Séminaire Lotharingien de Combinatoire **91B** (2024)
Article #15, 12 pp.

Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (Bochum)

Generalized Heawood graphs and triangulations of tori

Cesar Ceballos*[1] and Joseph Doolittle[1]

[1] *TU Graz, Institut für Geometrie, Kopernikusgasse 24, 8010 Graz, Austria*

Abstract. The Heawood graph is a remarkable graph that played a fundamental role in the development of the theory of graph colorings on surfaces in the 19th and 20th centuries. Based on permutahedral tilings, we introduce a generalization of the classical Heawood graph indexed by a sequence of positive integers. We show that the resulting generalized Heawood graphs are toroidal graphs, which are dual to higher dimensional triangulated tori. We also present explicit combinatorial formulas for their f-vectors and study their automorphism groups.

Keywords: Heawood graph, triangulations of tori, permutahedron, map coloring.

1 Introduction

The Heawood graph is a remarkable graph which played a fundamental role in the historical development of the theory of map colorings on surfaces. The four color theorem is an important result in this area, and perhaps one of the most well known results in mathematics in general. It states that for any map on a sphere, for example Europe, there is a coloring of that map with four colors, such that each region (or country) has one color and any two adjacent regions[1] have different colors. This problem has an interesting history dating back to 1852, but the theorem was only proved more than a hundred years later in 1976 by Kenneth Appel and Wolfgang Haken [1] after many false proofs and false counterexamples, and it is the first major result in mathematics that was proved using a computer.

One famous false proof of the four color theorem was given by Alfred Kempe in 1879 [4]. His proof was announced in *Nature* [5] and was regarded as an established fact for more than a decade. In 1890, Percy John Heawood found a gap in Kempe's proof, and modified his argument to show that five colors are sufficient to color a map on a sphere [3]. This became known as the five color theorem.

*cesar.ceballos@tugraz.at. Both authors were supported by the Austrian Science Fund FWF, Project P 33278. Our work was also supported by the ANR-FWF International Cooperation Project PAGCAP, funded by the FWF Project I 5788.

[1] Two regions are adjacent if they share a common boundary curve segment, not just a point.

In the same paper [3], Heawood investigated coloring of maps on other surfaces. He showed that $N_p = \left\lfloor \frac{7+\sqrt{1+48p}}{2} \right\rfloor$ colors are sufficient to color a map on the oriented surface of genus $p \geq 1$, where $\lfloor x \rfloor$ is the largest integer not greater than x. For instance, it is possible to color any map on a torus (genus $p = 1$ surface) using seven colors. Heawood also showed that for $p = 1$ the number seven is tight, by showing a map of the torus where seven colors are necessary: a map consisting of seven regions for which any two regions are adjacent to each other.

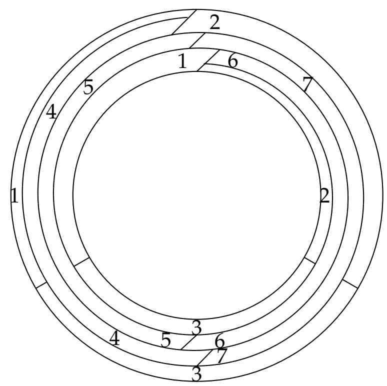


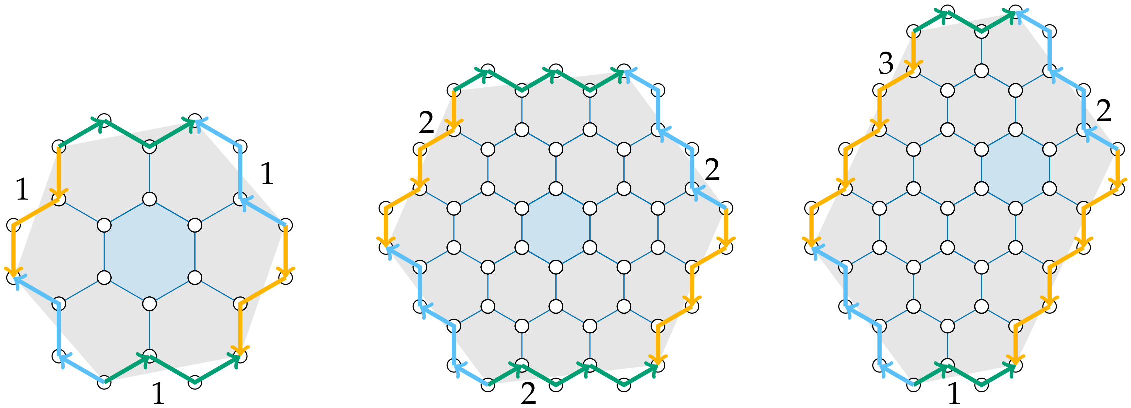
Figure 1: Reproduction of Heawood's map on a torus from 1890. The inner and outer circle are identified to produce a torus.

The fact that the number N_p is tight for a genus p orientable surface became known as Heawood's Conjecture, and was finally proved in 1968 [10]. The case $p = 1$ (the torus) is known as the seven color theorem, and has inspired beautiful math-art works.

