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Kromatic quasisymmetric functions

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Abstract. We provide a construction for the kromatic symmetric function $\overline{X}_G$ of a graph introduced by Crew, Pechenik, and Spirkl using combinatorial (linearly compact) Hopf algebras. As an application, we show that $\overline{X}_G$ has a positive expansion into multifundamental quasisymmetric functions. We also study two related quasisymmetric q-analogues of $\overline{X}_G$, which are K-theoretic generalizations of the quasisymmetric chromatic function of Shareshian and Wachs. We classify exactly when one of these analogues is symmetric. For the other, we derive a positive expansion into symmetric Grothendieck functions for graphs G that are natural unit interval orders.

Keywords: Chromatic symmetric functions, combinatorial Hopf algebras, linearly compact modules, multifundamental quasisymmetric functions

1 Introduction

The purpose of this note is to re-examine the algebraic origins of the *kromatic symmetric function* of a graph that was recently introduced by Crew, Pechenik, and Spirkl [3], and to study two quasisymmetric analogues of this power series.

Let $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{P} = \{1, 2, 3, \dots\}$, and $[n] = \{1, 2, \dots, n\}$ for $n \in \mathbb{N}$. All graphs are undirected by default, and are assumed to be simple with a finite set of vertices. We do not distinguish between isomorphic graphs.

If G is any graph then we write $V(G)$ for its set of vertices and $E(G)$ its set of edges. A *proper coloring* of G is a map $\kappa : V(G) \to \mathbb{P}$ with $\kappa(u) \neq \kappa(v)$ for all $\{u, v\} \in E(G)$. For maps $\kappa : V \to \mathbb{P}$ let $x^\kappa = \prod_{i \in V} x_{\kappa(i)}$ where $x_1, x_2, \dots$ are commuting variables.

Definition 1.1 (Stanley [12])**.** The *chromatic symmetric function* of G is the symmetric power series $X_G := \sum_\kappa x^\kappa$ where the sum is over all proper colorings κ of G.

Example 1.2. If $G = K_n$ is the *complete graph* with $V(G) = [n]$ then $X_G = n!e_n$ for the *elementary symmetric function* $e_n := \sum_{1 \le i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$.

A poset is *$(3+1)$-free* if it does not contain a 3-element chain $a < b < c$ whose elements are all incomparable to some fourth element d. The *Stanley–Stembridge conjecture* [13] proposes that if G is the incomparability graph of a $(3+1)$-free poset then X_G has a

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positive expansion into elementary symmetric functions. This conjecture has several refinements and generalizations, and has been resolved in a number of interesting special cases, but remains open in general.

Let G be an *ordered graph*, that is, a graph with a total order $<$ on its vertex set $V(G)$. An *ascent* (resp., *descent*) of a map $\kappa : V(G) \to \mathbb{P}$ is an edge $\{u, v\} \in E(G)$ with $u < v$ and $\kappa(u) < \kappa(v)$ (resp., $\kappa(u) > \kappa(v)$). Let $\mathrm{asc}_G(\kappa)$ and $\mathrm{des}_G(\kappa)$ be the number of ascents and descents. Shareshian and Wachs [10] introduced the following q-analogue of X_G:

Definition 1.3 ([10])**.** The *chromatic quasisymmetric function* of an ordered graph G is $X_G(q) = \sum_\kappa q^{\mathrm{asc}_G(\kappa)} x^\kappa \in \mathbb{N}[q][\![x_1, x_2, \dots]\!]$ where the sum is over all proper colorings.

Example 1.4. If $G = K_n$ then $X_G(q) = [n]_q! e_n$ where $[i]_q = \frac{1-q^i}{1-q}$ and $[n]_q! = \prod_{i=1}^n [i]_q$.

Let $\mathsf{Set}(\mathbb{P})$ be the set of finite nonempty subsets of $\mathbb{P}$. For a map $\kappa : V \to \mathsf{Set}(\mathbb{P})$ define $x^\kappa = \prod_{i \in V} \prod_{j \in \kappa(i)} x_j$. A *proper set-valued coloring* is a map $\kappa : V(G) \to \mathsf{Set}(\mathbb{P})$ with $\kappa(u) \cap \kappa(v) = \varnothing$ for all $\{u, v\} \in E(G)$. There is also a "K-theoretic" analogue of X_G:

Definition 1.5 (Crew, Pechenik, and Spirkl [3])**.** The *kromatic symmetric function* of a graph G is the sum $\overline{X}_G = \sum_\kappa x^\kappa \in \mathbb{Z}[\![x_1, x_2, \dots]\!]$ over all proper set-valued colorings of G.

Example 1.6. $\overline{X}_{K_n} = n! \sum_{r=n}^\infty \left\{ {r \atop n} \right\} e_r$ where $\left\{ {r \atop n} \right\}$ is the Stirling number of the second kind.

Remark 1.7. Given $\alpha : V \to \mathbb{N}$, let $\mathrm{Cl}_\alpha(V)$ be the set of pairs (v, i) with $v \in V$ and $i \in [\alpha(v)]$. If G is a graph and $\alpha : V(G) \to \mathbb{N}$ is any map, then the *α-clan graph* $\mathrm{Cl}_\alpha(G)$ has vertex set $\mathrm{Cl}_\alpha(V(G))$ and edges $\{(v, i), (w, j)\}$ whenever $\{v, w\} \in E(G)$ or both $v = w$ and $i \neq j$. As observed in [3], one has $\overline{X}_G = \sum_{\alpha : V(G) \to \mathbb{P}} \frac{1}{\alpha!} X_{\mathrm{Cl}_\alpha(G)}$ where $\alpha! := \prod_v \alpha(v)!$. Many properties of X_G extend to $\overline{X}_G$ via this identity, but some interesting features of $\overline{X}_G$ cannot be explained by this formula alone.

Our main results provide a natural construction for $\overline{X}_G$ using the theory of *combinatorial Hopf algebras*. This approach requires some care, as $\overline{X}_G$ is not a symmetric function of bounded degree. We explain things precisely in terms of *linearly compact Hopf algebras* after reviewing a similar, simpler construction of X_G in Section 2, following [1].

As an application of our approach, we show that $\overline{X}_G$ has a positive expansion into *multifundamental quasisymmetric functions*. We also study two related q-analogues of $\overline{X}_G$, which give K-theoretic generalizations of $X_G(q)$. We classify exactly when one of these analogues is symmetric. For the other, we extend a theorem of Crew, Pechenik, and Spirkl (also lifting a theorem of Shareshian and Wachs) to derive a positive expansion into symmetric Grothendieck functions for graphs G that are natural unit interval orders.

