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Character factorisations, z-asymmetric partitions, and plethysm

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Abstract. An old theorem of D. E. Littlewood asserts that the Schur function with variables "twisted" by a primitive t-th root of unity vanishes unless the t-core of the indexing partition is empty, in which case it factors as a product of Schur functions indexed by the t-quotient. Recently, Ayyer and Kumari generalised Littlewood's result to characters of the classical groups $\mathrm{O}(2n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$. We show that Ayyer and Kumari's results may be lifted to the universal characters of the associated groups, and in doing so give a uniform extension involving a determinant of Bressoud and Wei which was later generalised by Hamel and King. What facilitates this extension is a new property of the Littlewood decomposition, extending results of Garvan, Kim and Stanton. We also explain the connection between Littlewood's original result and an instance of plethysm.

Keywords: Littlewood's decomposition, Schur functions, t-core, t-quotient, universal characters, z-asymmetric partitions

1 Introduction

In his classic 1940 book on group characters D. E. Littlewood gives a factorisation for the Schur function with variables "twisted" (not his term) by a primitive t-th root of unity ζ [13, §7.3]. More precisely, he proves that the Schur function s_λ with tn variables $\zeta^j x_i$ for $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant t-1$ vanishes if the t-core of λ is nonempty. If the t-core is empty then, up to a sign, it factors as a product of Schur functions indexed by the partitions forming its t-quotient, each with variables $x_1^t, \dots, x_n^t$. His proof is based on a clever manipulation of the classical definition of s_λ as a ratio of alternants.

Inspired by a recent rediscovery of Littlewood's theorem, Ayyer and Kumari proved analogous factorisation theorems for the characters of the classical groups $\mathrm{O}(2n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$ using Littlewood's method [3]. As in the Schur case, when these twisted characters are nonzero they factor as a product of other group characters expressed in terms of the t-quotient of the indexing partition. However, the vanishing is now governed by the t-core having a particular form. Specifically, Ayyer and Kumari show that the twisted characters for $\mathrm{O}(2n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$ are nonzero

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if and only if t-core(λ) may be written in Frobenius notation as $(a+z \mid a)$ for $z = 1, -1$ and 0 respectively; see the next section for the relevant definitions.

In [2] we lifted the results of Ayyer and Kumari to the universal characters of the aforementioned groups as defined by Koike and Terada [11]. To describe how this works, let h_r denote the r-th complete homogeneous symmetric function, the set of which is algebraically independent over $\mathbb{Z}$ and generates Λ, the ring of symmetric functions. The notion of "twisting" by a root of unity is replaced by an endomorphism φ_t of Λ for each integer $t \geqslant 2$, defined by

$$\varphi_t h_r = \begin{cases} h_{r/t} & \text{if } t \text{ divides } r, \\ 0 & \text{otherwise.} \end{cases}$$

This operator is occasionally referred to as the *t-th Verschiebung operator*, see for instance [7, §2.9] and references therein.[1] It is a quite a natural operator on symmetric functions, being the adjoint of plethysm by a power sum p_t with respect to the Hall scalar product; see Subsection 3.1. The main results of [2] are the action of φ_t on the universal characters.

In the present work we outline a new approach to proving these universal character factorisations. Following Ayyer and Kumari we call partitions which for $z \in \mathbb{Z}$ may be expressed in Frobenius notation as $(a+z \mid a)$ *z-asymmetric*. Bressoud and Wei [4] and Hamel and King [8] have defined a general symmetric function $\mathfrak{X}_\lambda(z)$ which essentially reduces to the above universal characters for $z = 1, -1$ or 0. They also show that $\mathfrak{X}_\lambda(z)$ may be expressed as a signed sum over skew Schur functions with inner shape a z-asymmetric partition. Using this expression and the action of φ_t on the skew Schur functions (Theorem 4) we can compute $\varphi_t \mathfrak{X}_\lambda(z)$. This is one of our main results, which, in order to keep things as simple as possible, we state only for $0 \leqslant z \leqslant t-1$ as Theorem 6 below. The cases $z \geqslant t$ and $z < 0$ require slightly cumbersome modifications, but no new techniques, so we defer these to future work [1]. The main advantage of our approach is that it produces a parameterised family of such factorisations which may be stated and proved uniformly. A key tool in our proof is the Littlewood decomposition, a bijection which maps a partition to its t-core and t-quotient. Our first main result, Theorem 2, is a characterisation of the Littlewood decomposition of z-symmetric partitions through restrictions on the core and quotient, which reduces to results of Garvan, Kim and Stanton for $z = 0, 1$ [6].

