition probabilities can be easily seen to equal $G_{\lambda/\!/\mu}(\mathbf{x}_n; \boldsymbol{\beta})$ up to an explicit overall simple factor [8, Thm. 1.1]. Indeed, this can be seen in a number of different ways: Directly comparing the Jacobi–Trudi formula [9] with the natural symmetric function replacements in the determinants in [4], using extensions of the Schur operators to encode the dynamics [8, Sec. 4.2], or bijectively with set-valued tableaux [8, Sec. 5.3]. A similar statement holds for Case B with the weak Grothendieck polynomials.

A natural question is what particle process corresponds to the canonical Grothendieck polynomials [7, 16] (up to an analogous simple factor). However, it does not seem possible to build a particle process from naively combining the Case BC processes, which is similar to some of the combinatorial aspects of $G_{\lambda/\!/\mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$. Instead, we develop our stochastic model by using the Schur operators for $G_{\lambda/\!/\mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ developed in [8, Sec. 3] as they were shown to encode the particle movements when $\boldsymbol{\alpha} = 0$. This leads to a position inhomogeneous version of the Case BC process described above, which has been studied when $\boldsymbol{\beta} = 0$ in recent works [1, 11]. Our main result is that our new discrete time particle process has a transition kernel given by the canonical Grothendieck polynomials. Using this, we give a formula for the multi-point distribution for this process. All of our formulas can be described as determinants of contour integrals using [9].

This is an extended abstract based on [8, Sec. 8], and is organized as follows. In Section 2, we describe canonical Grothendieck polynomials. In Section 3, we give the necessary free fermion representations. In Section 4, we describe our particle process.

2 Grothendieck polynomials

Let $\mathcal{P}$ denote the set of all *partitions* $\lambda = (\lambda_1, \lambda_2, \ldots)$, weakly decreasing sequences of nonnegative integers with finite sum. We draw our Young diagrams using English convention. Let $\ell(\lambda)$ denote the largest index ℓ such that $\lambda_\ell > 0$, called the *length* of λ. Let λ' denote the conjugate partition. A *hook* is a partition $a1^m$ with *arm* $a-1$ and *leg* m.

Let $\mathbf{x} = (x_1, x_2, \ldots)$ denote a countably infinite sequence of indeterminates and denote $\mathbf{x}_n := (x_1, \ldots, x_n, 0, 0, \ldots)$. We make similar definitions for other such sequences. In particular, we take parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots)$.

A *hook-valued tableau* of skew shape λ/μ is a filling of the Young diagram by hook shaped tableau, fillings of a hook shape with entries weakly (resp. strictly) increasing along the arm (resp. leg), satisfying the local conditions

a	b
c	

$$\begin{array}{c} \max(\mathsf{a}) \leq \min(\mathsf{b}) \\ \wedge \\ \min(\mathsf{c}) \end{array}$$

Work	[7]	[16]	[3]	[14]
Specialization	$G_\lambda(\mathbf{x}; -\boldsymbol{\alpha}, \boldsymbol{\beta})$	$G_\lambda(\mathbf{x}; -\alpha, -\beta)$	$G_\lambda(\mathbf{x}; 0, \beta)$	$G_\lambda(\mathbf{x}; 0, -\boldsymbol{\beta})$

Table 1: The relationship between our sign choices and some other papers.

(provided the requisite box exists). Note that this is a generalization of the semistandard conditions, which reduce to the usual ones when $\mathsf{a}, \mathsf{b}, \mathsf{c}$ all consist of a single entry.

For $\mu \subseteq \lambda$, the *canonical Grothendieck function* (we omit the word "refined" to simplify our nomenclature from [7]) is the generating function

$$G_{\lambda/\mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_T \prod_{\mathsf{b} \in T} (-\alpha_i)^{a(\mathsf{b})} (-\beta_j)^{b(\mathsf{b})} \operatorname{wt}(\mathsf{b}),$$

where we sum over all hook-valued tableaux T of shape λ/μ, product over all entries b in T with $a(\mathsf{b})$ (resp. $b(\mathsf{b})$) the arm (resp. leg) of the shape of b and i (resp. j) the row (resp. column) of the entry. We indicate various specializations and relation with some of the literature in Table 1, which $G_{\lambda/\mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ also specializes those in [2, 12]. While technically we should work in a completion of the ring of symmetric functions, this does not affect our results, so we suppress this here. The set $\{G_\lambda(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})\}_{\lambda \in \mathcal{P}}$ is a basis for (the completion of) symmetric functions (see, *e.g.*, [6, 7]).

The skew shape description is not natural from the algebraic perspective. Hence, refining [2, Eq. (6.4)] and [16, Prop. 8.8], we define [9, Sec. 4.1]

$$G_{\lambda/\!/\mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta}) := \sum_{\nu \subseteq \mu} \prod_{(i,j) \in \mu/\nu} -(\alpha_i + \beta_j) G_{\lambda/\nu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta}), \tag{2.1}$$

where ν is formed by removing some corners of μ (boxes (i, μ_i) such that $\mu_i > \mu_{i+1}$).