The Heawood graph is defined as the graph of Heawood's map: its vertices are the common points of three pairwise adjacent regions, and the edges are the lines connecting these points. It is a toroidal and distance-transitive graph on 14 vertices and 21 edges. Our favorite representation of Heawood's graph is illustrated in Figure 2a, which is based on a highly symmetric representation due to Leech in [8, Figure 2]. Note that here, the graph is the graph induced by the edge graph of the seven hexagons, where the boundary is identified by gluing the opposite colored lines as illustrated.

The main purpose of this paper is to introduce a generalization $H_{\mathbf{k}}$ of Heawood's graph that extends Leech's representation. Our generalization is indexed by a sequence $\mathbf{k} = (k_1, \ldots, k_{d+1}) \in \mathbb{N}^{d+1}$ of positive integers for some $d \geq 2$, and recovers the classical Heawood graph when $\mathbf{k} = (1, 1, 1)$. As in the classical case, we show that $H_{\mathbf{k}}$ is a toroidal graph which is naturally embedded in a d-dimensional torus.

When there are three parameters, the generalized Heawood graph $H_{(k_1,k_2,k_3)}$ is a 2-dimensional generalization of the classical Heawood graph. It is obtained by gluing together $\prod(k_i + 1) - \prod k_i$ regular hexagons: From a "central" hexagon one adds k_i



(a) $\mathbf{k} = (1,1,1)$ (b) $\mathbf{k} = (2,2,2)$ (c) $\mathbf{k} = (3,1,2)$

Figure 2: Examples of the Heawood graph $H_{\mathbf{k}}$ in dimension 2. The opposite sides (with the same color) are identified, making this graph a toroidal graph. The torus is the gray hexagon with opposite edges identified.

hexagons pointing in the direction at angle $(i-2)\frac{2\pi}{3}$ for $i = 1,2,3$; then fill the "big hexagon" that they generate with other small hexagons. Several examples are illustrated in Figure 2. We also provide three different choices of fundamental domain in Figure 3, where the torus can be visualized in its more common rectangular presentation.

The case $d = 3$ gives 3-dimensional generalizations of the Heawood graph. The smallest choice of parameters is $H_{(1,1,1,1)}$, which is obtained by gluing $15 = 2^4 - 1^4$ polytopes that are 3-dimensional permutahedra, see Figure 4. The boundary of the result is identified to itself to form the complex into a 3-dimensional torus (see Section 4.2).

One special object of interest is the dual triangulation of $H_{(1,1,\ldots,1)}$. This triangulation consists of $2^{d+1} - 1$ vertices and appeared in the work of Wolfgang Kühnel and Gunter Lassmann from the 1980's in [6, 7]. Interestingly, it is conjectured to be a minimal triangulation of the d-dimensional torus [9, Conjecture 21].

A longer version of this extended abstract with more details and proofs is available at [2].

2 The generalized Heawood graph

The generalized Heawood graph $H_{\mathbf{k}}$ is indexed by a sequence $\mathbf{k} = (k_1, \ldots, k_{d+1}) \in \mathbb{N}^{d+1}$ of positive integers for some $d \geq 2$. It is obtained by making some identifications on an infinite graph $\widetilde{G}_d$, which is the graph of the d-dimensional permutahedral tiling. Before explaining this connection, we provide a direct definition in this section.

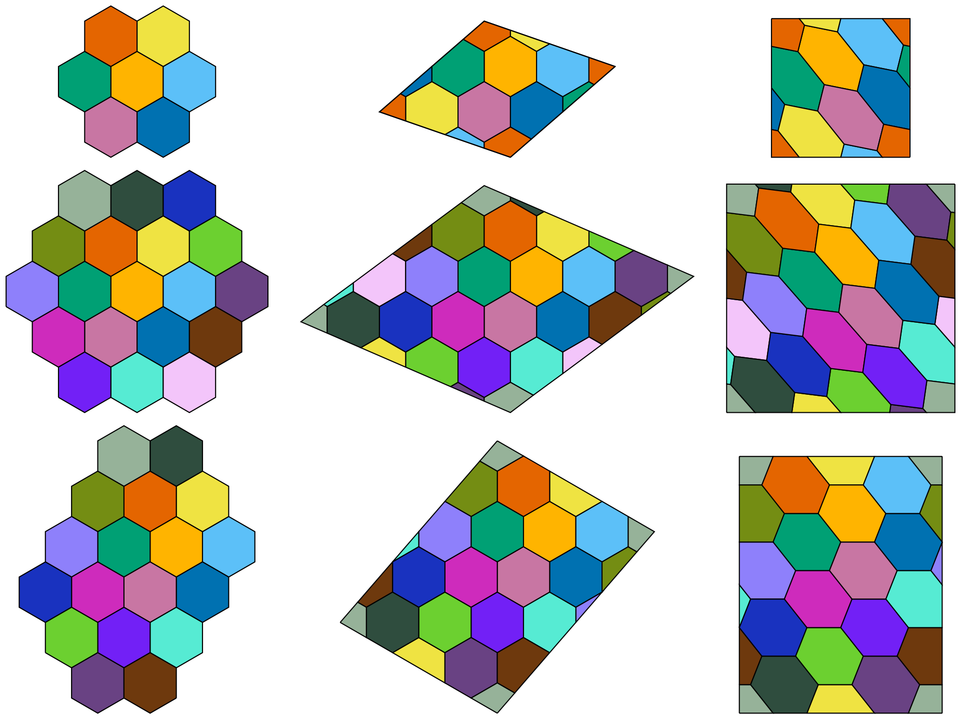


Figure 3: Different presentations of the fundamental domain for the Heawood graphs $H_{(1,1,1)}$, $H_{(2,2,2)}$ and $H_{(3,1,2)}$.