2 Background

Let $\mathbb{K}$ be an integral domain; in practice, one can assume this is $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}[q]$, or $\mathbb{Q}(q)$.

2.1 Hopf algebras

Write $\otimes = \otimes_{\mathbb{K}}$ for the tensor product over $\mathbb{K}$. A *$\mathbb{K}$-algebra* is a $\mathbb{K}$-module A with $\mathbb{K}$-linear product $\nabla : A \otimes A \to A$ and unit $\iota : \mathbb{K} \to A$ maps. Dually, a *$\mathbb{K}$-coalgebra* is a $\mathbb{K}$-module A with $\mathbb{K}$-linear coproduct $\Delta : A \to A \otimes A$ and counit $\epsilon : A \to \mathbb{K}$ maps. The (co)product and (co)unit maps must satisfy several associativity axioms; see [5, §1].

A $\mathbb{K}$-module A that is both a $\mathbb{K}$-algebra and a $\mathbb{K}$-coalgebra is a *$\mathbb{K}$-bialgebra* if the coproduct and counit maps are algebra morphisms. A bialgebra $A = \bigoplus_{n\in\mathbb{N}} A_n$ is *graded* if its (co)product and (co)unit are graded maps; in this case A is *connected* if $A_0 = \mathbb{K}$.

Let $\mathrm{End}(A)$ denote the set of $\mathbb{K}$-linear maps $A \to A$. This set is a $\mathbb{K}$-algebra for the product $f * g := \nabla \circ (f \otimes g) \circ \Delta$. The unit of this *convolution algebra* is the composition $\iota \circ \epsilon$ of the unit and counit of A. A bialgebra A is a *Hopf algebra* if $\mathrm{id} : A \to A$ has a two-sided inverse $\mathsf{S} : A \to A$ in $\mathrm{End}(A)$. When it exists, we call S the *antipode* of A.

Example 2.1. Let Graphs_n for $n \in \mathbb{N}$ be the free $\mathbb{K}$-module spanned by all isomorphism classes of undirected graphs with n vertices, and set $\mathsf{Graphs} = \bigoplus_{n\in\mathbb{N}} \mathsf{Graphs}_n$. One views Graphs as a connected, graded Hopf algebra with product $\nabla(G \otimes H) = G \sqcup H$ and coproduct $\Delta(G) = \sum_{S\sqcup T=V(G)} G|_S \otimes G|_T$ for graphs G and H, where $\sqcup$ denotes disjoint union and $G|_S$ denotes the subgraph of G induced on S.

A *lower set* in a directed acyclic graph $D = (V, E)$ is a set $S \subseteq V$ such that if a directed path connects $v \in V$ to $s \in S$ then $v \in S$. An *upper set* is the complement of a lower set.

Example 2.2. Let DAGs_n for $n \in \mathbb{N}$ be the free $\mathbb{K}$-module spanned by all isomorphism classes of directed acyclic graphs with n vertices, and set $\mathsf{DAGs} = \bigoplus_{n\in\mathbb{N}} \mathsf{DAGs}_n$. One views DAGs as a connected, graded Hopf algebra with product $\nabla(C \otimes D) = C \sqcup D$ and coproduct $\Delta(D) = \sum D|_S \otimes D|_T$ for directed acyclic graphs graphs C and D, where the sum is over all disjoint unions $S \sqcup T = V(D)$ with S a lower set and T an upper set.

A *labeled poset* is a pair (D, γ) consisting of a directed acyclic graph D and an injective map $\gamma : V(D) \to \mathbb{Z}$. We consider $(D, \gamma) = (D', \gamma')$ if there is an isomorphism $D \xrightarrow{\sim} D'$, written $v \mapsto v'$, such that $\gamma(u) - \gamma(v)$ and $\gamma'(u') - \gamma'(v')$ have the same sign for all edges $u \to v \in E(D)$. If (D_1, γ_1) and (D_2, γ_2) are labeled posets then let $\gamma_1 \sqcup \gamma_2 : V(D_1 \sqcup D_2) \to \mathbb{Z}$ be any injective map such that $(\gamma_1 \sqcup \gamma_2)(u) - (\gamma_1 \sqcup \gamma_2)(v)$ has the same sign as $\gamma_i(u) - \gamma_i(v)$ for all $u, v \in V(D_i)$.

Example 2.3. Let $\mathsf{LPosets}_n$ be the free $\mathbb{K}$-module spanned by all labeled poset with n vertices, and set $\mathsf{LPosets} = \bigoplus_{n\in\mathbb{N}} \mathsf{LPosets}_n$. This is a connected, graded Hopf algebra with product $\nabla((D_1, \gamma_1) \otimes (D_2, \gamma_2)) = (D_1 \sqcup D_2, \gamma_1 \sqcup \gamma_2)$ and coproduct $\Delta((D, \gamma)) = \sum (D|_S, \gamma|_S) \otimes (D|_T, \gamma|_T)$ where the sum is over all disjoint decompositions $S \sqcup T = V(D)$ with S a lower set and T an upper set.

A *(strict) composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ is a finite sequence of positive integers, called its *parts*. We say that α is a composition of $|\alpha| := \sum_i \alpha_i \in \mathbb{N}$.

Example 2.4. Fix a composition α and let $x_1, x_2, \dots$ be a countable sequence of commuting variables. The *monomial quasisymmetric function* of α is the power series $M_\alpha = \sum_{1\leq i_1<i_2<\cdots<i_l} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_l}^{\alpha_l}$. Let $\mathsf{QSym} = \mathbb{K}\text{-span}\{M_\alpha : \alpha \text{ any composition}\}$ be the ring of quasisymmetric functions of bounded degree. This ring is a graded connected Hopf algebras for the coproduct $\Delta(M_\alpha) = \sum_{\alpha=\alpha'\alpha''} M_{\alpha'} \otimes M_{\alpha''}$ where $\alpha'\alpha''$ denotes concatenation of compositions, and the counit that acts on power series by setting $x_1 = x_2 = \cdots = 0$.

A *partition* is a composition sorted into decreasing order. We write $\lambda = 1^{m_1}2^{m_2}\cdots$ to denote the partition with exactly m_i parts equal to i.

Example 2.5. The *elementary symmetric function* of a partition λ is the product $e_\lambda := e_{\lambda_1}e_{\lambda_2}\cdots$ where $e_n := M_{1^n}$. These power series are a basis for the Hopf subalgebra $\mathsf{Sym} \subset \mathsf{QSym}$ of symmetric functions of bounded degree.