2 Littlewood's decomposition and z-asymmetric partitions

2.1 Preliminaries

A *partition* is a weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ such that the *size* $|\lambda| := \lambda_1 + \lambda_2 + \lambda_3 + \cdots$ is finite. The nonzero λ_i are called *parts*

[1] *Verschiebung* is German for *shift*.

and the number of parts the *length*, denoted $l(\lambda)$. The set of all partitions is written $\mathscr{P}$ and the empty partition, the unique partition of 0, is denoted by $\varnothing$. We write (m^ℓ) for the partition with ℓ parts equal to m, and the difference $\lambda - (m^\ell)$ is then the partition obtained by subtracting m from the first ℓ parts of λ. We identify a partition with its *Young diagram*, which (in English notation) is the left-justified array of cells consisting of λ_i cells in row i with i increasing downward. An example is given in Figure 1. We define the *conjugate* partition λ' by reflecting the diagram of λ in the main diagonal, so that the conjugate of $(6,5,5,1)$ is $(4,3,3,3,3,1)$.

The *Frobenius rank* of a partition, $d(\lambda)$, is defined as the number of cells along its main diagonal. Another way to notate partitions is with *Frobenius notation*, which records the number of cells to the right of and below each cell on the main diagonal, which we write in terms of the partition λ as $\lambda = (\lambda_1 - 1, \ldots, \lambda_{d(\lambda)} - d(\lambda) \mid \lambda'_1 - 1, \ldots, \lambda'_{d(\lambda)} - d(\lambda))$; again, see Figure 1 for an example. Any two strictly decreasing nonnegative integer sequences a, b with the same number of elements, say k, thus give a unique partition $\lambda = (a \mid b)$ of Frobenius rank k. Clearly self-conjugate partitions are those of the form $(a \mid a)$. Now let $a + z := (a_1 + z, \ldots, a_k + z)$ for any $z \in \mathbb{Z}$. Ayyer and Kumari [3] define what they call *z-asymmetric partitions* to be those of the form $(a + z \mid a)$ for any sequence a (of any length) and fixed $z \in \mathbb{Z}$. The set of z-asymmetric partitions is denoted by $\mathscr{P}_z$ and $(6,5,5,1)$ in Figure 1 is 2-asymmetric.

Figure 1: The partition $\lambda = (6,5,5,1) = (5,3,2 \mid 3,1,0)$ with its main diagonal shaded (left) and the same partition with hook length of each cell inscribed (right). We have $|\lambda| = 17$, $l(\lambda) = 4$ and $d(\lambda) = 3$.

Given a cell s in the Young diagram of λ its *hook length* is one more than the sum of the number of cells below and to the right of s; see Figure 1. For an integer $t \geqslant 2$ we say a partition is a *t-core* if it contains no cell with hook length t (or, equivalently, no cell with hook length divisible by t). For a pair of partitions λ, μ we say μ is *contained* in λ, written $\mu \subseteq \lambda$, if its Young diagram may be drawn inside the Young diagram of λ. The corresponding *skew shape* is the arrangement of cells formed by removing μ's diagram from λ's. A skew shape is a *ribbon* if it is edge-connected and contains no 2×2 square of cells, and a *t-ribbon* is a ribbon containing t cells.[2] The *height* of a ribbon R, $\mathrm{ht}(R)$, is one less than the number of rows it occupies; see Figure 2.

[2]Elsewhere in the literature ribbons are variously called *border strips*, *rim hooks* or *skew hooks*.

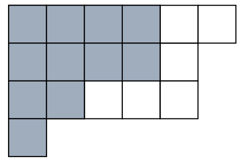

Figure 2: The pair of partitions $(4,4,2,1) \subseteq (6,5,5,1)$. The unshaded cells form a 6-ribbon of height 2.

We say a skew shape λ/μ is *t-tileable* if there exists a sequence of partitions

$$\mu =: \nu^{(0)} \subseteq \nu^{(1)} \subseteq \cdots \subseteq \nu^{(m-1)} \subseteq \nu^{(m)} := \lambda$$

such that the skew shapes $\nu^{(r)}/\nu^{(r-1)}$ are each t-ribbons for $1 \leqslant r \leqslant m$. It is a non-trivial fact, see, e.g. [17, Lemma 4.1], that the sign

$$\operatorname{sgn}_t(\lambda/\mu) := (-1)^{\sum_{r=1}^{m} \operatorname{ht}(\nu^{(r)}/\nu^{(r-1)})} \tag{2.1}$$

is constant over the set of all t-ribbon decompositions of λ/μ (so, indeed, the above is well-defined).