The vertices $\mathrm{Vert}(\widetilde{G}_d)$ of the *graph* $\widetilde{G}_d$ are the elements of the affine subspace

$$\{\mathbf{x} = (x_1, \dots, x_{d+1}) : \ x_1 + \cdots + x_{d+1} = 1 + \cdots + (d+1)\} \subset \mathbb{R}^{d+1}$$

whose entries are integers containing all the numbers $1, 2, \dots, d+1$ mod $(d+1)$. For instance, all permutations of $[d+1]$ satisfy this property. Two vertices $\mathbf{x}, \mathbf{y}$ of $\widetilde{G}_d$ are connected by an edge if $\mathbf{y} - \mathbf{x} = e_j - e_i$ for some $i \neq j$, where $e_1, \dots, e_{d+1}$ denote the standard basis vectors in $\mathbb{R}^{d+1}$. Figure 5 shows a portion of the graph $\widetilde{G}_2$, where the blue hexagon is the convex hull of all permutations of $[3]$.

For $\mathbf{k} = (k_1, \dots, k_{d+1}) \in \mathbb{N}^{d+1}$ we denote by $M_\mathbf{k}$ the matrix

$$M_\mathbf{k} = \begin{pmatrix} k_1+1 & -k_2 & & & & \\ & k_2+1 & -k_3 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & k_d+1 & -k_{d+1} \\ -k_1 & & & & & k_{d+1}+1 \end{pmatrix} \tag{2.1}$$

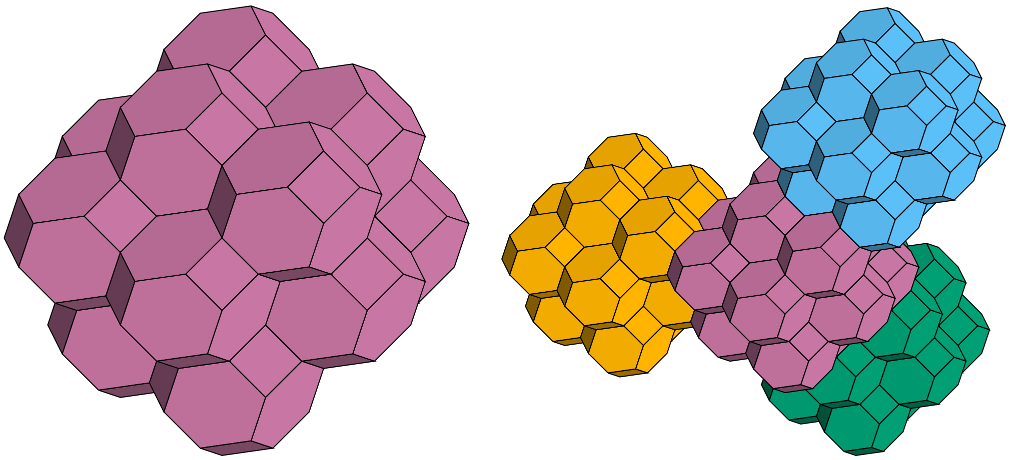


Figure 4: The Heawood graph $H_{(1,1,1,1)}$ is the edge graph of this portion of the 3-dimensional permutahedral tiling after properly identifying its boundary by translations (see Section 4.2), making it into a toroidal graph.

and let $w_1, \ldots, w_{d+1} \in \mathbb{Z}^{d+1}$ be the vectors

$$w_i = (d+1)e_i - \sum_{j=1}^{d+1} e_j. \tag{2.2}$$

Equivalently, w_i has ith coordinate equal to d and all other coordinates equal to -1.

Note that if $\mathbf{x} \in \mathrm{Vert}(\widetilde{G}_d)$ then $\mathbf{x} + w_i \in \mathrm{Vert}(\widetilde{G}_d)$. Moreover, if $\mathbf{x}, \mathbf{y} \in \mathrm{Vert}(\widetilde{G}_d)$ are connected by an edge then $\mathbf{x} + w_i$ and $\mathbf{y} + w_i$ are connected by an edge as well. In other words, the graph $\widetilde{G}_d$ is invariant under translations by the vectors $w_1, \ldots, w_{d+1}$.