Proposition 2.1 (Branching rules [9, Prop. 4.5])**.** *We have*

$$G_{\lambda/\mu}(\mathbf{x}, \mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{\nu \subseteq \lambda} G_{\lambda/\!/\nu}(\mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\beta}) G_{\nu/\mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta}),$$
$$G_{\lambda/\!/\mu}(\mathbf{x}, \mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{\mu \subseteq \nu \subseteq \lambda} G_{\lambda/\!/\nu}(\mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\beta}) G_{\nu/\!/\mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta}).$$

The *dual canonical Grothendieck functions* $\{g_\lambda(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})\}_{\lambda \in \mathcal{P}}$ are defined as the dual basis to the canonical Grothendieck functions under the Hall inner product, defined by $\{s_\lambda(\mathbf{x})\}_{\lambda \in \mathcal{P}}$, where $s_\lambda(\mathbf{x}) = G_\lambda(\mathbf{x}; 0, 0)$ are the Schur functions, is an orthonormal basis. A combinatorial definition of $g_\lambda(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ was given in [7], which is a refinement of the rim border tableaux description of [16].

We have the skew Cauchy formula [9, Thm. 4.6] (a non-skew version is in [7] or implied from [3, Rem. 3.9]). This is a refined version of [17, Thm. 1.1].

Theorem 2.2 (Skew Cauchy formula)**.** *We have*

$$\sum_{\lambda} G_{\lambda/\!/\mu}(\mathbf{x};\boldsymbol{\alpha},\boldsymbol{\beta}) g_{\lambda/\nu}(\mathbf{y};\boldsymbol{\alpha},\boldsymbol{\beta}) = \prod_{i,j} \frac{1}{1-x_i y_j} \sum_{\eta} G_{\nu/\!/\eta}(\mathbf{x};\boldsymbol{\alpha},\boldsymbol{\beta}) g_{\mu/\eta}(\mathbf{y};\boldsymbol{\alpha},\boldsymbol{\beta}).$$

3 Free fermions and Schur-type operators

We describe the free-fermion presentation of the (dual) canonical Grothendieck polynomials from [9]. For more details, we refer the reader to [10]. The unital associative Clifford algebra (over $\mathbb{C}$) is generated by $\{\psi_n, \psi_n^* \mid n \in \mathbb{Z}\}$ with relations

$$\psi_m \psi_n + \psi_n \psi_m = \psi_m^* \psi_n^* + \psi_n^* \psi_m^* = 0, \qquad \psi_m \psi_n^* + \psi_n^* \psi_m = \delta_{m,n},$$

known as the canonical anti-commuting relations. The *current operators* are defined as $a_k := \sum_{i\in\mathbb{Z}} \psi_i \psi_{i+k}^*$, (care is needed for $k=0$, but we will not use this) and satisfy the Heisenberg algebra relations $[a_m, a_k] = m\delta_{m,-k}$, We will use the *Hamiltonian operators*

$$H(\mathbf{x}/\mathbf{y}) := \sum_{k>0} \frac{p_k(\mathbf{x}/\mathbf{y})}{k} a_k, \quad H^*(\mathbf{x}/\mathbf{y}) := \sum_{k>0} \frac{p_k(\mathbf{x}/\mathbf{y})}{k} a_{-k}, \quad \text{where } p_k(\mathbf{x}/\mathbf{y}) = \sum_{i=1}^{\infty} x_i^k - y_i^k,$$

and the corresponding *half vertex operators* $e^{H(\mathbf{x}/\mathbf{y})}$ and $e^{H^*(\mathbf{x}/\mathbf{y})}$. These satisfy the relations

$$e^{H(\mathbf{x}/\mathbf{y})} \psi_k e^{-H(\mathbf{x}/\mathbf{y})} = \sum_{i=0}^{\infty} h_i(\mathbf{x}/\mathbf{y}) \psi_{k-i}, \qquad e^{-H(\mathbf{x}/\mathbf{y})} \psi_k^* e^{H(\mathbf{x}/\mathbf{y})} = \sum_{i=0}^{\infty} h_i(\mathbf{x}/\mathbf{y}) \psi_{k+i}^*,$$

where $h_i(\mathbf{x}/\mathbf{y})$ is the homogeneous supersymmetric function.

Fermionic Fock space is the Clifford algebra representation $\mathcal{F}$ generated by the *shifted vacuum vectors* with relations

$$|m\rangle = \begin{cases} \psi_{m-1} \cdots \psi_0 |0\rangle & \text{if } m \geq 0, \\ \psi_m^* \cdots \psi_{-1}^* |0\rangle & \text{if } m < 0, \end{cases} \qquad \langle m| = \begin{cases} \langle 0| \psi_0^* \cdots \psi_{m-1}^* & \text{if } m \geq 0, \\ \langle 0| \psi_{-1} \cdots \psi_m & \text{if } m < 0. \end{cases}$$

Note that $e^{H(\mathbf{x}/\mathbf{y})}|m\rangle = |m\rangle$ and $\langle m| e^{H^*(\mathbf{x}/\mathbf{y})} = \langle m|$ for all m. We will use the vectors

$$|\lambda\rangle_{[\boldsymbol{\alpha},\boldsymbol{\beta}]} := \prod_{1\leq i\leq \ell}^{\rightarrow} \left(e^{-H(A_{\lambda_i-1})} \psi_{\lambda_i - i} e^{H(\beta_i)} e^{H(A_{\lambda_i-1})} \right) |-\ell\rangle,$$

$$|\lambda\rangle^{[\boldsymbol{\alpha},\boldsymbol{\beta}]} := \prod_{1\leq i\leq \ell}^{\rightarrow} \left(e^{H^*(A_{\lambda_i})} \psi_{\lambda_i - i} e^{-H^*(\beta_i)} e^{-H^*(A_{\lambda_i})} \right) e^{H^*(A_{\lambda_\ell})} |-\ell\rangle,$$

here $A_k = -\boldsymbol{\alpha}_k = (-\alpha_1, \ldots, -\alpha_k)$ and the product is ordered $\Psi_1 \cdots \Psi_\ell$. We restrict ourselves to the subspace $\mathcal{F}^0$ and the bases [9, Thm. 3.10] $\{|\lambda\rangle_{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\}_{\lambda\in\mathcal{P}}$ and $\{|\lambda\rangle^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\}_{\lambda\in\mathcal{P}}$.