2.2 Combinatorial Hopf algebras

Following [1], a *combinatorial Hopf algebra* (H, ζ) is a graded, connected Hopf algebra H of finite graded dimension with an algebra homomorphism $\zeta : H \to \mathbb{K}$.

Example 2.6. The pair $(\mathsf{QSym}, \zeta_{\mathsf{Q}})$ is an example of a combinatorial Hopf algebra, where $\zeta_{\mathsf{Q}} : \mathsf{QSym} \to \mathbb{K}$ is the map $\zeta_{\mathsf{Q}}(f) = f(1,0,0,\dots)$, which sends $M_{(n)} \mapsto 1$ and $M_\alpha \mapsto 0$ for all α with at least two parts.

For a graph G define $\zeta_{\mathsf{Graphs}}(G) = 0^{|E(G)|}$ where throughout we interpret $0^0 := 1$. For a directed acyclic graph D likewise set $\zeta_{\mathsf{DAGs}}(D) = 0^{|E(D)|}$ for each directed acyclic graph D. These formulas extend to linear maps on Graphs and DAGs. Finally let $\zeta_{\mathsf{LPosets}} : \mathsf{LPosets} \to \mathbb{K}$ be the linear map with $\zeta_{\mathsf{LPosets}}((D,\gamma)) = 1$ if $\gamma(u) < \gamma(v)$ for all edges $u \to v \in E(D)$ with $\zeta_{\mathsf{LPosets}}((D,\gamma)) = 0$ otherwise.

Example 2.7. The pairs $(\mathsf{Graphs}, \zeta_{\mathsf{Graphs}})$, $(\mathsf{DAGs}, \zeta_{\mathsf{DAGs}})$, and $(\mathsf{LPosets}, \zeta_{\mathsf{LPosets}})$ are all combinatorial Hopf algebras.

A morphism $\Psi : (H,\zeta) \to (H',\zeta')$ is a graded Hopf algebra morphism $\Psi : H \to H'$ with $\zeta = \zeta' \circ \Psi$. Results in [1] show that there exists a unique morphism from any combinatorial Hopf algebra to $(\mathsf{QSym}, \zeta_{\mathsf{Q}})$. Moreover, the image of Ψ is contained in the Hopf subalgebra $\mathsf{Sym} \subset \mathsf{QSym}$ if H is cocommutative. There is an explicit formula for this morphism in [1], which translates to the following maps for our examples above.

For a graph G, let $\mathsf{AO}(G)$ be its set of acyclic orientations. For a directed acyclic graph D, let $(D, \gamma^{\mathsf{op}})$ be the labeled poset with $\gamma^{\mathsf{op}}(u) > \gamma^{\mathsf{op}}(v)$ for all edges $u \to v \in E(D)$. Also set $\Gamma(D) = \sum_\kappa x^\kappa \in \mathbb{N}[\![x_1, x_2, \dots]\!]$ where the sum is over all maps $\kappa : V(D) \to \mathbb{P}$ with $\kappa(u) < \kappa(v)$ whenever $u \to v \in E(D)$.

More generally, for a labeled poset (D,γ) define $\Gamma(D,\gamma) = \sum_\kappa x^\kappa$ where the sum is over all maps $\kappa : V(D) \to \mathbb{P}$ with $\kappa(u) \leq \kappa(v)$ whenever $u \to v \in E(D)$ and $\gamma(u) < \gamma(v)$, and with $\kappa(u) < \kappa(v)$ whenever $u \to v \in E(D)$ and $\gamma(u) > \gamma(v)$. Such maps κ are called *P-partitions* for $P = (D, \gamma)$ [11].

Proposition 2.8. There is a commutative diagram of combinatorial Hopf algebras

$$\begin{array}{ccccc} (\mathsf{Graphs}, \zeta_{\mathsf{Graphs}}) & \hookrightarrow & (\mathsf{DAGs}, \zeta_{\mathsf{DAGs}}) & \hookrightarrow & (\mathsf{LPosets}, \zeta_{\mathsf{LPosets}}) \\ & \searrow & \downarrow & \swarrow & \\ & & (\mathsf{QSym}, \zeta_{\mathsf{Q}}) & & \end{array}$$

in which the horizontal maps send $G \mapsto \sum_{D \in \mathsf{AO}(G)} D$ and $D \mapsto (D, \gamma^{\mathsf{op}})$, and the QSym-valued maps send $G \mapsto X_G$, $D \mapsto \Gamma(D)$, and $(D, \gamma) \mapsto \Gamma(D, \gamma)$, respectively.

3 K-theoretic generalizations

We now explain how the results in the previous can be extended "K-theoretically" to construct interesting quasisymmetric functions of unbounded degree, including $\overline{X}_G$. This requires a brief discussion of monoidal structures on *linearly compact modules*.

3.1 Linearly compact modules

Let A and B be $\mathbb{K}$-modules with a $\mathbb{K}$-bilinear form $\langle \cdot, \cdot \rangle : A \times B \to \mathbb{K}$. Assume A is free and $\langle \cdot, \cdot \rangle$ is *nondegenerate* in the sense that $b \mapsto \langle \cdot, b \rangle$ is a bijection $B \to \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$.

Fix a basis $\{a_i\}_{i \in I}$ for A. For each $i \in I$, there exists a unique $b_i \in B$ with $\langle a_i, b_j \rangle = \delta_{ij}$ for all $i, j \in I$, and we identify $b \in B$ with the formal linear combination $\sum_{i \in I} \langle a_i, b \rangle b_i$. We call $\{b_i\}_{i \in I}$ a *pseudobasis* for B.

We give $\mathbb{K}$ the discrete topology. Then the *linearly compact topology* [4, §I.2] on B is the coarsest topology in which the maps $\langle a_i, \cdot \rangle : B \to \mathbb{K}$ are all continuous. This topology depends on $\langle \cdot, \cdot \rangle$ but not on the choice of basis for A. For a basis of open sets in the linearly compact topology, see [9, Eq. (3.1)].

Definition 3.1. A *linearly compact* (or *LC* for short) $\mathbb{K}$-module is a $\mathbb{K}$-module B with a nondegenerate bilinear form $A \times B \to \mathbb{K}$ for some free $\mathbb{K}$-module A, given the linearly compact topology; in this case we say that B is the *dual* of A. Morphisms between such modules are continuous $\mathbb{K}$-linear maps.

Let B and B' be linearly compact $\mathbb{K}$-modules dual to free $\mathbb{K}$-modules A and A'. Let $\langle \cdot, \cdot \rangle$ denote both of the associated forms. Every linear map $\phi : A' \to A$ has a unique adjoint $\psi : B \to B'$ such that $\langle \phi(a), b \rangle = \langle a, \psi(b) \rangle$. A linear map $B \to B'$ is continuous when it is the adjoint of some linear map $A' \to A$.