We denote by $\mathcal{L}_d$ the lattice of integer linear combinations of the w_i

$$\mathcal{L}_d := \{a_1 w_1 + \cdots + a_{d+1} w_{d+1} : \ a_1, \ldots, a_{d+1} \in \mathbb{Z}^{d+1}\}, \tag{2.3}$$

and by $\mathcal{S}_\mathbf{k} \subset \mathcal{L}_d$ the sublattice

$$\mathcal{S}_\mathbf{k} := \left\{ a_1 w_1 + \cdots + a_{d+1} w_{d+1} : \begin{array}{l} (a_1, \ldots, a_{d+1}) = (b_1, \ldots, b_{d+1}) M_\mathbf{k} \\ \text{for some } b_1, \ldots, b_{d+1} \in \mathbb{Z} \end{array} \right\}. \tag{2.4}$$

That is, $\mathcal{S}_\mathbf{k}$ is the set of linear combinations of $w_1, \ldots, w_{d+1}$ whose coefficient vector $(a_1, \ldots, a_{d+1})$ is an integer linear combination of the rows of $M_\mathbf{k}$.

We say that $\mathbf{x}, \mathbf{y} \in \mathrm{Vert}(\widetilde{G}_d)$ are **k**-*equivalent*, in which case we write $\mathbf{x} \sim_\mathbf{k} \mathbf{y}$, if

$$\mathbf{y} = \mathbf{x} + v \text{ for some } v \in \mathcal{S}_\mathbf{k}. \tag{2.5}$$

Two edges of $\widetilde{G}_d$ are **k**-*equivalent* if one is a translation of the other by a vector in $\mathcal{S}_\mathbf{k}$.

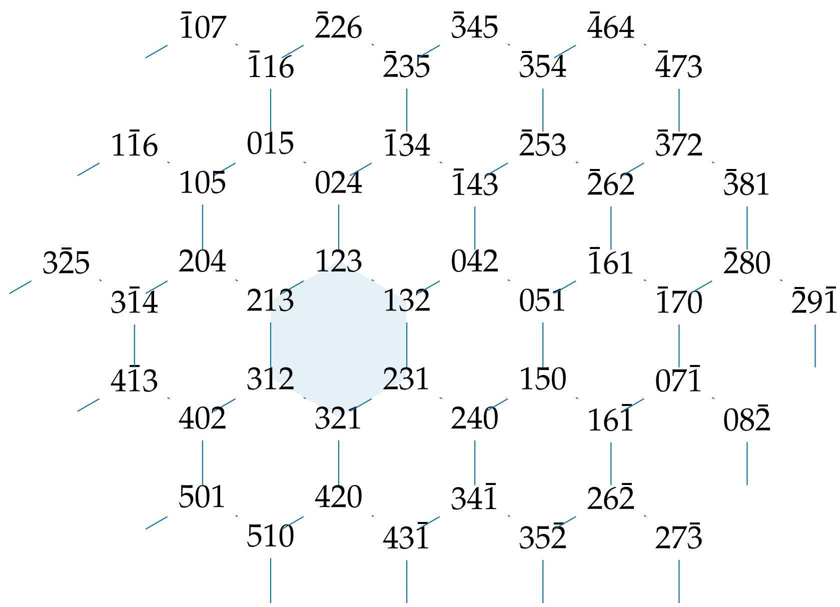


Figure 5: The graph $\widetilde{G}_2$ of the permutahedral tiling for $d = 2$. Commas and parenthesis are omited for simplicity. An overlined number $\bar{k}$ represent the negative number $-k$. For instance, $\bar{1}43$ represents the vertex $(-1, 4, 3)$.

Definition 2.1 (Generalized Heawood graph)**.** Let $\mathbf{k} = (k_1, \ldots, k_{d+1}) \in \mathbb{N}^{d+1}$ be a sequence of positive integers for some $d \geq 2$. The *Heawood graph* $H_{\mathbf{k}}$ is the graph whose vertices and edges are the $\mathbf{k}$-equivalent classes of vertices and edges of $\widetilde{G}_d$, respectively. In other words, $H_{\mathbf{k}}$ is the graph obtained by identifying vertices and edges of $\widetilde{G}_d$ up to translation by vectors in $\mathcal{S}_{\mathbf{k}}$.

Example 2.2 (Classical Heawood graph)**.** The classical Heawood graph is obtained when $d = 2$ and $\mathbf{k} = (1, 1, 1)$, and is illustrated in Figure 6. The lattice $\mathcal{L}_2$ consists of integer linear combinations of the vectors $w_1 = (2, -1, -1)$, $w_2 = (-1, 2, -1)$, $w_3 = (-1, -1, 2)$.

The associated matrix is

$$M_{(1,1,1)} = \begin{pmatrix} 2 & -1 & \\ & 2 & -1 \\ -1 & & 2 \end{pmatrix}$$

The sublattice $\mathcal{S}_{(1,1,1)}$ consists of integer linear combinations of the rows of this matrix, when considered as vectors of coefficients of the w_i's, i.e. integer linear combinations of the vectors $2w_1 - w_2$, $2w_2 - w_3$, $2w_3 - w_1$.

Figure 6 shows a tiling of the plane, where each fundamental tile consists of seven hexagons: one hexagon in the center together with its six surrounding hexagons. The barycenters of the central hexagons correspond exactly to elements of the sublattice $\mathcal{S}_{(1,1,1)}$. The equivalence relation $\cong_{\mathbf{k}}$ then identifies vertices and edges via translations that transform one fundamental tile into another.