There is also the dual representation $\mathcal{F}^*$, which has a canonical bilinear pairing called the *vacuum expectation value* that satisfies

$$\langle k|m\rangle = \delta_{km}, \qquad (\langle w|X)|v\rangle = \langle w|(X|v\rangle)$$

for all $k, m \in \mathbb{Z}$, operator X, $\langle w| \in \mathcal{F}^*$, and $|v\rangle \in \mathcal{F}$. Note that $|k\rangle^* = \langle k|$. Define by the anti-involution $\psi_i \leftrightarrow \psi_i^*$ the vectors ${}_{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\lambda| := (|\lambda\rangle^{[\boldsymbol{\alpha},\boldsymbol{\beta}]})^*$ and ${}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\lambda| := (|\lambda\rangle_{[\boldsymbol{\alpha},\boldsymbol{\beta}]})^*$. We have the orthonormal bases [9, Thm. 3.10]

$${}_{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\lambda|\mu\rangle_{[\boldsymbol{\alpha},\boldsymbol{\beta}]} = {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\lambda|\mu\rangle^{[\boldsymbol{\alpha},\boldsymbol{\beta}]} = \delta_{\lambda\mu}. \tag{3.1}$$

Moreover, there is the *boson-fermion correspondence* from $\mathcal{F}^0$ to symmetric functions defined by $|v\rangle \mapsto \langle 0|e^{H(\mathbf{x}/\mathbf{y})}|v\rangle$, which satisfies [9, Cor. 4.2, Eq. (4.1)]

$$G_{\lambda/\!/\mu}(\mathbf{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\mu|e^{H(\mathbf{x})}|\lambda\rangle^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}, \qquad g_{\lambda/\mu}(\mathbf{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = {}_{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\mu|e^{H(\mathbf{x})}|\lambda\rangle_{[\boldsymbol{\alpha},\boldsymbol{\beta}]}. \tag{3.2}$$

We denote $\kappa_i \colon \mathbf{k}[\mathcal{P}] \to \mathbf{k}[\mathcal{P}]$ the i-th (row) *Schur operator* that adds a box to the i-th row of a partition λ if $\lambda_i < \lambda_{i-1}$ (that is, we can add the box and obtain a partition) and is 0 otherwise. We define the linear operator $U_i^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ by

$$U_i^{(\boldsymbol{\alpha},\boldsymbol{\beta})} := \kappa_i + \Theta_i, \quad \text{where} \quad \Theta_i \cdot \lambda := \begin{cases} -\alpha_{\lambda_i}\lambda & \text{if } \lambda_i < \lambda_{i-1}, \\ \beta_{i-1}\lambda & \text{if } \lambda_i = \lambda_{i-1}, \end{cases}$$

for any $\lambda \in \mathcal{P}$. We consider $\lambda_0 = \infty$ and $\alpha_0 = 0$ (although our proofs could have α_0 be an arbitrary parameter). When there is no ambiguity, we will simply write $U_i := U_i^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$.

Lemma 3.1 ([8, Lemma 3.2])**.** *The operators $\mathbf{U} = \{U_i\}_{i=1}^{\infty}$ satisfy the weak Knuth relations.*

Lemma 3.1 implies we can use $\mathbf{U}$ with noncommutative symmetric functions [5].

Theorem 3.2 ([8, Thm. 3.3])**.** *We have ${}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\lambda|S_\mu(a_1, a_2, \ldots) = {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle s_\mu(\mathbf{U}/\boldsymbol{\beta}) \cdot \lambda|$, where $S_\lambda(p_1(\mathbf{x}), p_2(\mathbf{x}), \ldots) = s_\lambda(\mathbf{x})$.*

4 Particle Process

Now we describe a particle process whose transition kernel naturally uses the canonical Grothendieck polynomials. We start by explicitly defining the stochastic process, and then we will show how to interpret it using the noncommutative operators $\mathbf{U}$. Let $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots)$ be a sequence of parameters such that $0 \le \pi_i x_j < 1$ for all i and j.