Definition 3.2. Define $B \mathbin{\overline{\otimes}} B' := \mathrm{Hom}_{\mathbb{K}}(A \otimes A', \mathbb{K})$ and give this the LC-topology from the pairing $(A \otimes A') \times \mathrm{Hom}_{\mathbb{K}}(A \otimes A', \mathbb{K}) \to \mathbb{K}$.

If $\{b_i\}_{i\in I}$ and $\{b'_j\}_{j\in J}$ are pseudobases for B and B', then we can realize the *completed tensor product* $B \mathbin{\overline{\otimes}} B'$ concretely as the linearly compact $\mathbb{K}$-module with the set of tensors $\{b_i \otimes b'_j\}_{(i,j)\in I\times J}$ as a pseudobasis.

Suppose $\nabla : B \mathbin{\overline{\otimes}} B \to B$ and $\iota : B \to \mathbb{K}$ are continuous linear maps which are the adjoints of linear maps $\epsilon : \mathbb{K} \to A$ and $\Delta : A \to A \otimes A$. We say that (B, ∇, ι) is an *LC-algebra* if (A, Δ, ϵ) is a $\mathbb{K}$-coalgebra. Similarly, we say that $\Delta : B \to B \mathbin{\overline{\otimes}} B$ and $\epsilon : B \to \mathbb{K}$ make B into an *LC-coalgebra* if Δ and ϵ are the adjoints of the product and unit maps of a $\mathbb{K}$-algebra on A. We define *LC-bialgebras* and *LC-Hopf algebras* analogously; see [9]. If B is an LC-Hopf algebra then its *antipode* is the adjoint of the antipode of A.

3.2 Combinatorial LC-Hopf algebras

Following [9], we define a *combinatorial LC-Hopf algebra* to be a pair (H, ζ) consisting of an LC-Hopf algebra H with a continuous linear map $\zeta : H \to \mathbb{K}[\![t]\!]$ such that $\zeta(\cdot)|_{t\mapsto 0}$ is the counit of H. A morphism of combinatorial LC-Hopf algebras $\Psi : (H, \zeta) \to (H', \zeta')$ is a LC-Hopf algebra morphism $\Psi : H \to H'$ with $\zeta = \zeta' \circ \Psi$.

Example 3.3. Let $\mathfrak{m}\mathsf{QSym}$ be the set of all quasisymmetric power series in $\mathbb{K}[\![x_1, x_2, \dots,]\!]$ of possibly unbounded degree. The (co)product, (co)unit, and antipode QSym all extend to continuous $\mathbb{K}$-linear maps that make $\mathfrak{m}\mathsf{QSym}$ into an LC-Hopf algebra, with $\{M_\alpha\}$ as a pseudobasis. Then $(\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$ is a combinatorial LC-Hopf algebra when $\overline{\zeta}_{\mathsf{Q}}$ is the map $\overline{\zeta}_{\mathsf{Q}} : f \mapsto f(t, 0, 0, \dots)$.

The preceding example is an instance of a general construction. If A is a free $\mathbb{K}$-module with basis S, then its *completion* $\overline{A}$ is the set of arbitrary $\mathbb{K}$-linear combinations of S. We view $\overline{A}$ as a linearly compact $\mathbb{K}$-module with S as a pseudobasis, relative to the nondegenerate bilinear form $A \times \overline{A} \to \mathbb{K}$ making S orthonormal.

If (H, ζ) is a combinatorial Hopf algebra then then there is a unique way of extending its (co)unit and (co)product to continuous linear maps on $\overline{H}$. As the Hopf algebra $H = \bigoplus_{n\in\mathbb{N}}$ is graded, we can also extend $\zeta : H \to \mathbb{K}$ to a continuous linear map $\overline{\zeta} : \overline{H} \to \mathbb{K}[\![t]\!]$ by the formula $\overline{\zeta}(h) = \zeta(h)t^n$ for $n \in \mathbb{N}$ and $h \in H_n$.

Proposition 3.4. If (H, ζ) is combinatorial Hopf algebra then the extended structures just given make $(\overline{H}, \overline{\zeta})$ into a combinatorial LC-Hopf algebra, and the unique morphism $(H, \zeta) \to (\mathsf{QSym}, \zeta_{\mathsf{Q}})$ extends to a morphism $(\overline{H}, \overline{\zeta}) \to (\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$.

The pair $(\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$ is a final object in the category of combinatorial LC-Hopf algebras, meaning there is a unique morphism $(H, \zeta) \to (\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$ for any combinatorial LC-Hopf algebra. More specifically, if H has coproduct Δ, then define $\Delta^{(0)} = \mathrm{id}_H$ and $\Delta^{(k)} = (\Delta^{(k-1)} \mathbin{\overline{\otimes}} \mathrm{id}) \circ \Delta : H \to H^{\overline{\otimes}(k+1)}$ for $k \in \mathbb{P}$. For compositions $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, let $\zeta_\alpha : H \to \mathbb{K}$ be the map sending $h \in H$ to the coefficient of $t^{\alpha_1} \otimes t^{\alpha_2} \otimes \cdots \otimes t^{\alpha_k}$ in $\zeta^{\otimes k} \circ \Delta^{(k-1)}(h) \in \mathbb{K}[\![t]\!]$. When $\alpha = \varnothing$ is empty let $\zeta_\varnothing = \zeta(\cdot)|_{t\mapsto 0}$ be the counit of H.

Theorem 3.5 ([8]). If (H, ζ) is a combinatorial LC-Hopf algebra then the map $\Psi_{H,\zeta} : h \mapsto \sum_\alpha \zeta_\alpha(h) M_\alpha$ is the unique morphism $(H, \zeta) \to (\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$.

Let $\mathfrak{m}\mathsf{Sym}$ be the LC-Hopf subalgebra of symmetric functions in $\mathfrak{m}\mathsf{QSym}$. When H cocommutative, the morphism $\Psi_{H,\zeta}$ evidently has its image in $\mathfrak{m}\mathsf{Sym}$.

3.3 Set-valued P-partitions

For a directed acyclic graph D, let $\overline{\Gamma}(D) = \sum_\kappa x^\kappa$ where the sum is over all maps $\kappa : V(D) \to \mathsf{Set}(\mathbb{P})$ with $\kappa(u) \prec \kappa(v)$ whenever $u \to v \in E(D)$.