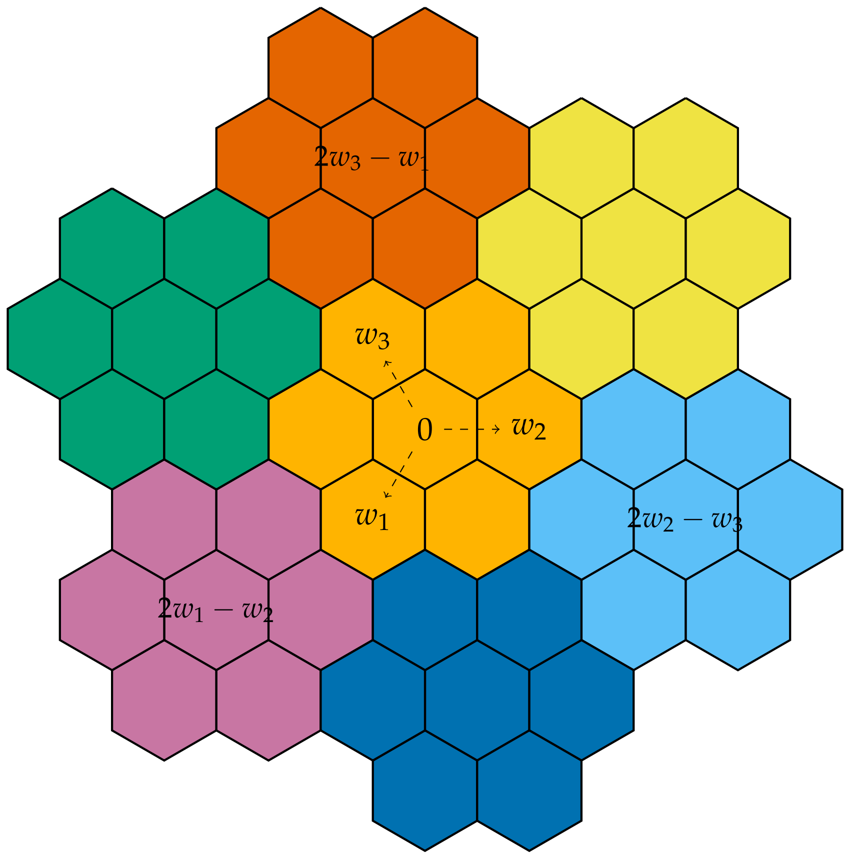


Figure 6: The classical Heawood graph $H_{(1,1,1)}$ as a quotient of the graph of the permutahedral tiling in dimension two.

Our aim is to prove some structural and enumerative properties of the generalized Heawood graph. Our first result is the following.

Theorem 2.3. *The generalized Heawood graph $H_{\mathbf{k}}$ is a vertex-transitive graph with $d!D_{\mathbf{k}}$ many vertices and $\frac{(d+1)!}{2}D_{\mathbf{k}}$ many edges, where*

$$D_{\mathbf{k}} = \det M_{\mathbf{k}} = \prod(k_i + 1) - \prod k_i. \tag{2.6}$$

Similarly to the classical case, the generalized Heawood graph is the dual graph of a triangulated torus, for which a simple combinatorial formula for its number of faces can be explicitly given.

We denote by $\left\{ {n \atop k} \right\}$ the Stirling number of the second kind, which counts the number of ways to partition a set of n objects into k non-empty subsets. These numbers can be explicitly calculated as

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} i^n. \tag{2.7}$$

Theorem 2.4. *The generalized Heawood graph $H_{\mathbf{k}}$ is the dual graph of a triangulation of a d-dimensional torus with f-vector $(f_0, f_1, \ldots, f_d)$ determined by*

$$f_i = i! \left\{ {d+1 \atop i+1} \right\} D_{\mathbf{k}}. \tag{2.8}$$

In particular,

$$f_0 = D_{\mathbf{k}}, \qquad f_d = d!\, D_{\mathbf{k}}, \qquad f_{d-1} = \frac{(d+1)!}{2} D_{\mathbf{k}}. \tag{2.9}$$

Table 1 shows the factor $c(i,d) := f_i / D_{\mathbf{k}}$ for some small values.

$d \backslash i$	0	1	2	3	4	5
2	1	3	2			
3	1	7	12	6		
4	1	15	50	60	24	
5	1	31	180	390	360	120

Table 1: The factor $c(i,d)$ for some small values of i and d.

Example 2.5 ($d = 2$)**.** We consider the classical Heawood graph, when $\mathbf{k} = (1,1,1)$. The factor $D_{(1,1,1)} = 2^3 - 1^3 = 7$ counts the number of hexagons in Figure 2a. The f-vector of its dual 2-dimensional triangulated torus is

$$(1 \cdot 7, 3 \cdot 7, 2 \cdot 7) = (7, 21, 14).$$

Interpreting this in the graph setting, we have 7 hexagons, 21 edges, and 14 vertices.