2.2 Littlewood's decomposition

The Littlewood decomposition is, for each integer $t \geqslant 2$, a bijection which decomposes a partition λ into a pair $(t\text{-core}(\lambda), \boldsymbol{\lambda})$, where t-core(λ) is the unique *t-core of* λ and $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(t-1)})$ is a t-tuple of partitions called the *t-quotient* [14]. Here we describe the Littlewood decomposition through the lens of Maya diagrams, which is essentially the abacus model of James and Kerber [9, §2.7]. A purely algebraic description may be found in [16, p. 12].

Given a partition λ its *Maya diagram* is the following subset of the set of half integers, sometimes called the *beta set*

$$\beta(\lambda) := \Big\{\lambda_i - i + \frac{1}{2} : i \geqslant 1\Big\}.$$

This is visualised as a configuration of "beads" on the real line placed at the positions indicated by $\beta(\lambda)$. The map from partitions to Maya diagrams is clearly a bijection, and one way to reconstruct λ from $\beta(\lambda)$ is to count the number of empty spaces to the left of each bead starting from the right. From the Maya diagram we extract t sub-diagrams formed by the beads at positions x such that $x - 1/2$ is r modulo t for $0 \leqslant r \leqslant t-1$, which we dub the *t-Maya diagram*. An example of this procedure is given in Figure 3. The corresponding partitions are denoted by $\lambda^{(r)}$ according to the residues modulo t of the original positions, and these precisely form Littlewood's *t-quotient*.

The key observation behind the definition of the t-core is that moving a bead one space to the left in the t-Maya diagram is equivalent to the removal of a t-ribbon from λ such that what remains is still a partition. Since such ribbons are in correspondence with hooks of length t in λ, pushing all beads to the left leaves a t-core. The t-Maya diagram shows that this is independent of the order in which such ribbons are removed, and so the resulting unique partition is denoted $t\text{-core}(\lambda)$.

Theorem 1 (Littlewood's decomposition)**.** *For any integer $t \geqslant 2$ the above procedure encodes a bijection*

$$\begin{aligned} \mathscr{P} &\longrightarrow \mathscr{C}_t \times \mathscr{P}^t \\ \lambda &\longmapsto \big(t\text{-core}(\lambda), (\lambda^{(0)}, \dots, \lambda^{(t-1)})\big) \end{aligned}$$

such that $|\lambda| = |t\text{-core}(\lambda)| + t(|\lambda^{(0)}| + \cdots + |\lambda^{(t-1)}|)$.

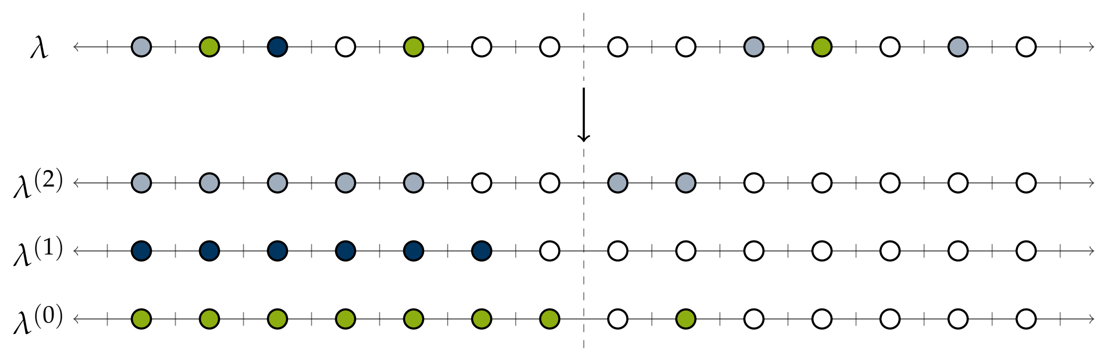

Figure 3: The Maya diagram of $\lambda = (6,5,5,1)$ (top) and the 3-Maya diagram of the same partition (bottom). We have that $3\text{-core}(\lambda) = (1,1)$, $\kappa_3((1,1)) = (1,-1,0)$ and $(\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}) = ((1), \varnothing, (2,2))$.