Example 3.6. If $D = (1 \to 2 \to 3 \to \cdots \to n)$ is an n-element chain then define $\bar{e}_n := \overline{\Gamma}(D) = \sum_{k=0}^{\infty} \binom{n-1+k}{n-1} e_{n+k}$. For each partition λ let $\bar{e}_\lambda := \bar{e}_{\lambda_1}\bar{e}_{\lambda_2}\cdots$. These functions are a pseudobasis for $\mathfrak{m}\mathsf{Sym}$.

For a labeled poset (D, γ) define $\overline{\Gamma}(D, \gamma) = \sum_\kappa x^\kappa$ where the sum is over all maps $\kappa : V(D) \to \mathsf{Set}(\mathbb{P})$ with $\kappa(u) \preceq \kappa(v)$ whenever $u \to v \in E(D)$ and $\gamma(u) < \gamma(v)$, and with $\kappa(u) \prec \kappa(v)$ whenever $u \to v \in E(D)$ and $\gamma(u) > \gamma(v)$. Such maps κ are called *set-valued P-partitions* for $P = (D, \gamma)$ in [7, 8].

Example 3.7. If $D = (1 \to 2 \to 3 \to \cdots \to n)$ is an n-element chain and S is the set of $i \in [n-1]$ with $\gamma(i) > \gamma(i+1)$ then the we define $\overline{L}_{n,S} := \overline{\Gamma}(D, \gamma)$. Following [7], the *multifundamental quasisymmetric function* of a composition α is defined by $\overline{L}_\alpha := \overline{L}_{n,S}$ where $n = |\alpha|$ and $S = I(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \dots\} \setminus \{n\}$. These power series form another pseudobasis for $\mathfrak{m}\mathsf{QSym}$ [7]. An element of $\mathfrak{m}\mathsf{QSym}$ is *multifundamental positive* if its expansion in this pseudobasis involves only nonnegative coefficients.

A *multilinear extension* of a directed acyclic graph D with n vertices is a sequence $w = (w_1, w_2, \dots, w_N)$ with $V(D) = \{w_1, w_2, \dots, w_N\}$ such that $i < j$ whenever $w_i \to w_j \in E(D)$, and $w_i \neq w_{i+1}$ for all $i \in [N-1]$. If $\mathcal{M}(D)$ is the set of all multilinear extensions of D and $\gamma : V(D) \to \mathbb{Z}$ is injective, then $\overline{\Gamma}(D, \gamma) = \sum_{w \in \mathcal{M}(D)} \overline{L}_{\ell(w), \mathrm{Des}(w,\gamma)}$ where $\mathrm{Des}(w, \gamma) := \{i \in [\ell(w) - 1] : \gamma(w_i) > \gamma(w_{i+1})\}$ for $w \in \mathcal{M}(D)$ [7].

3.4 Acyclic multi-orientations

Let G be a graph. An *acyclic multi-orientation* of G is an acyclic orientation of the α-clan graph $\mathrm{Cl}_\alpha(G)$ from Remark 1.7 for some $\alpha : V(G) \to \mathbb{P}$, such that for each $v \in V(G)$ both (a) if $i, j \in [\alpha(v)]$ have $i > j$ then $(v, i) \to (v, j)$ is a directed edge; and (b) if $i \in [\alpha(v) - 1]$ then there exists a directed path involving no edges of the form $(v, j) \to (v, k)$ that connects $(v, i+1)$ to (v, i). Let $\mathfrak{m}\mathsf{AO}(G)$ be the set of all acyclic multi-orientations of G.

One can relate the $\bar{e}$-expansion of the symmetric function $\overline{X}_G$ to the source counts of its acyclic multi-orientations, generalizing a result of Stanley [12, Thm. 3.3].

Theorem 3.8. Let G be a graph and suppose $\overline{X}_G = \sum_\lambda c_\lambda \overline{e}_\lambda$ for some coefficients $c_\lambda \in \mathbb{Z}$. Then the number of acyclic multi-orientations of G with exactly j sources and k vertices is $\sum_{\ell(\lambda)=j,|\lambda|=k} c_\lambda \in \mathbb{N}$.

As noted in [3], in general, the coefficients c_λ appearing in $\overline{X}_G = \sum_\lambda c_\lambda \overline{e}_\lambda$ can be negative, even when $G = \mathsf{inc}(P)$ is the *incomparability graph* of a $(3+1)$-free poset P.

3.5 Morphisms

For each graph G let $\blacktriangle(G) = \sum_{S\cup T=V(G)} G|_S \otimes G|_T$. This only differs from our other coproduct in allowing vertex decompositions that are not disjoint. Likewise, for each directed acyclic graph D and labeled poset $P = (D,\Gamma)$, define $\blacktriangle(D) = \sum D|_S \otimes D|_T$ and $\blacktriangle(P) = \sum (D|_S, \gamma|_S) \otimes (D|_T, \gamma|_T)$, where both sums are over all (not necessarily disjoint) vertex decompositions $S \cup T = V(D)$ in which S is a lower set, T is an upper set, and $S \cap T$ is an antichain.

Use the continuous linear extensions of these operations to replace the coproducts in the completions of Graphs, DAGs, and LPosets, and denote the resulting structures as $\mathfrak{m}$Graphs, $\mathfrak{m}$DAGs, and $\mathfrak{m}$LPosets to distinguish them from $\overline{\mathsf{Graphs}}$, $\overline{\mathsf{DAGs}}$, and $\overline{\mathsf{LPosets}}$.

Theorem 3.9. The pairs $(\mathfrak{m}\mathsf{Graphs}, \overline{\zeta}_{\mathsf{Graphs}})$, $(\mathfrak{m}\mathsf{DAGs}, \overline{\zeta}_{\mathsf{DAGs}})$, and $(\mathfrak{m}\mathsf{LPosets}, \overline{\zeta}_{\mathsf{LPosets}})$ are all combinatorial LC-Hopf algebras, and there is a commutative diagram

$$\begin{array}{ccccc} (\mathfrak{m}\mathsf{Graphs}, \overline{\zeta}_{\mathsf{Graphs}}) & \hookrightarrow & (\mathfrak{m}\mathsf{DAGs}, \overline{\zeta}_{\mathsf{DAGs}}) & \hookrightarrow & (\mathfrak{m}\mathsf{LPosets}, \overline{\zeta}_{\mathsf{LPosets}}) \\ & \searrow & \downarrow & \swarrow & \\ & & (\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}}) & & \end{array}$$

in which the horizontal maps send $G \mapsto \sum_{D\in\mathfrak{m}\mathsf{AO}(G)} D$ and $D \mapsto (D, \gamma^{\mathsf{op}})$, and the $\mathfrak{m}$QSym-valued maps send $G \mapsto \overline{X}_G$, $D \mapsto \overline{\Gamma}(D)$, and $(D,\gamma) \mapsto \overline{\Gamma}(D,\gamma)$.