When $d = 2$, with a general $\mathbf{k}$, we have $D_{\mathbf{k}}$ many hexagons, $3D_{\mathbf{k}}$ many edges, and $2D_{\mathbf{k}}$ many vertices. Table 2 shows these numbers for all the examples in Figure 2.

3 The affine arrangement and the permutahedral tiling

In order to prove these results, it is useful to build on the connection with permutahedral tilings and their dual affine arrangements. We consider the collection of affine hyperplanes

$$H_{ij}^k = \{\mathbf{x} \in \mathbb{R}^{d+1} : \ x_j - x_i = k\}$$

$\mathbf{k}$ \ i	0	1	2
$(1,1,1)$	$1 \cdot 7$	$3 \cdot 7$	$2 \cdot 7$
$(2,2,2)$	$1 \cdot 19$	$3 \cdot 19$	$2 \cdot 19$
$(3,1,2)$	$1 \cdot 18$	$3 \cdot 18$	$2 \cdot 18$

Table 2: Number of hexagons, edges, and vertices for the Heawood graphs in Figure 2.

for $1 \leq i < j \leq d+1$ and $k \in \mathbb{Z}$.

The *affine Coxeter arrangement* $\widetilde{\mathcal{H}}_d$ of type $\widetilde{A}_d$ is the restriction of this arrangement to the hyperplane $V = \{\mathbf{x} \in \mathbb{R}^{d+1} : x_1 + \cdots + x_{d+1} = 0\}$. For $d = 2$, this is the arrangement of affine hyperplanes of a triangular lattice, which is illustrated on the left of Figure 7.

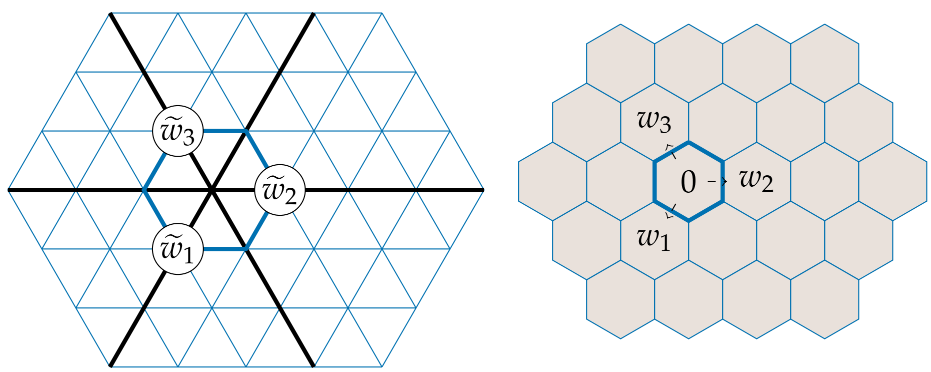


Figure 7: A finite piece of the simplicial complex $\widetilde{\mathcal{C}}_2$ of the affine Coxeter arrangement of type $\widetilde{A}_2$ (left). A finite piece of its dual tiling of space by permutahedra $\mathcal{PT}_2$ (right).

In general, the arrangement $\widetilde{\mathcal{H}}_d$ decomposes the space V into an infinite number of simplices, giving rise to an infinite simplicial complex that we denote by $\widetilde{\mathcal{C}}_d$. The vertices of this complex are the elements of

$$\widetilde{\mathcal{L}}_d := \{\mathbf{x} \in V : \ x_j - x_i \in \mathbb{Z}, \text{ for all } 1 \leq i < j \leq d+1\}. \tag{3.1}$$

This set is a lattice, which is known as the *weight lattice* of type A_d. It is generated by integer linear combinations of the vectors $\widetilde{w}_1, \ldots, \widetilde{w}_{d+1}$ determined by $(d+1)\widetilde{w}_i = w_i$, as defined in (2.2). They satisfy the relation

$$\widetilde{w}_1 + \cdots + \widetilde{w}_{d+1} = 0. \tag{3.2}$$

For $d = 2$, the dual of the triangular lattice is the hexagonal lattice, which is illustrated on the right of Figure 7. This generalizes to higher dimensions, where the dual of the complex $\widetilde{\mathcal{C}}_d$ is a combinatorial structure known as the permutahedral tiling.

The *permutahedron* Perm_d is the convex hull of all permutations of $[d+1]$:

$$\mathrm{Perm}_d = \mathrm{conv}\left\{(i_1,\ldots,i_{d+1}) : \text{ for } \{i_1,\ldots,i_{d+1}\} = [d+1]\right\} \subseteq \mathbb{R}^{d+1}. \tag{3.3}$$

The *permutahedral tiling* $\mathcal{PT}_d$ is the infinite tiling of the affine subspace

$$\{\mathbf{x} \in \mathbb{R}^{d+1} : \ x_1 + \cdots + x_{d+1} = 1 + \cdots + (d+1)\} \tag{3.4}$$

whose tiles are translates $\mathrm{Perm}_d + v$ of the permutahedron, for $v \in \mathcal{L}_d$. An example for $d = 2$ is shown on the right of Figure 7.