We will also need a different characterisation of t-cores. Call a Maya diagram *balanced* if it contains as many beads to the right of 0 as empty spaces to the left. The way we defined Maya diagrams ensures they are always balanced, but Figure 3 shows that the constituent diagrams of the quotient need not be. Let c_r^+ (resp. c_r^-) denote the number of beads to the right of 0 (resp. number of empty spaces to the left of 0) in row $\lambda^{(r)}$ of the t-Maya diagram. Now the sequence of integers $(c_0, \dots, c_{t-1})$ defined by $c_r := c_r^+ - c_r^-$ has total sum zero, and is invariant under valid bead movements. As observed by Garvan, Kim and Stanton, this encodes a bijection [6, Bijection 2]

$$\kappa_t : \mathscr{C}_t \longrightarrow \{(c_0, \dots, c_{t-1}) \in \mathbb{Z}^t : c_0 + \cdots + c_{t-1} = 0\} \tag{2.2}$$

such that for $\mu \in \mathscr{C}_t$

$$|\mu| = \sum_{r=0}^{t-1} \left(\frac{tc_r^2}{2} + rc_r \right).$$

The conjugate of a partition λ can be read off its Maya diagram by interchanging beads and empty spaces and then reflecting the picture about 0. Using this fact one may show that $t\text{-core}(\lambda') = t\text{-core}(\lambda)'$ which, if $\kappa_t(t\text{-core}(\lambda)) = (c_0, \dots, c_{t-1})$, translates to $\kappa_t(t\text{-core}(\lambda')) = (-c_{t-1}, \dots, -c_0)$ in terms of (2.2). Moreover, the quotient of λ' is given by $((\lambda^{(t-1)})', \dots, (\lambda^{(0)})')$. From these properties it is easy to see that the Littlewood decomposition of a self-conjugate partition must satisfy $t\text{-core}(\lambda) \in \mathscr{P}_0$, i.e., $c_r + c_{t-r-1} = 0$ for $0 \leqslant r \leqslant t-1$ and $\lambda^{(r)} = (\lambda^{(t-r-1)})'$ for r in the same range. Garvan, Kim and Stanton [6, §8] show that something similar holds for 1-asymmetric partitions. That is, if $\lambda \in \mathscr{P}_1$ then $t\text{-core}(\lambda), \lambda^{(0)} \in \mathscr{P}_1$ and the remaining entries in the quotient satisfy $\lambda^{(r)} = (\lambda^{(t-r)})'$ for $1 \leqslant r \leqslant t-1$.

Our first main result is a generalisation of the theorems of Garvan, Kim and Stanton to z-asymmetric partitions. To keep things simple we restrict to $0 \leqslant z \leqslant t-1$, with negative z being obtained by conjugation and larger values of z requiring only a minor, but slightly cumbersome, modification. To fix some notation, let $\mathcal{C}_{z;t} \subset \mathbb{Z}^t$ consist of those sequences for which $c_r + c_{z-r-1} = 0$ for $0 \leqslant r \leqslant z-1$ and $c_s + c_{t+z-s-1} = 0$ for $z \leqslant s \leqslant t-1$. Also let $d_c(\lambda)$ denote the Frobenius rank of the partition obtained by removing the first c rows of λ.

Theorem 2. *Let $t \geqslant 2$ and z be integers and λ a partition such that $0 \leqslant z \leqslant t-1$ and $\lambda \in \mathscr{P}_z$. Then $\kappa_t(t\text{-core}(\lambda)) \in \mathcal{C}_{z;t}$ and the quotient $(\lambda^{(0)}, \dots, \lambda^{(t-1)})$ is such that for $0 \leqslant r \leqslant z-1$ with $c_r \geqslant 0$ there exists a partition $\nu^{(r)}$ with*[3]

$$\lambda^{(r)} = \nu^{(r)} + (1^{c_r + d_{c_r}(\nu^{(r)})}) \quad \textit{and} \quad \lambda^{(z-r-1)} = (\nu^{(r)})' + (1^{d_{c_r}(\nu^{(r)})}). \tag{2.3a}$$

Moreover, for $z \leqslant s \leqslant t-1$.

$$\lambda^{(s)} = (\lambda^{(t+z-s-1)})'. \tag{2.3b}$$

Before we comment on the proof of this characterisation an example is in order. Let $t = 5$, $z = 3$ and $\lambda = (21, 17, 16, 15, 12, 11, 6, 6, 5, 4, 4, 4, 3, 2, 1, 1, 1, 1)$, or, in Frobenius notation, $\lambda = (20\ 15\ 13\ 11\ 7\ 5 \mid 17\ 12\ 10\ 8\ 4\ 2)$. Then $5\text{-core}(\lambda) = (6, 5, 3, 2, 1, 1, 1, 1)$ which has associated integer vector $(2, 0, -2, 1, -1) \in \mathcal{C}_{3,5}$ and the quotient is given by

$$\big(\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)}\big) = \big((3,3,2), (3,1), (4,2), (2), (1,1)\big). \tag{2.4}$$

The reader may check the conditions (2.3) are satisfied by looking at Figure 4.