Corollary 3.10. The unique morphism $(\mathfrak{m}\mathsf{Graphs}, \overline{\zeta}_{\mathsf{Graphs}}) \to (\mathfrak{m}\mathsf{QSym}, \overline{\zeta}_{\mathsf{Q}})$ assigns a graph G to its kromatic symmetric function, which is symmetric as $\mathfrak{m}$Graphs is cocommutative. One can express $\overline{X}_G = \sum_{D\in\mathfrak{m}\mathsf{AO}(G)} \overline{\Gamma}(D)$ and thus $\overline{X}_G$ is multifundamental positive.

Fix a directed acyclic graph D. When $\alpha : V(D) \to \mathbb{N}$ is any map, define $\mathrm{Cl}^{\mathsf{dag}}_\alpha(D)$ to be the directed acyclic graph with vertices $\mathrm{Cl}_\alpha(V(D))$ and with edges $(v,i) \to (w,j)$ whenever $v \to w \in E(D)$ or both $v = w$ and $i < j$. When $\gamma : V(D) \to \mathbb{Z}$ is injective, so that (D,γ) is a labeled poset, define $\mathrm{Cl}^{\mathsf{dag}}_\alpha(D,\gamma) = (\mathrm{Cl}^{\mathsf{dag}}_\alpha(D), \tilde{\gamma})$ to be the labeled poset where $\tilde{\gamma}(v,i) < \tilde{\gamma}(w,j)$ if and only if $\gamma(v) < \gamma(w)$ or both $v = w$ and $i > j$.

Theorem 3.11. Assume $\mathbb{Q} \subseteq \mathbb{K}$. Then there is a commutative diagram

$$\begin{array}{ccccc} (\mathfrak{m}\mathsf{Graphs}, \overline{\zeta}_{\mathsf{Graphs}}) & \hookrightarrow & (\mathfrak{m}\mathsf{DAGs}, \overline{\zeta}_{\mathsf{DAGs}}) & \hookrightarrow & (\mathfrak{m}\mathsf{LPosets}, \overline{\zeta}_{\mathsf{LPosets}}) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ (\overline{\mathsf{Graphs}}, \overline{\zeta}_{\mathsf{Graphs}}) & \hookrightarrow & (\overline{\mathsf{DAGs}}, \overline{\zeta}_{\mathsf{DAGs}}) & \hookrightarrow & (\overline{\mathsf{LPosets}}, \overline{\zeta}_{\mathsf{LPosets}}) \end{array}$$

with horizontal maps extending Proposition 2.8 and Theorem 3.9, where the vertical isomorphisms are the continuous linear maps sending $G \mapsto \sum_{\alpha:V(G)\to\mathbb{P}} \frac{1}{\alpha!}\mathrm{Cl}_\alpha(G)$, $D \mapsto \sum_{\alpha:V(D)\to\mathbb{P}} \mathrm{Cl}^{\mathsf{dag}}_\alpha(D)$, and $(D,\gamma) \mapsto \sum_{\alpha:V(D)\to\mathbb{P}} \mathrm{Cl}^{\mathsf{dag}}_\alpha(D,\gamma)$, respectively.

3.6 Kromatic quasisymmetric functions

For the rest of this note we assume $\mathbb{K} \supseteq \mathbb{Z}$ and let q be a formal parameter. We will consider the polynomial and power series rings $\mathsf{Sym}[q] \subset \mathfrak{m}\mathsf{QSym}[q] \subset \mathfrak{m}\mathsf{QSym}[\![q]\!]$.

Let G be an *ordered graph*, that is, a graph with a total order $<$ on its vertex set $V(G)$. One can think of the ordering on $V(G)$ as defining an acyclic orientation on the edges of G, and we do not distinguish between G and another ordered graph H if the corresponding directed acyclic graphs are isomorphic. The following power series is a K-theoretic generalization of $X_G(q)$ and q-analogue of $\overline{X}_G$:

Definition 3.12. For an ordered graph G define $\overline{L}_G(q) = \sum_\kappa q^{\mathrm{asc}_G(\max\circ\kappa)} x^\kappa \in \mathfrak{m}\mathsf{QSym}[q]$ where the sum is over all proper set-valued colorings.

Example 3.13. If $G = K_n$ is the complete graph on the vertex set $[n]$ then $\overline{L}_G(q) = [n]_q! \sum_{r=n}^\infty \left\{ {r \atop n} \right\} e_r = [n]_q! \sum_{r=n}^\infty \left\{ {r-1 \atop n-1} \right\} \bar{e}_r$ where $\left\{ {r \atop n} \right\}$ is the Stirling number of the second kind.

Let us clarify the apparent asymmetry in Definition 3.12. Define $\overline{L}_G^{\mathrm{des,min}}(q)$ by replacing "asc" by "des" and "max" by "min" in Definition 3.12. Construct $\overline{L}_G^{\mathrm{asc,min}}(q)$ and $\overline{L}_G^{\mathrm{des,max}}(q)$ analogously. Let ρ be the continuous involution of $\mathfrak{m}\mathsf{QSym}[q]$ sending $M_{(\alpha_1,\dots,\alpha_k)} \mapsto M_{(\alpha_k,\dots,\alpha_1)}$. Let τ be the involution of $\mathfrak{m}\mathsf{QSym}[q]$ sending $f \mapsto q^{\deg_q(f)} f(q^{-1})$.

Proposition 3.14. We have $\overline{L}_G(q) = \rho\left(\overline{L}_G^{\mathrm{des,min}}(q)\right) = \tau\left(\overline{L}_G^{\mathrm{des,max}}(q)\right) = \rho\circ\tau\left(\overline{L}_G^{\mathrm{asc,min}}(q)\right)$.

Recall that a *cluster graph* is a disjoint union of complete graphs.

Theorem 3.15. We have $\overline{L}_G(q) \in \mathfrak{m}\mathsf{Sym}[q]$ if and only if G is a cluster graph.

Fix $D \in \mathfrak{m}\mathsf{AO}(G)$. Each vertex in D has the form (v,i) for some $v \in V(G)$ and $i \in \mathbb{P}$. Define $\mathsf{align}(D) := |\{(u,i) \to (v,j) \in E(D) : u < v \text{ and } i = j = 1\}|$.