4 The triangulated torus and the Heawood complex

4.1 The triangulated torus

We consider the *sublattice* $\widetilde{\mathcal{S}}_{\mathbf{k}} \subset \widetilde{\mathcal{L}}_d$ defined by

$$\widetilde{\mathcal{S}}_{\mathbf{k}} := \left\{ a_1\widetilde{w}_1 + \cdots + a_{d+1}\widetilde{w}_{d+1} : \begin{array}{l} (a_1,\ldots,a_{d+1}) = (b_1,\ldots,b_{d+1})M_{\mathbf{k}} \\ \text{for some } b_1,\ldots,b_{d+1} \in \mathbb{Z} \end{array} \right\}. \tag{4.1}$$

Its elements are integer linear combinations of the vectors $\widetilde{w}_1,\ldots,\widetilde{w}_{d+1}$, whose vector of coefficients is an integer linear combination of the rows of the matrix $M_{\mathbf{k}}$.

We say that two faces $F, F' \in \widetilde{\mathcal{C}}_d$ are ***k**-equivalent*, and write $F \sim_{\mathbf{k}} F'$, if $F' = F + v$ for some $v \in \widetilde{\mathcal{S}}_{\mathbf{k}}$. That is, the face F' is a translation of F by $v \in \widetilde{\mathcal{S}}_d$.

Definition 4.1 (The torus)**.** The *quotient complex* $\mathcal{T}_{\mathbf{k}} = \widetilde{\mathcal{C}}_d / \widetilde{\mathcal{S}}_{\mathbf{k}}$ is the simplicial complex of $\mathbf{k}$-equivalent classes of faces of $\widetilde{\mathcal{C}}_d$. In other words, $\mathcal{T}_{\mathbf{k}}$ is the simplicial complex of faces of $\widetilde{\mathcal{C}}_d$ up to translation by vectors in $\widetilde{\mathcal{S}}_{\mathbf{k}}$.

We define the *fundamental vectors* with respect to $\mathbf{k}$ as the elements of the set

$$\widetilde{F}_{\mathbf{k}} = \left\{ a_1\widetilde{w}_1 + \cdots + a_{d+1}\widetilde{w}_{d+1} : \begin{array}{c} 0 \leq a_i \leq k_i \in \mathbb{Z}^{d+1} \\ \text{at least one } a_i = 0 \end{array} \right\}. \tag{4.2}$$

Lemma 4.2. *The quotient $\widetilde{\mathcal{L}}_d / \widetilde{\mathcal{S}}_{\mathbf{k}}$ is finite. Its cardinality is $\det M_{\mathbf{k}} = \prod(k_i+1) - \prod k_i$. The fundamental vectors in $\widetilde{F}_{\mathbf{k}}$ are element representatives of the classes of $\widetilde{\mathcal{L}}_d / \widetilde{\mathcal{S}}_{\mathbf{k}}$.*

Proposition 4.3. *$\mathcal{T}_{\mathbf{k}}$ is a triangulated d-dimensional torus on $D_{\mathbf{k}} = \det M_{\mathbf{k}}$ many vertices.*

The proof of this proposition is based on a parallelepiped domain of $\mathcal{T}_{\mathbf{k}}$, which we explain in the longer version of this manuscript [2], see the first illustration in Figure 8.

We also provide a permutahedral domain, see the second illustration in Figure 8, which leads to the following independent result.

Proposition 4.4. *The permutahedron Perm_d with opposite facets identified by translation is a topological d-dimensional torus.*

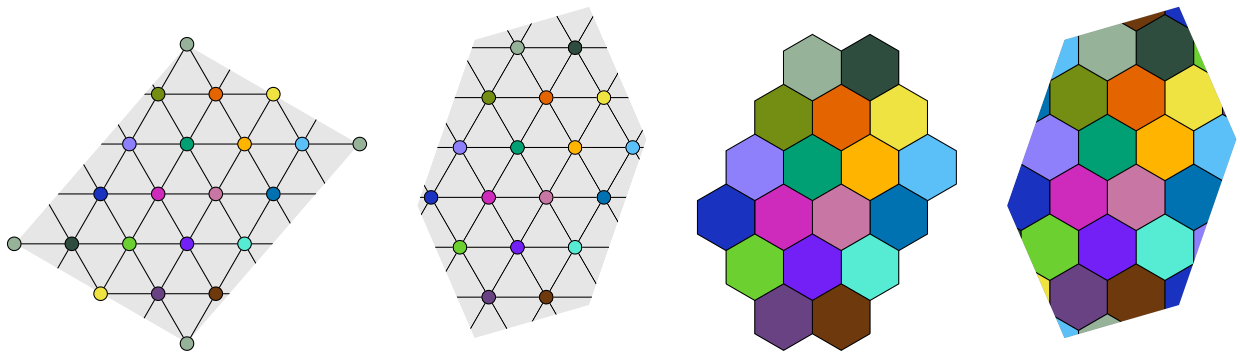


Figure 8: The parallelepiped domain and the permutahedron domain of $\mathcal{T}_{(3,1,2)}$, and the fundamental tile and the permutahedron domain of $\mathcal{HC}_{(3,1,2)}$.