Let $G(j,i)$ denote the position of the j-th particle at time i, which is determined by

$$G(j,i) = \min\bigl(G(j,i-1) + w_{ji}, G(j-1,i-1)\bigr), \tag{4.1}$$

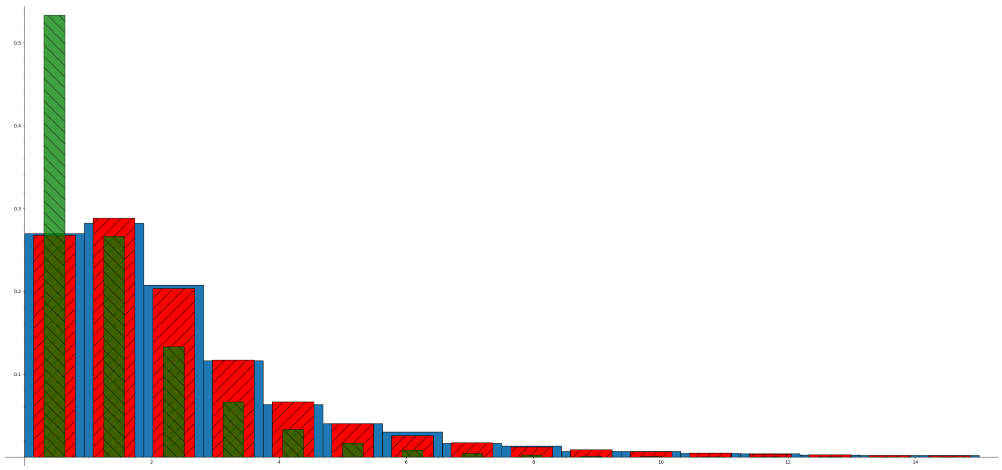

Figure 1: A sampling using 10000 samples of the inhomogeneous geometric distribution $\mathsf{P}_{\mathcal{G}}$ for $x_i = 1$, $\pi_j = .5$, and $\alpha_k = 1 - ke^{-k/2}$ (blue) under the exact distribution (red), which is under the geometric distribution with parameter $\pi_j x_i$ (green).

by convention $G(0, i-1) := \infty$, where the random variable w_{ij} (which depends on $G(j, i-1)$) is determined by the *inhomogeneous geometric distribution* defined by

$$\mathsf{P}_{\mathcal{G}}(w_{ji} = m' \mid G(j, i-1) = m) := \frac{1 - \pi_j x_i}{1 + \alpha_{m+m'} x_i} \prod_{k=m}^{m+m'-1} \frac{(\alpha_k + \pi_j) x_i}{1 + \alpha_k x_i}. \tag{4.2}$$

In other words, the j-th particle at time i attempts to jump w_{ji} steps, but can be blocked by the $(j-1)$-th particle, which updates its position after the j-th particle moves.

Let us digress slightly on why (4.2) is called an inhomogeneous geometric distribution. We can realize it as the waiting time for a failure in sequence of Bernoulli variables (*i.e.*, weighted coin flips), but the k-th trial given a probability of success $(\alpha_k + \pi_j) x_i (1 + \alpha_k x_i)^{-1}$. Indeed, we note that the probability of a failure is

$$1 - \frac{\alpha_k x_i + \pi_j x_i}{1 + \alpha_k x_i} = \frac{1 - \pi_j x_i}{1 + \alpha_k x_i}.$$

Hence, this gives us a sampling algorithm for the distribution $\mathsf{P}_{\mathcal{G}}$. We illustrate the effectiveness of this sampling in Figure 1. This perspective also allows us to easily see that we have a probability measure on $\mathbb{Z}_{\geq m}$ for any fixed m. The case $\boldsymbol{\pi} = 0$ can also be seen as a projection of the Warren–Windridge dynamics [15]; see also [1, Sec. 2.2].

We will give some remarks on the meaning of the $\boldsymbol{\alpha}$ parameters. From the behavior of the operators $\mathbf{U}$, it would be tempting to consider the $\boldsymbol{\alpha}$ parameters as a viscosity, but for $\boldsymbol{\alpha} > 0$, we have $\mathsf{P}_{\mathcal{G}}(w_{ji} = k) > \mathsf{P}_{Ge}(w_{ji} = k)$, where P_{Ge} denotes the usual geometric distribution with parameter $\pi_j x_i$. Thus, in this case, the $\boldsymbol{\alpha}$ parameters act as a current being applied to the system, the strength (and direction) of which can vary at each position. On the other hand, when $\boldsymbol{\alpha} < 0$, we have $\mathsf{P}_{\mathcal{G}}(w_{ji} = k) < \mathsf{P}_{Ge}(w_{ji} = k)$, and so indeed $\boldsymbol{\alpha}$ then acts as (position-based) viscosity. We can also introduce locations where certain particles must stop by having $-\alpha_k = \pi_j$ since this would have $\mathcal{P}_{\mathcal{G}}(w_{ij} = k') = 0$ for all k' that would move the j-th particle past position k.

To see how to obtain this process using the noncommutative operators $\mathbf{U}$, we initiate by taking the skew Cauchy formula (Theorem 2.2) with $\nu = \emptyset$ and with the specializations $\mathbf{y} = \boldsymbol{\pi}_1$ and $\beta_j = \pi_{j+1}$, yielding

$$\sum_{\lambda} G_{\lambda /\!/ \mu}(\mathbf{x}_n; \boldsymbol{\alpha}, \boldsymbol{\beta}) g_{\lambda}(\boldsymbol{\pi}_1; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \prod_{i} (1 - \pi_1 x_i)^{-1} g_{\mu}(\boldsymbol{\pi}_1; \boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{4.3}$$