Proposition 3.16. If G is an ordered graph then $\overline{L}_G(q) = \sum_{D\in\mathfrak{m}\mathsf{AO}(G)} q^{\mathsf{align}(D)} \overline{\Gamma}(D)$. This power series is multifundamental positive in the sense of being a possibly infinite $\mathbb{N}[q]$-linear combination of multifundamental quasisymmetric functions.

We can make this more explicit, generalizing a result in [10]. Following [7], a *multipermutation* of $n \in \mathbb{N}$ is a word $w = w_1w_2\cdots w_m$ with $\{w_1,w_2,\dots,w_m\} = \{1,2,\dots,n\}$ and $w_i \neq w_{i+1}$ for all $i \in [m-1]$. Let $\overline{S}_n$ be the set of all multipermutations of n.

For each $w = w_1w_2\cdots w_m \in \overline{S}_n$ let $\mathrm{Inv}(w)$ be the set of pairs (w_i, w_j) with $i < j$ and $w_i > w_j$ and $\{w_1,w_2,\dots,w_{i-1}\} \cap \{w_i\} = \{w_1,w_2,\dots,w_{j-1}\} \cap \{w_j\} = \varnothing$. If P is a poset on $[n]$ and $G = \mathsf{inc}(P)$ is its incomparability graph, then we set $\mathrm{inv}_G(w) := |\{(a,b) \in \mathrm{Inv}(w) : \{a,b\} \in E(G)\}|$ and $S(w,P) := \{m-i : i \in [m-1] \text{ and } w_i \not>_P w_{i+1}\}$.

Theorem 3.17. If $G = \mathsf{inc}(P)$ for a poset P on $[n]$ then $\overline{L}_G(q) = \sum_{w \in \overline{S}_n} q^{\mathrm{inv}_G(w)} \overline{L}_{\ell(w),S(w,P)}$.

The homogeneous component of $\overline{L}_G(q)$ of lowest x-degree recovers $X_G(q)$. The latter power series, like X_G, naturally arises as the image of a morphism of combinatorial Hopf algebras. In detail, assume $\mathbb{K} = \mathbb{Z}[q]$ and let $\mathsf{OGraphs}_n$ be the free $\mathbb{K}$-module spanned by all isomorphism classes of ordered graphs with n vertices. Then the direct sum $\mathsf{OGraphs} := \bigoplus_{n\in\mathbb{N}} \mathsf{OGraphs}_n$ has a graded connected Hopf algebra structure in which the product is disjoint union and the coproduct Δ_q satisfies

$$\Delta_q(G) = \textstyle\sum_{S \sqcup T = V(G)} q^{\mathrm{asc}_G(S,T)} G|_S \otimes G|_T \quad \text{for each ordered graph } G, \tag{3.1}$$

where $\mathrm{asc}_G(S,T) := |\{(s,t) \in S \times T : \{s < t\} \in E(G)\}|$. If ζ_{OGraphs} is the algebra morphism $\mathsf{OGraphs} \to \mathbb{K}$ sending $G \mapsto 0^{|E(G)|}$, then $(\mathsf{OGraphs}, \zeta_{\mathsf{OGraphs}})$ is a combinatorial Hopf algebra and the morphism $(\mathsf{OGraphs}, \zeta_{\mathsf{OGraphs}}) \to (\mathsf{QSym}, \zeta_{\mathsf{Q}})$ sends $G \mapsto X_G(q)$.

We do not know how to give the completion $\mathfrak{m}\mathsf{OGraphs} \supset \mathsf{OGraphs}$ a combinatorial LC-Hopf algebra structure that lets us construct $\overline{L}_G(q)$ in a similar way. In particular, we have not been able to find a K-theoretic generalization of the coproduct Δ_q. Unlike the $q = 1$ case, simply replacing $\sqcup$ in (3.1) by arbitrary union $\cup$ does not lead to a co-associative map. This problem remains if we change the q-power exponent $\mathrm{asc}_G(S,T)$ to other forms like $\mathrm{asc}_G(S-T,T)$, $\mathrm{asc}_G(S,T-S)$, or $\mathrm{asc}_G(S-T,T-S)$.

3.7 Another quasisymmetric analogue

The preceding results indicate that $\overline{L}_G(q)$ is an interesting quasisymmetric q-analogue of $\overline{X}_G$ and K-theoretic extension of $X_G(q)$. However, there is another natural candidate for such a generalization. Continue to let G be an ordered graph. Following [6], an *ascent* of a set-valued map $\kappa : V(G) \to \mathsf{Set}(\mathbb{P})$ is a tuple (u,v,i,j) with $\{u,v\} \in E(G)$, $i \in \kappa(u)$, $j \in \kappa(v)$, and both $u < v$ and $i < j$. Let $\mathrm{asc}_G(\kappa)$ denote the number of such ascents.

Definition 3.18. For an ordered graph G, define $\overline{X}_G(q) = \sum_\kappa q^{\mathrm{asc}_G(\kappa)} x^\kappa \in \mathfrak{m}\mathsf{QSym}[\![q]\!]$ where the sum is over all proper set-valued colorings $\kappa : V(G) \to \mathsf{Set}(\mathbb{P})$.

This definition is closely related to the quasisymmetric functions $X_G(\mathbf{x}, q, \mu)$ studied in [6]. For each map $\mu : V(G) \to \mathbb{N}$, Hwang [6] defines $X_G(\mathbf{x}, q, \mu) := \sum_\kappa q^{\mathrm{asc}_G(\kappa)} x^\kappa$ where the sum is over all proper set-valued colorings κ of G with $|\kappa(v)| = \mu(v)$. Evidently $\overline{X}_G(q) = \sum_{\mu : V(G) \to \mathbb{P}} X_G(\mathbf{x}, q, \mu)$, and as noted in [6, Rem. 2.2] one has $X_G(\mathbf{x}, q, \mu) = \frac{1}{[\mu]_q!} X_{\mathrm{Cl}_\mu(G)}(q)$ where $[\mu]_q! := \prod_{v \in V(G)} [\mu(v)]_q!$. Here, we view $\mathrm{Cl}_\mu(G)$ as an ordered graph with $(v,i) < (w,j)$ if either $v < w$ or $v = w$ and $i < j$.

Using these observations, various positive or alternating expansions of $X_G(q)$ (e.g., into fundamental quasisymmetric functions [10, Thm. 3.1], Schur functions [10, Thm. 6.3], power sum symmetric functions [2, Thm. 3.1], or elementary symmetric functions [10, Conj. 5.1]) can be extended in a straightforward way to $X_G(\mathbf{x}, q, \mu)$ and $\overline{X}_G(q)$. See Hwang's results [6, Thms. 3.3, 4.10, and 4.19] and his conjecture [6, Conj. 3.10].