Note that if z is even then the $r = z/2$ case of (2.3a) just says that $\lambda^{(z/2)}$ is 1-asymmetric. This this it is clear that the 0- and 1-asymmetric cases are contained in the theorem. However, as the above example shows, not all cores with image in $\mathcal{C}_{z;t}$ are themselves z-asymmetric. The following corollary clarifies when the t-core of a z-asymmetric partition is again z-asymmetric, the first part of which is essentially contained in [3, Lemma 3.6].

[3] If $c_r = 0$ then $\nu^{(r)} = (\nu^{(z-r-1)})'$ is forced by (2.3a).

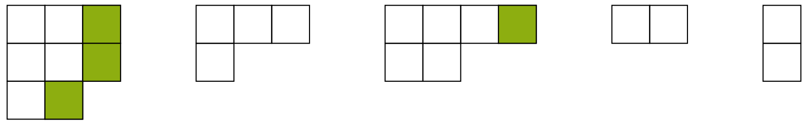

Figure 4: Young diagrams representing the 5-quotient (2.4). Note that $c_0 = 2$ so $d_{c_0}(\lambda^{(0)}) = 1$, $\nu^{(0)} = (2,2,1)$ and $\nu^{(1)} = (2,1)$. The highlighted cells in the first and third partitions denote those subtracted when verifying (2.3a).

Corollary 3. *A t-core μ is z-asymmetric if and only if $\kappa_t(\mu)$ satisfies $c_r = 0$ for $0 \leqslant r \leqslant z-1$. Moreover, for any sequence $\mathbf{c} \in \mathcal{C}_{z;t}$ the unique z-asymmetric partition $\mu_{\mathbf{c}}$ with $\kappa_t(t\text{-core}(\mu_{\mathbf{c}})) = \mathbf{c}$ and minimal $|\mu_{\mathbf{c}}|$ has quotient $\mu_{\mathbf{c}}^{(r)} = (1^{c_r})$ for those r with $0 \leqslant r \leqslant z-1$ and $c_r > 0$.*

Proof. By Theorem 2 a z-asymmetric partition μ must have $\kappa_t(t\text{-core}(\mu)) \in \mathcal{C}_{z;t}$ and $\lambda^{(r)} = \varnothing$ for all $0 \leqslant r \leqslant t-1$. However, the restrictions (2.3a) admit the empty partition as a solution if and only if $c_r = 0$. The second part of the corollary is then immediate. □

Theorem 2 may be proved by induction on z. For $z \geqslant 1$ there is an obvious bijection from $\mathscr{P}_{z-1}$ to $\mathscr{P}_z$ which adds one to the first $d(\lambda)$ parts of λ. We imagine the t-Maya diagram is wrapped around a cylinder, so that this bijection pushes the beads at positive positions up one row, and additionally moves the beads passing from row $t-1$ to row 0 one space to the right. This leads to an extension of Theorem 2 for $z \geqslant 0$, and to this end we say a pair of partitions satisfying (2.3a) are 1-*conjugate*. Then one writes $z = at + b$ for $a \geqslant 0$ and $0 \leqslant b \leqslant t-1$, so that the generalisation of Theorem 2 claims that $t\text{-core}(\lambda) \in \mathcal{C}_{b;t}$, and the partitions in (2.3a) will now be $(a+1)$-conjugate and those in (2.3b) a-conjugate [1].

3 Factorisations of universal characters

3.1 Symmetric functions and plethysm

As mentioned in the introduction, the ring of symmetric functions Λ has an algebraic basis given by the *complete homogeneous symmetric functions*, which for a countably infinite alphabet $X = (x_1, x_2, x_3, \dots)$ may be defined by the generating function

$$\prod_{i \geqslant 1} \frac{1}{1 - ux_i} = \sum_{k \geqslant 0} u^k h_k(X).$$

The most important linear basis for Λ is given by the *Schur functions* s_λ, which we define at the generality of skew shapes by the Jacobi–Trudi determinant

$$s_{\lambda/\mu} := \det_{1 \leqslant i,j \leqslant l(\lambda)} (h_{\lambda_i - \mu_j - i + j}), \tag{3.1}$$

where $h_{-k} := 0$ for $k \geqslant 1$. There is an inner product on Λ, the *Hall inner product*, under which the s_λ are orthonormal, so

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$$

with $\delta_{\lambda\mu}$ the usual Kronecker delta.