In particular, if we let $\widehat{\lambda}_i = \lambda_i - 1$ for all $1 \leq i \leq \ell(\lambda)$, then from the combinatorial description of [7, Thm. 7.2], we have $g_{\lambda}(\boldsymbol{\pi}_1; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \boldsymbol{\pi}^{1^{\ell(\lambda)}} \prod_{(i,j) \in \widehat{\lambda}} (\alpha_i + \pi_j)$. Hence, Equation (4.3) can be considered a Littlewood-type identity for canonical Grothendieck polynomials. Dividing this by the factor on the right hand side and taking the term corresponding to λ, we obtain a probability distribution for n step random growth process (since we must have $\mu \subseteq \lambda$ and currently the interpretation we have described is only on partitions) given by

$$\mathsf{P}_{\mathcal{C},n}(\lambda|\mu) = \prod_{i=1}^{n} (1 - \pi_1 x_i) \boldsymbol{\pi}^{1^{\ell(\lambda)}/1^{\ell(\mu)}} \prod_{(i,j) \in \widehat{\lambda}/\widehat{\mu}} (\alpha_i + \pi_j) G_{\lambda /\!/ \mu}(\mathbf{x}_n; \boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{4.4}$$

Note that Equation (4.3) is equivalent to $\sum_{\lambda} \mathsf{P}_{\mathcal{C},n}(\lambda|\mu) = 1$ for any fixed μ and n.

Rephrasing Equation (4.4) and adding an $\alpha_0 = 0$ parameter in order to simplify the product in $g_{\lambda}(\boldsymbol{\pi}_1; \boldsymbol{\alpha}, \boldsymbol{\beta})$, what we have computed are coefficients

$$C_{\lambda\mu} = \prod_{i=1}^{n} (1 - \pi_1 x_i) (\vec{\boldsymbol{\alpha}} + \boldsymbol{\beta})^{\lambda/\mu}, \qquad \text{where } (\vec{\boldsymbol{\alpha}} + \boldsymbol{\beta})^{\lambda/\mu} := \prod_{(i,j) \in \lambda/\mu} (\alpha_{i-1} + \pi_j)$$

that is defined to be 0 if $\lambda \not\supseteq \mu$, such that

$$C_{\lambda\mu} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \mu | e^{H(\mathbf{x}_n)} | \lambda \rangle^{[\boldsymbol{\alpha},\boldsymbol{\beta}]} = \mathsf{P}_{\mathcal{C},n}(\lambda|\mu) \iff {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \mu | e^{H(\mathbf{x}_n)} = \sum_{\lambda \supseteq \mu} \frac{\mathsf{P}_{\mathcal{C},n}(\lambda|\mu)}{C_{\lambda\mu}} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \lambda |, \tag{4.5}$$

where the equivalence of the two formulas is given by the orthonormality (3.1).

We now restrict ourselves to a single timestep at time i in order to encode the growth process as a particle process by using the operators $\mathbf{U}$. This incurs no loss of generality as $\mathsf{P}_{\mathcal{C},n+n'}(\lambda|\mu) = \sum_{\nu} \mathsf{P}_{\mathcal{C},n}(\lambda|\nu) \mathsf{P}_{\mathcal{C},n'}(\nu|\mu)$ by the branching rules (Proposition 2.1) and we have a Markov process. Define the time evolution operator

$$\mathcal{T}_{\mathcal{C}} = \sum_{k=0}^{\infty} h_k(x_i \mathbf{U}) = \sum_{k=0}^{\infty} x_i^k h_k(\mathbf{U}).$$

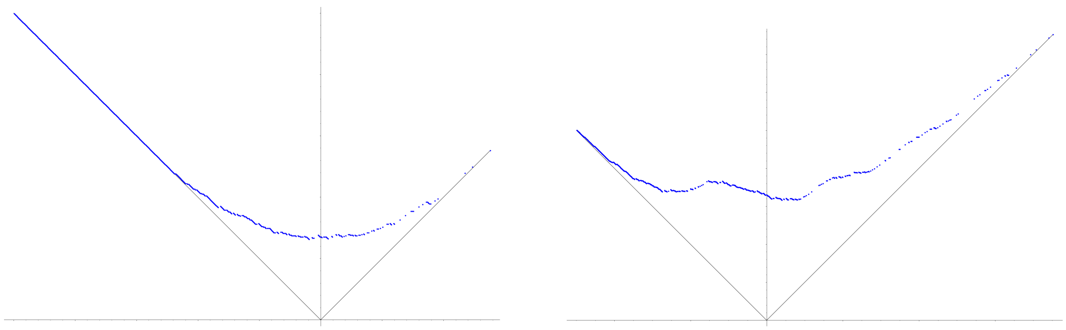

Figure 2: Samples of our process with $\ell = 500$ particles after $n = 50000$ time steps with (left) $\boldsymbol{\pi} = 1$, $\mathbf{x} = 0.01$, and $\boldsymbol{\alpha} = -0.5$; (right) $\boldsymbol{\pi} = 0.5$, $\mathbf{x} = .2$, and $\alpha_k = 0.5\sin(k/50)^6$.