4.2 The Heawood complex

We say that two faces $\mathbf{B}, \mathbf{B}'$ of the permutahedral tiling $\mathcal{PT}_d$ are $\mathbf{k}$-*equivalent*, and write $\mathbf{B} \sim_{\mathbf{k}} \mathbf{B}'$, if $\mathbf{B}' = \mathbf{B} + v$ for some $v \in \mathcal{S}_{\mathbf{k}}$. That is, the face $\mathbf{B}'$ is a translation of $\mathbf{B}$ by a vector $v \in \mathcal{S}_{\mathbf{k}}$.

Definition 4.5 (The Heawood complex)**.** The *Heawood complex* $\mathcal{HC}_{\mathbf{k}} = \mathcal{PT}_d / \mathcal{S}_{\mathbf{k}}$ is the polytopal complex of $\mathbf{k}$-equivalent classes of faces of $\mathcal{PT}_d$. That is, $\mathcal{HC}_{\mathbf{k}}$ is the polytopal complex of faces of $\mathcal{PT}_d$ up to translation by vectors in $\mathcal{S}_{\mathbf{k}}$.

We define

$$F_{\mathbf{k}} = \left\{ a_1 w_1 + \cdots + a_{d+1} w_{d+1} : \begin{array}{c} 0 \leq a_i \leq k_i \in \mathbb{Z}^{d+1} \\ \text{at least one } a_i = 0 \end{array} \right\}. \tag{4.3}$$

Since $w_i = (d+1)\widetilde{w}_i$, then the vectors in $F_{\mathbf{k}}$ are just the fundamental vectors in $\widetilde{F}_{\mathbf{k}}$, dilated by a factor of $d+1$. The *fundamental tile* $P_{\mathbf{k}}$ is the union of the permutahedra of the form $\mathrm{Perm}_d + v$ with $v \in F_{\mathbf{k}}$. An example of the fundamental tile $P_{(1,1,1)}$, including six translations of it, is illustrated in Figure 6.

In general, translations of the fundamental tile $P_{\mathbf{k}}$ by elements of the sublattice $\mathcal{S}_{\mathbf{k}}$ tile space. Thus, the Heawood complex is the complex of faces of this fundamental tile, where the boundary is identified according to how the translations glue together, see also Figure 4.

Proposition 4.6. *The following hold:*

1. *The Heawood graph* $H_{\mathbf{k}}$ *is the edge graph of the Heawood complex* $\mathcal{HC}_{\mathbf{k}}$.

2. *The Heawood complex and the torus are dual complexes:* $\mathcal{HC}_{\mathbf{k}} \cong \mathcal{T}^*_{\mathbf{k}}$.

3. *The Heawood graph $H_{\mathbf{k}}$ is the dual graph of the torus $\mathcal{T}_{\mathbf{k}}$.*

This, together with Proposition 4.3, finishes part of the proof of our main Theorem 2.4. The proof of the remaining enumerative part can be found in the longer version of this manuscript [2]. There, we also describe the automorphism groups of the triangulated torus $\mathcal{T}_{\mathbf{k}}$ and the generalized Heawood graph $H_{\mathbf{k}}$, and discuss about potential generalizations including the hyperbolic setting.

In view of Proposition 4.4, we finish with the following open question.

Question 4.7. What is the topology of other families of polytopes with opposite facets identified by translation?

Natural families that fit into this context are Permutahedra arising from finite Coxeter groups, Postnikov's generalized permutahedra obtained by removing some pairs of opposite facets of the classical permutahedron, and Zonotopes in general. A small example of the first and the last is an octagon. Identifying opposite edges of an octagon leads to a topological 2-hole torus.

References

[1] K. Appel and W. Haken. "The solution of the four-color-map problem". *Sci. Amer.* **237**.4 (1977), pp. 108–121, 152. DOI.

[2] C. Ceballos and J. Doolittle. "Generalized Heawood Graphs and Triangulations of Tori". arXiv:2307.11859.

[3] P. J. Heawood. "Map Colour Theorem". *Quant. J. Math.* **24** (1890), pp. 332–338.

[4] A. B. Kempe. "On the Geographical Problem of the Four Colours". *Amer. J. Math.* **2**.3 (1879), pp. 193–200. DOI.

[5] A. B. Kempe. "How to Colour a Map with Four Colours". *Nature* **21** (1880), 399–400. DOI.

[6] W. Kühnel and G. Lassmann. "The rhombidodecahedral tessellation of 3-space and a particular 15-vertex triangulation of the 3-dimensional torus". *Manuscripta Math* **49** (1984), pp. 61–77. DOI.

[7] W. Kühnel and G. Lassmann. "Combinatorial d-tori with a large symmetry group". *Discrete Comput Geom* **3** (1988), pp. 169–176. DOI.

[8] J. Leech. "Seven region maps on a torus". *Math. Gaz.* **39** (1955), pp. 102–105. DOI.

[9] F. H. Lutz. "Triangulated Manifolds with Few Vertices: Combinatorial Manifolds". arXiv: math/0506372.

[10] G. Ringel and J. W. T. Youngs. "Solution of the Heawood map-coloring problem". *Proceedings of the National Academy of Sciences* **60**.2 (1968), pp. 438–445.