Plethysm is a composition of symmetric functions first introduced by Littlewood which we denote by $f \circ g$ for $f, g \in \Lambda$; see, e.g., [16, §1.9]. We only require the case where $g = p_t(X) := \sum_{i \geqslant 1} x_i^t$, the t-th power sum. This may most easily be defined by expanding f as a sum of monomials in X and then replacing each x_i by x_i^t. This particular plethysm satisfies $f \circ p_t = p_t \circ f$ and $p_s \circ p_t = p_{st}$ for $s, t \in \mathbb{N}$. Another way to define the operator φ_t is as the adjoint of the plethysm by a power sum with respect to the Hall inner product, i.e., for any $f, g \in \Lambda$,

$$\langle f \circ p_t, g \rangle = \langle f, \varphi_t g \rangle. \tag{3.2}$$

This may also be verified directly using, for instance, the orthonormality of the complete homogeneous and monomial symmetric functions.

As alluded to in the introduction, the content of [13, §7.3] is the computation of the action of φ_t on the Schur basis.[4] Later on, Farahat generalised this result to skew Schur functions of the form $s_{\lambda/t\text{-core}(\lambda)}$ [5], and the full skew Schur function case may be found in [16, p. 91]. Here the notion of "empty t-core" is replaced by the requirement that λ/μ is t-tileable. This is equivalent to $t\text{-core}(\lambda) = t\text{-core}(\mu)$ and $\lambda^{(r)} \supseteq \mu^{(r)}$ for each $0 \leqslant r \leqslant t-1$ [2, Lemma 2.1].

Theorem 4. *For any integer $t \geqslant 2$ and skew shape λ/μ we have that $\varphi_t s_{\lambda/\mu} = 0$ unless λ/μ is t-tileable, in which case*

$$\varphi_t s_{\lambda/\mu} = \operatorname{sgn}_t(\lambda/\mu) \prod_{r=0}^{t-1} s_{\lambda^{(r)}/\mu^{(r)}},$$

where the sign is defined in (2.1).

It is our opinion that this theorem has been somewhat neglected, and deserves to be better known. Although we will not give a full account its history here, the interested reader may find a few historical remarks in [2, §3] and its references. Using the Jacobi–Trudi formula (3.1) and the algebraic description of the t-core and t-quotient the proof is relatively straightforward, with the only difficulty being the identification of the sign.

One of the first applications of Littlewood's core and quotient construction is to the plethysm $s_\lambda \circ p_t$ which is now referred to as the *SXP rule* [14, p. 351].

[4] In his book [13] Littlewood does not use the language of cores and quotients, nor the map φ_t, and they appear only implicitly.

Theorem 5. *Let $c^{\lambda}_{\nu^{(0)},\dots,\nu^{(t-1)}}$ be the coefficient of s_λ in the Schur expansion of the product $s_{\nu^{(0)}}\cdots s_{\nu^{(t-1)}}$. Then for any $t \geqslant 2$,*

$$s_\lambda \circ p_t = \sum_{\substack{\nu \\ t\text{-core}(\nu)=\varnothing}} \operatorname{sgn}_t(\nu) c^{\lambda}_{\nu^{(0)},\dots,\nu^{(t-1)}} s_\nu.$$

By the adjoint relation (3.2) this is equivalent to Theorem 4 with $\mu = \varnothing$. Wildon has given a generalisation of the SXP rule for the expression $s_\tau(s_{\lambda/\mu} \circ p_t)$ [19], which may be derived from the full Theorem 4 in the same manner. Littlewood proved versions of the SXP rule for orthogonal and symplectic characters in the cases $t = 2, 3$ [15], and these were given lifts to the universal characters by Scharf and Thibon [18]. Lecouvey then greatly extended this by proving SXP rules for the universal symplectic and orthogonal characters for arbitrary $t \geqslant 2$ [12]. Using the adjoint relation (3.2), one can show that these expressions are equivalent to special cases of our Theorem 6 below.

3.2 Generalised universal characters

For a finite set of n variables the Schur function $s_\lambda(x_1, \dots, x_n)$ may be regarded as the character of the irreducible polynomial representation of $\mathrm{GL}(n, \mathbb{C})$ indexed by λ. The classical groups $\mathrm{O}(2n, \mathbb{C})$, $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{SO}(2n+1, \mathbb{C})$ also carry irreducible representations indexed by partitions. The characters of these representations are rather symmetric Laurent polynomials in n variables, however they may still be expressed as determinants in the complete homogeneous symmetric functions $h_r(x_1, 1/x_1, \dots, x_n, 1/x_n)$. Using these expressions, Koike and Terada defined the *universal characters* of the above groups, which are lifts of the characters to symmetric functions, and proved some expansions in terms of skew Schur functions [11, Theorem 2.3.1]. For example the universal character of $\mathrm{O}(2n, \mathbb{C})$ satisfies

$$\mathrm{o}_\lambda := \det_{1 \leqslant i,j \leqslant l(\lambda)} (h_{\lambda_i - i + j} - h_{\lambda_i - i - j}) = \sum_{\mu \in \mathscr{P}_1} (-1)^{|\mu|/2} s_{\lambda/\mu}. \tag{3.3}$$

with similar identities for the universal characters sp_λ and so_λ as sums over -1- and 0-asymmetric partitions respectively.