Some of these statements require G to be isomorphic to the incomparability graph of a *natural unit interval order*, meaning a poset P on a finite subset of $\mathbb{P}$ such that if $x <_P z$ then $x < z$ and every y incomparable in P to both x and z has $x < y < z$ [10, Prop. 4.1].[1] If G has this property, then so do all of its α-clans. Therefore $\overline{X}_G(q)$ is symmetric if G is the incomparability graph of a natural unit order interval [6, Thm. 3.8].

Example 3.19. If K_n is the complete graph on $[n]$ then $\overline{X}_{K_n}(q) = \sum_{r=n}^{\infty} F_r^{(n)} e_r$ for $F_r^{(n)} := \sum_{\substack{k_1,k_2,\dots,k_n \in \mathbb{P} \\ k_1+k_2+\dots+k_n = r}} \binom{r}{k_1,k_2,\dots,k_n}_q$ where $(q)_n := \prod_{i\in[n]}(1-q^i)$ and $\binom{r}{k_1,k_2,\dots,k_n}_q = \frac{(q)_r}{(q)_{k_1}(q)_{k_2}\cdots(q)_{k_n}}$.

When q is a prime power, $F_r^{(n)}$ counts the *strictly increasing flags* of $\mathbb{F}_q$-subspaces $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{F}_q^r$. Vinroot [14] derived a recurrence for the *generalized Galois numbers* $G_r^{(n)} := \sum_{i=0}^{n} \binom{n}{i} F_r^{(i)}$. This can be used to show (setting $F_r^{(n)} = 0$ if $r < 0$) that:

Proposition 3.20. One has $F_{r+1}^{(n)} = \sum_{i=0}^{n-1} \sum_{j=n-1-i}^{n} \binom{n}{j}\binom{j}{n-1-i}(-1)^i \frac{(q)_r}{(q)_{r-i}} F_{r-i}^{(j)}$.

Like $\overline{L}_G(q)$, the power series $\overline{X}_G(q)$ also does not seem to arise naturally as the image in $\mathfrak{m}\mathsf{QSym}$ of a combinatorial LC-Hopf algebra. Unlike $\overline{L}_G(q)$, however, $\overline{X}_G(q)$ is not generally multifundamental-positive (or $\bar{e}$-positive). However, $\overline{X}_G(q)$ does have a nontrivial positivity property that is not shared by $X_G(\mathbf{x}, q, \mu)$ or $\overline{L}_G(q)$.

A *set-valued tableau* T of shape λ is an assignment of sets $T_{ij} \in \mathsf{Set}(\mathbb{P})$ to the cells (i, j) in $\mathsf{D}_\lambda = \{(i, j) \in \mathbb{P} \times \mathbb{P} : 1 \leq j \leq \lambda_i\}$ of a partition λ. We write $(i, j) \in T$ to indicate that (i, j) belongs to the shape of T. A set-valued tableau T is *semistandard* if $T_{ij} \preceq T_{i,j+1}$ and $T_{ij} \prec T_{i+1,j}$ for all relevant positions. Let $x^T := \prod_{(i,j)\in T} \prod_{k\in T_{ij}} x_k$ and $|T| := \sum_{(i,j)\in T} |T_{ij}|$.

Definition 3.21. The *symmetric Grothendieck function* of a partition λ is the power series $\bar{s}_\lambda := \sum_{T\in\mathsf{SetSSYT}(\lambda)} (-1)^{|T|-|\lambda|} x^T \in \mathbb{Z}[\![x_1, x_2, \dots]\!]$ where $\mathsf{SetSSYT}(\lambda)$ is the set of all semistandard set-valued tableaux of shape λ.

Each $\bar{s}_\lambda$ is in $\mathfrak{m}\mathsf{Sym}$ and the set of all symmetric Grothendieck functions is another pseudobasis for $\mathfrak{m}\mathsf{Sym}$. We write $\mu \subseteq \lambda$ for two partitions with $\mathsf{D}_\mu \subseteq \mathsf{D}_\lambda$ and set $\mathsf{D}_{\lambda/\mu} := \mathsf{D}_\lambda \setminus \mathsf{D}_\mu$. A *semistandard tableau* of shape λ/μ is a filling of $\mathsf{D}_{\lambda/\mu}$ by positive integers such that each row is weakly increasing and each column is strict increasing.

Definition 3.22 ([3, Def. 3.8])**.** Suppose P is a finite poset and λ is a partition. A *Grothendieck P-tableau* of shape λ is a pair $T = (U, V)$ with these two properties: (a) U is a filling of D_μ by elements of P for some partition $\mu \subseteq \lambda$, such that each element of P is in at least one cell, and for each $(i, j) \in \mathsf{D}_\mu$ one has $U_{ij} <_P U_{i,j+1}$ if $(i, j+1) \in \mathsf{D}_\mu$ and $U_{ij} \not>_P U_{i+1,j}$ if $(i+1, j) \in \mathsf{D}_\mu$; and (b) V is a semistandard tableau of shape λ/μ, whose entries in each row i are all less than i (so $\mathsf{D}_{\lambda/\mu}$ must have no cells in the first row).

[1] A finite poset is isomorphic to one with these properties iff it is $(3+1)$- and $(2+2)$-free [10, §4].

Let $\mathscr{G}_P$ be the set of Grothendieck P-tableaux. Let $\lambda(T)$ be the shape of $T \in \mathscr{G}_P$. One of the main results of [3] establishes that if $G = \mathsf{inc}(P)$ is the incomparability graph a $(3+1)$-free poset P then $\overline{X}_G = \sum_{T\in\mathscr{G}_P} \overline{s}_{\lambda(T)}$. This theorem has a q-analogue.

Suppose P is a finite poset on a subset of $\mathbb{P}$, and let $G = \mathsf{inc}(P)$. Choose some $T = (U, V) \in \mathscr{G}_P$ and let μ be the partition shape of the tableau U. Define a *G-inversion* of T to be a pair of cells $(i, j), (k, l) \in \mathsf{D}_\mu$ with $i > k$ such that $U_{ij} < U_{kl}$ but $U_{ij} \not<_P U_{kl}$ and $U_{ij} \not>_P U_{kl}$. Finally, let $\mathrm{inv}_G(T)$ be the number of all G-inversions of T.

Theorem 3.23. If P is a natural unit interval order then $\overline{X}_G = \sum_{T\in\mathscr{G}_P} q^{\mathrm{inv}_G(T)}\overline{s}_{\lambda(T)}$.

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