By Theorem 3.2, by some algebraic and plethystic manipulations as in [8, Sec. 4.2]

$$^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \mu | e^{H(x_i)} = \prod_{j=2}^{\infty} \frac{1}{1-\pi_j x_i} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \mathcal{T}_{\mathcal{C}} \cdot \mu |.$$

Thus, if we consider the expansion $\langle \mathcal{T}_{\mathcal{C}} \cdot \mu | = \sum_{\lambda} B_{\lambda\mu} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \lambda |$, and matching coefficients in (4.5) (equivalently, pairing with $|\lambda\rangle^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}$), we obtain

$$\mathsf{P}_{\mathcal{C}}(\lambda|\mu) = \frac{B_{\lambda\mu}}{(\vec{\boldsymbol{\alpha}}+\boldsymbol{\beta})^{\lambda/\mu}} \prod_{j=1}^{\infty} (1-\pi_j x_i)^{-1}.$$

Example 4.1. Consider $\mu = (1,1)$ and set $\pi_j = 0$ for all $j > 3$. Using

$$h_1(\mathbf{u}_3) = u_1 + u_2 + u_3, \qquad h_2(\mathbf{u}_3) = u_1^2 + u_1u_2 + u_1u_3 + u_2^2 + u_2u_3 + u_3^2,$$
$$h_3(\mathbf{u}_3) = u_1^3 + u_1^2u_2 + u_1^2u_3 + u_1u_2^2 + u_1u_2u_3 + u_1u_3^2 + u_2^3 + u_2^2u_3 + u_2u_3^2 + u_3^3,$$

and recalling we consider $\alpha_0 = 0$, we compute

$$\begin{aligned}
h_1(\mathbf{U}_3)\cdot\mu &= \left(-\alpha_1 \ydiagram{1,1} + \ydiagram{2,1}\right) + \pi_1 \ydiagram{1,1} + \ydiagram{1,1,1},\\
h_2(\mathbf{U}_3)\cdot\mu &= \left(\alpha_1^2 \ydiagram{1,1} - (\alpha_1+\alpha_2)\ydiagram{2,1} + \ydiagram{3,1}\right) + \pi_1\left(-\alpha_1 \ydiagram{1,1} + \ydiagram{2,1}\right) + \left(-\alpha_1 \ydiagram{1,1,1} + \ydiagram{2,1,1}\right)\\
&\quad + \pi_1^2 \ydiagram{1,1} + \pi_1 \ydiagram{1,1,1} + \pi_2 \ydiagram{1,1,1},\\
h_3(\mathbf{U}_3)\cdot\mu &= \left(-\alpha_1^3 \ydiagram{1,1} + (\alpha_1^2+\alpha_1\alpha_2+\alpha_2^2)\ydiagram{2,1} - (\alpha_1+\alpha_2+\alpha_3)\ydiagram{3,1} + \ydiagram{4,1}\right)\\
&\quad + \pi_1\left(\alpha_1^2 \ydiagram{1,1} - (\alpha_1+\alpha_2)\ydiagram{2,1} + \ydiagram{3,1}\right) + \left(\alpha_1^2 \ydiagram{1,1,1} - (\alpha_1+\alpha_2)\ydiagram{2,1,1} + \ydiagram{3,1,1}\right)\\
&\quad + \pi_1^2\left(-\alpha_1 \ydiagram{1,1} + \ydiagram{2,1}\right) + \pi_1\left(-\alpha_1 \ydiagram{1,1,1} + \ydiagram{2,1,1}\right) + \pi_2\left(-\alpha_1 \ydiagram{1,1,1} + \ydiagram{2,1,1}\right)\\
&\quad + \pi_1^3 \ydiagram{1,1} + \pi_1^2 \ydiagram{1,1,1} + \pi_1\pi_2 \ydiagram{1,1,1} + \pi_2^2 \ydiagram{1,1,1}.
\end{aligned}$$

Recall that $A_k = -\boldsymbol{\alpha}_k$. Therefore, we have

$$\begin{aligned}
{}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \mathcal{T}_{\mathcal{C}} \cdot \mu| &= (1 + h_1(\boldsymbol{\beta}_1 \sqcup A_1)x_i + h_2(\boldsymbol{\beta}_1 \sqcup A_1)x_i^2 + h_3(\boldsymbol{\beta}_1 \sqcup A_1)x_i^3 + \cdots) \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 1,1| \\
&\quad + x_i(1 + h_1(\boldsymbol{\beta}_1 \sqcup A_2)x_i + h_2(\boldsymbol{\beta}_1 \sqcup A_2)x_i^2 + \cdots) \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 2,1| \\
&\quad + x_i(1 + h_1(\boldsymbol{\beta}_2 \sqcup A_1)x_i + h_2(\boldsymbol{\beta}_2 \sqcup A_1)x_i^2 + \cdots) \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 1,1,1| \\
&\quad + x_i^2(1 + h_1(\boldsymbol{\beta}_1 \sqcup A_3)x_i + \cdots) \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 3,1| \\
&\quad + x_i^2(1 + h_1(\boldsymbol{\beta}_2 \sqcup A_2)x_i \cdots) \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 2,1,1| + \cdots \\
&= \frac{(1+\alpha_1 x_i)^{-1}}{1-\pi_2 x_i} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 1,1| + \frac{(\alpha_1 x_i + \pi_1 x_i)(1+\alpha_1 x_i)^{-1}(1+\alpha_2 x_i)^{-1}}{(1-\pi_2 x_i)(\vec{\boldsymbol{\alpha}} + \boldsymbol{\pi})^{(2,1)/\mu}} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 2,1| \\
&\quad + \frac{(\alpha_0 x_i + \pi_3 x_i)(1+\alpha_1 x_i)^{-1}}{(1-\pi_2 x_i)(1-\pi_3 x_i)(\vec{\boldsymbol{\alpha}} + \boldsymbol{\pi})^{(1,1,1)/\mu}} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 1,1,1| \\
&\quad + \frac{(\alpha_1 x_i + \pi_1 x_i)(\alpha_2 x_i + \pi_1 x_i)(1+\alpha_1 x_i)^{-1}(1+\alpha_2 x_i)^{-1}(1+\alpha_3 x_i)^{-1}}{(1-\pi_2 x_i)(\vec{\boldsymbol{\alpha}} + \boldsymbol{\pi})^{(3,1)/\mu}} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 3,1| \\
&\quad + \frac{(\alpha_1 x_i + \pi_1 x_i)(\alpha_0 x_i + \pi_3 x_i)(1+\alpha_1 x_i)^{-1}(1+\alpha_2 x_i)^{-1}}{(1-\pi_2 x_i)(1-\pi_3 x_i)(\vec{\boldsymbol{\alpha}} + \boldsymbol{\pi})^{(2,1,1)/\mu}} {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 2,1,1| + \cdots .
\end{aligned}$$