In [4] Bressoud and Wei proved an extension of (3.3) involving an integer $z \geqslant -1$ which reproduces the classical cases for $z \in \{-1, 0, 1\}$. This was generalised further by Hamel and King to an expression valid for all $z \in \mathbb{Z}$ and including an additional parameter q [8]. Let $[S]$ denote the *Iverson bracket*: $[S] = 1$ if the statement S is true and zero otherwise. Then the identity of Hamel and King is

$$\mathfrak{X}_\lambda(z; q) := \det_{1 \leqslant i,j \leqslant l(\lambda)} \big(h_{\lambda_i - i + j} + [j > -z] q h_{\lambda_i - i - j + 1 - z}\big) \tag{3.4a}$$

$$= \sum_{\mu \in \mathscr{P}_z} (-1)^{(|\mu| - d(\mu)(z+1))/2} q^{d(\mu)} s_{\lambda/\mu}. \tag{3.4b}$$

The odd and even orthogonal cases are recovered by setting $q = (-1)^z$ and then choosing $z = 0, 1$ respectively. The expression in terms of skew Schur functions immediately implies the following duality with respect to the involution ω on Λ which acts as $\omega s_{\lambda/\mu} = s_{\lambda'/\mu'}$:

$$\omega \mathfrak{X}_\lambda(z; q) = \mathfrak{X}_{\lambda'}(-z; (-1)^z q).$$

Again setting $q = (-1)^z$ and then $z = 1$ on the right then recovers the symplectic case of (3.3), thus extending $\omega \mathrm{o}_\lambda = \mathrm{sp}_{\lambda'}$ [11, Theorem 2.3.2]. Since our results require only minor modification to account for negative z we now restrict to $0 \leqslant z \leqslant t-1$.

Our main result is the action of φ_t on $\mathfrak{X}_\lambda(z; (-1)^z)$ (which was denoted simply $\mathfrak{X}_\lambda(z)$ in the introduction). To state this we need one more symmetric function, defined for $a, b, c \in \mathbb{N}$ by the following sum

$$\mathrm{rs}_{\lambda,\mu}(a, b; c) := \sum_\nu (-1)^{|\nu|} s_{\lambda/(\nu + (a^{c+d_c(\nu)}))} s_{\mu/(\nu' + (b^{d_c(\nu)}))},$$

where $d_c(\nu)$ is the modified Frobenius rank from Theorem 2. This symmetric function not only arises naturally in the proof of our main theorem, but also has Jacobi–Trudi-type determinantal expressions. For $c = 0$ this was also considered by Hamel and King, who also gave a Jacobi–Trudi-type expression [8]. If $a = b = c = 0$ it is essentially the universal character of the rational representation of $\mathrm{GL}(n, \mathbb{C})$ indexed by the pair (λ, μ) as defined by Koike [10]. (In fact, Koike's universal character was the inspiration for the extension of Hamel and King.) The function $\mathrm{rs}_{\lambda,\mu}(a, a; 0)$ is symmetric in λ and μ, however the same does not hold for $c \neq 0$. To make the statement below compact we adopt the convention that $\mathrm{rs}_{\lambda^{(r)},\lambda^{(z-r-1)}}(a, a; c_r) = \mathrm{rs}_{\lambda^{(z-r-1)},\lambda^{(r)}}(a, a; -c_r)$ if $c_r < 0$. Finally, recall that for $\mathbf{c} \in \mathcal{C}_{z;t}$, $\mu_{\mathbf{c}}$ is the unique smallest z-asymmetric partition with $\kappa_t(t\text{-core}(\mu_{\mathbf{c}})) = \mathbf{c}$ provided by Corollary 3. With this established, we are ready to state our second main result.