If we include α_0 in the $\mathbf{U}$ operators, then all terms will be multiplied by $(1 + \alpha_0 x_i)^{-1}$ since the third particle can move from position 0. With this, some probabilities are

$$\begin{aligned}
\mathsf{P}_{\mathcal{C}}(1,1|\mu) &= \frac{(1-\pi_1 x_i)(1-\pi_3 x_i)}{(1+\alpha_0 x_i)(1+\alpha_1 x_i)}, \\
\mathsf{P}_{\mathcal{C}}(2,1|\mu) &= \frac{(\alpha_1 x_i + \pi_1 x_i)(1-\pi_1 x_i)(1-\pi_3 x_i)}{(1+\alpha_0 x_i)(1+\alpha_1 x_i)(1+\alpha_2 x_i)}, \\
\mathsf{P}_{\mathcal{C}}(1,1,1|\mu) &= \frac{(\alpha_0 x_i + \pi_3 x_i)(1-\pi_1 x_i)}{(1+\alpha_0 x_i)(1+\alpha_1 x_i)}, \\
\mathsf{P}_{\mathcal{C}}(3,1|\mu) &= \frac{(\alpha_1 x_i + \pi_1 x_i)(\alpha_2 x_i + \pi_1 x_i)(1-\pi_1 x_i)(1-\pi_3 x_i)}{(1+\alpha_0 x_i)(1+\alpha_1 x_i)(1+\alpha_2 x_i)(1+\alpha_3 x_i)}, \\
\mathsf{P}_{\mathcal{C}}(2,1,1|\mu) &= \frac{(\alpha_1 x_i + \pi_1 x_i)(\alpha_0 x_i + \pi_3 x_i)(1-\pi_1 x_i)}{(1+\alpha_0 x_i)(1+\alpha_1 x_i)(1+\alpha_2 x_i)}.
\end{aligned}$$

Any individual (free) particle motion is (up to changing $\pi_j \mapsto \pi_1$) equivalent to the first particle's motion. Thus, let us consider λ with $\ell(\lambda) = 1$, and a straightforward computation (say, at time i) using either the operators $\mathbf{U}$ or the combinatorial description of $G_{\lambda/\!/\mu}(x_i; \boldsymbol{\alpha}, \boldsymbol{\beta})$ yields

$$\mathsf{P}_{\mathcal{C}}(m'|m) = \frac{1-\pi_j x_i}{1+\alpha_{m'+m} x_i} \prod_{k=m}^{m+m'-1} \frac{(\alpha_k + \pi_j)x_i}{1+\alpha_k x_i},$$

which is precisely the measure specified in (4.4). By (4.4), for any fixed m this is a probability measure for all $\alpha_k + \pi_j \geq 0$ with the natural assumptions $0 \leq \pi_j x_i < 1$ and

$\alpha_k x_i \geq -1$. This can also be extended to include $(\alpha_k)_{k\in\mathbb{Z}}$ by shifting the parameters $\alpha_k \mapsto \alpha_{k\pm1}$. Hence, the same analysis as in [8, Sec. 4.2] yields the following.

Theorem 4.2. *Suppose $\ell(\lambda) \leq \ell$, $\pi_j x_i \in (0,1)$, $\alpha_k x_i > -1$, and $\alpha_k + \pi_j \geq 0$ for all i, j, k. Set $\beta_j = \pi_{j+1}$. Let $\mathsf{P}_{\mathcal{C},n}(\lambda|\mu)$ denote the n-step transition probability for particle system using the distribution (4.2) for the jump probability of the particles with interactions as given by (4.1). Then the n-step transition probability is given by*

$$\mathsf{P}_{\mathcal{C},n}(\lambda|\mu) = \prod_{i=1}^{n}(1-\pi_1 x_i)(\vec{\boldsymbol{\alpha}}+\boldsymbol{\pi})^{\lambda/\mu} G_{\lambda/\!/\mu}(\mathbf{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta}).$$

Remark 4.3. Since the $\boldsymbol{\alpha}$ parameters used, and hence the probabilities, now depend on the positions of the particles, we can only work with the bosonic model, where multiple particles can occupy the same site. If we instead switch to a fermionic model by mapping the j-th particle at position λ_j to $\lambda_j - j$, then we are required to introduce additional parameters α_k for $k < 0$, in which case Theorem 4.2 no longer holds, or to account for the shifting of positions by replacing $\alpha_k \mapsto \alpha_{k+j}$ for the j-th particle distribution $\mathsf{P}_{\mathcal{G}}$.

We could also prove Theorem 4.2 by using the combinatorics of hook-valued tableaux as in [8, Sec. 5.3], where the positions of the particles is dictated by the smallest value in each entry of the hook-valued tableaux. The key observation is that we have a factor $x_i(1-\alpha_k x_i)^{-1}$ for every box in the k-th column that would normally contain an i in the set-valued tableaux (over all k), or where there is no arm. The leg (the column part except for the corner) corresponds to the choice between 1 and $-\pi_i x_j$ in the numerator of the normalization constant as in [8, Sec. 5.3]. The arm (the row part except for the corner) comes from waiting at that particular position and contributes an $-\alpha x_i$, which contributes a factor of $(1+\alpha x_i)^{-1}$ as in the Case B combinatorial proof [8, Sec. 5.4].

From [9, Thm. 4.1], we obtain determinant formulas for $\mathsf{P}_{\mathcal{C},n}(\lambda|\mu)$, where we can write the entries of the matrix as contour integrals [9, Thm. 4.19]. We can also redo [8, Thm. 6.8] at this level of generality to obtain the multi-point distribution.

Theorem 4.4. *The multi-point distribution is given by*

$$\begin{aligned}\mathsf{P}_{\geq,n}(\nu|\mu) &:= \mathsf{P}(G(\ell,n)\geq \nu_\ell,\ldots,G(1,n)\geq\nu_1 \mid G(\ell,0)=\mu_\ell,\ldots,G(1,0)=\mu_1)\\ &= \prod_{j=2}^{\ell}\prod_{i=1}^{n}(1-\pi_j x_i)^{-1}\det\left[h_{\nu_i-\mu_j-i+j}\big(\mathbf{x}/\!/(A_{(\mu_j,\nu_i]}\sqcup\boldsymbol{\pi}_i/\boldsymbol{\beta}_j)\big)\right]_{i,j=1}^{\ell}.\end{aligned}$$

We can give another simpler proof of Theorem 4.2 for the case when $\boldsymbol{\alpha}=\alpha$. This will follow from a straightforward generalization of the unrefined case [16, Prop. 3.4], noting our sign convention means we need to substitute $-\alpha$.

Proposition 4.5. *$G_\lambda(\mathbf{x};\alpha,\boldsymbol{\beta}) = G_\lambda(\mathbf{x}/(1+\alpha\mathbf{x});0,\alpha+\boldsymbol{\beta})$ by $x_i\mapsto x_i/(1+\alpha x_i)$, $\beta_i\mapsto\alpha+\beta_i$.*

Indeed, under this substitution, we have $\pi_j x_i \longmapsto \frac{(\alpha+\pi_j)x_i}{1+\alpha x_i}$. Hence, the geometric distribution P_{Ge} transforms to the distribution $\mathsf{P}_{\mathcal{G}}$ in (4.2) with $\boldsymbol{\alpha} = \alpha$. Moreover, in our formula in Case C of [8, Thm. 1.1], the total $\mathbf{x}$ degree and total $\boldsymbol{\pi}$ degree in each term of $\boldsymbol{\pi}^{\lambda/\mu} G_{\lambda/\!/\mu}(\mathbf{x};\boldsymbol{\beta})$ are equal, and so we can perform this substitution.

Remark 4.6. Let us discuss the relationship between this model and the doubly geometric inhomogeneous corner growth model defined in [11]. In their corresponding TASEP model, there is an additional set of position-dependent parameters $\boldsymbol{\nu}$ that are only involved after the initial movement of the particle (akin to static friction). Yet, if we set $\boldsymbol{\nu} = 0$, then the model in [11] is the fermionic realization of our model (*cf.* Remark 4.3) at $\boldsymbol{\beta} = 0$ with their parameters $(\mathbf{a}, \boldsymbol{\beta})$ equaling our parameters $(\boldsymbol{\alpha}, \mathbf{x})$. Hence, we end up with another TASEP version that is equivalent to Case B. It would be interesting to see if the model in [11] can be recovered from the free fermionic description.

We also remark that our model with $\boldsymbol{\pi} = 0$ was studied in [1], but using very different techniques based on Toeplitz matrices and Markov semigroups. Therefore, from the specialization of the canonical Grothendieck polynomials, it is essentially Case B as before, with a more probabilistic link being made by [1, Thm. 2.43].

We can similarly define a Bernoulli process with the position-dependent probability

$$\mathsf{P}_{\mathcal{B}}(w_{ji} = 1 \mid G(j, i-1) = m) := \frac{(\rho_j + \beta_m)x_i}{1 + \rho_j x_i}. \tag{4.6}$$

Analogously to Theorem 4.2 (including its proof), we have the following.

Theorem 4.7. *Suppose $\lambda_1 \leq \ell$, $\beta_k x_i \in (0,1)$, $\rho_j x_i > -1$, and $\rho_j + \beta_k \geq 0$ for all i, j, k. Set $\alpha_j = \rho_{j+1}$. The n-step transition probability for the particle system using Bernoulli jumps according to the distribution (4.6) is given by*

$$\mathsf{P}_{\mathcal{B},n}(\lambda|\mu) = \frac{(\vec{\beta} + \boldsymbol{\rho})^{\lambda/\mu}}{\prod_{i=1}^{n}(1+\rho_1 x_i)} G_{\lambda'/\!/\mu'}(\mathbf{x}_n; \boldsymbol{\alpha}, \boldsymbol{\beta}).$$

If we set $\boldsymbol{\alpha} = 0$ in this position-dependent version of [8, Case B], then we end up with a Bernoulli random variable version of [11] at $\boldsymbol{\nu} = 0$.

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