Theorem 6. *Let $t \geqslant 2$ and z be integers such that $0 \leqslant z \leqslant t-1$. Then $\varphi_t \mathfrak{X}_\lambda(z)$ vanishes unless $\kappa_t(t\text{-core}(\lambda)) := \mathbf{c} \in \mathcal{C}_{z;t}$ and $\lambda \supseteq \mu_{\mathbf{c}}$. If these conditions are satisfied, then*

$$\varphi_t \mathfrak{X}_\lambda(z; (-1)^z) = \varepsilon \prod_{r=0}^{\lfloor (z-2)/2 \rfloor} \mathrm{rs}_{\lambda^{(r)},\lambda^{(z-r-1)}}(1, 1; c_r) \prod_{s=z}^{\lfloor (t+z-2)/2 \rfloor} \mathrm{rs}_{\lambda^{(s)},\lambda^{(t+z-s-1)}}(0, 0; 0)$$
$$\times \begin{cases} 1 & \text{if } z \text{ even, } t \text{ even,} \\ \mathfrak{X}_{\lambda^{((z-1)/2)}}(1; -1) & \text{if } z \text{ odd, } t \text{ odd,} \\ \mathfrak{X}_{\lambda^{((t+z-1)/2)}}(0; 1) & \text{if } z \text{ even, } t \text{ odd,} \\ \mathfrak{X}_{\lambda^{((z-1)/2)}}(1; -1) \mathfrak{X}_{\lambda^{((t+z-1)/2)}}(0; -1) & \text{if } z \text{ odd, } t \text{ even,} \end{cases}$$

where the sign ε may be expressed as

$$\varepsilon = (-1)^{(|\mu_{\mathbf{c}}| + (z-1)d(\mu_{\mathbf{c}}))/2} \, \mathrm{sgn}_t(\lambda/\mu_{\mathbf{c}}).$$

For $z = 1$ the theorem states that $\varphi_t \mathrm{o}_\lambda$ vanishes unless t-core(λ) is 1-asymmetric, in which case

$$\varphi_t \mathrm{o}_\lambda = (-1)^{|t\text{-core}(\lambda)|/2} \operatorname{sgn}_t(\lambda/t\text{-core}(\lambda)) \mathrm{o}_{\lambda^{(0)}} \prod_{r=1}^{\lfloor (t-1)/2 \rfloor} \mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r)}} \times \begin{cases} \mathrm{so}^-_{\lambda^{(t/2)}} & t \text{ even,} \\ 1 & t \text{ odd,} \end{cases}$$

where $\mathrm{so}^-_\nu := \mathfrak{X}_\nu(0,-1)$. This is precisely [2, Theorem 3.2], which generalises [3, Theorem 2.15]. One notable improvement on both our previous results and those of Ayyer and Kumari is that the sign in the above has a much nicer, combinatorial, expression. Another improvement is that Theorem 6 admits a uniform statement and uniform proof for $0 \leqslant z \leqslant t-1$. To obtain the symplectic case one must compute $\varphi_t \mathfrak{X}_\lambda(-z; (-1)^{z-1})$, which is completely analogous to the proof of the above, even though it is (unfortunately) not contained in Theorem 6. There is also a more general version of the theorem where $z \in \mathbb{N}$, as for Theorem 2, which we defer to future work [1].

Let us briefly sketch the proof of Theorem 6. The first step is to apply the map φ_t to each term in the skew Schur expansion (3.4b) using Theorem 4. Doing so gives the expression

$$\varphi_t \mathfrak{X}_\lambda(z; (-1)^z) = \sum_{\substack{\mu \in \mathscr{P}_z \\ \lambda/\mu \ t\text{-tileable}}} (-1)^{(|\mu| + (z-1)d(\mu))/2} \operatorname{sgn}_t(\lambda/\mu) \prod_{r=0}^{t-1} s_{\lambda^{(r)}/\mu^{(r)}}.$$

From this vantage point the vanishing is already visible. Since λ/μ being t-tileable means that, in particular, t-core$(\lambda) = t$-core(μ) the first part of Theorem 2 implies that $\kappa_t(t\text{-core}(\lambda)) \in \mathcal{C}_{z;t}$ for the sum to be non-vanishing. The second part of the same theorem combined with Corollary 3 tells us that $(1^{c_r}) \subseteq \mu^{(r)} \subseteq \lambda^{(r)}$ (since λ/μ is t-tileable) for those $0 \leqslant r \leqslant z-1$ such that $c_r > 0$, which is equivalent to the requirement that $l(\lambda^{(r)}) \geqslant c_r$ for $0 \leqslant r \leqslant z-1$. The next step is to show that the sum decouples as a product, and each factor in this product corresponds to the symmetric functions present in the factorisation. This is the meat of the proof, requiring a careful analysis of the sign and Frobenius rank as they relate to the Littlewood decomposition. Unfortunately, at this stage we are unable to include the parameter q present in (3.4) precisely because the Frobenius rank does not decompose nicely in terms of the Littlewood decomposition.

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