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Multispecies TAZRP and modified Macdonald polynomials

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Abstract. Recently, a formula for the symmetric Macdonald polynomials $P_\lambda(X;q,t)$ was given in terms of objects called *multiline queues*, which also compute probabilities of a statistical mechanics model called the *multispecies ASEP* on a ring. It is natural to ask whether the modified Macdonald polynomials $\widetilde{H}_\lambda(X;q,t)$ can be obtained using a combinatorial gadget for some other statistical mechanics model. We answer this question in the affirmative. In this article, we give a new formula for $\widetilde{H}_\lambda(X;q,t)$ in terms of fillings of tableaux called *polyqueue tableaux*. We define a *multispecies TAZRP* on a ring with parameter t, whose (unnormalized) stationary probabilities are computed by polyqueue tableaux, and whose partition function is equal to $\widetilde{H}_\lambda(X;1,t)$.

1 Introduction

The theory of symmetric functions is very classical, having its origins in invariant theory, Galois theory, group theory and, of course, combinatorics. As Stanley [18, Notes in Chapter 7] remarks, the first published work on symmetric functions was a derivation of the well-known Newton–Girard identity by A. Girard [11] in 1629 (independently rediscovered by Newton around 1666). On the other hand, the theory of interacting particle systems is relatively modern. It was an influential paper of Spitzer [17] in 1970 that initiated the subject and set out the important questions in the field. In particular, the simple exclusion process and the zero-range process were first defined there.

Over the last couple of decades, the theory of special functions and symmetric functions have found unexpected connections to diverse interacting particle systems. The asymmetric simple exclusion process (ASEP) has played a central role in this connection. See for example [3, 4, 8, 7, 5].

The *modified Macdonald polynomials* $\widetilde{H}_\lambda(X;q,t)$ defined by Garsia and Haiman [10] are a special form of the well-known Macdonald polynomials P_λ's [16] with coefficients

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in $\mathbb{Z}[q,t]$ obtained through a formal operation called *plethysm* from a scalar multiple of $P_\lambda(X;q,t)$. Understanding the combinatorics of these polynomials has been a fundamental area of interest in algebraic combinatorics. The most commonly used combinatorial description of $\widetilde{H}_\lambda(X;q,t)$ is a tableaux formula due to Haglund, Haiman, and Loehr [12]. Recently, a different combinatorial model for these polynomials has been given by Garbali and Wheeler [9]. Independently, the second author together with Corteel, Haglund, Mason, and Williams found formulas for $\widetilde{H}_\lambda(X;q,t)$ that were based on the combinatorial interpretation of plethysm applied to multiline queues [6]. One such formula was a compact version of the original tableaux formula of Haglund, Haiman, and Loehr, but the other was a conjectural formula in terms of certain tableaux called *polyqueues*. One of the main results of this extended abstract is a proof of this conjecture.

Theorem 1.1. *The modified Macdonald polynomial is given by*

$$\widetilde{H}_\lambda(X;q,t) = \sum_{\sigma\in \mathrm{PQT}(\lambda)} t^{\mathrm{quinv}(\sigma)} q^{\mathrm{maj}(\sigma)} x^\sigma,$$

where the sum is over polyqueue tableaux corresponding to λ, denoted by $\mathrm{PQT}(\lambda)$*.*

In this extended abstract, we only sketch the main ideas involved in the proof. For the complete details, see the full version [2].

Upon the discovery of the link between Macdonald polynomials $P_\lambda(X;q,t)$ and the probabilities of the multispecies ASEP on a circle, a natural question was whether there exists a related statistical mechanics model for which some specialization of $\widetilde{H}_\lambda(X;q,t)$ is equal to its partition function. As a part of this project, we consider the *multispecies totally asymmetric zero-range process* (mTAZRP), which had been introduced in [19]. The stationary probabilities of this model turn out to be polynomials with coefficients in $\mathbb{N}$, and we denote by $\mathcal{Z}_{(\lambda,n)}$ the sum of the unnormalized probabilities over all states of the mTAZRP on n sites with particles types described by the partition λ. In statistical mechanics contexts, $\mathcal{Z}_{(\lambda,n)}$ is called the *partition function* of the mTAZRP, and is notable in particular because it encodes much important information about the behavior of the model. In the upcoming sequel [1], we will prove the following result.

Theorem 1.2. *The partition function $\mathcal{Z}_{(\lambda,n)}$ of the mTAZRP on n sites with site parameters $x_1,\dots,x_n$, priority parameter t and particle types described by λ equals $\widetilde{H}_\lambda(x_1,\dots,x_n;1,t)$.*

The plan of this article is as follows. Section 1.2 contains our main results. Section 2 gives the necessary background. The modified Macdonald polynomials are characterized by the three properties given in Section 2.1. In Section 3.1 we prove the first of these properties. In Section 3.2, we sketch a proof of the second property, which proves to be the main hurdle. This uses a bijection on words respecting a coinversion-type statistic given in Section 4, which is of independent interest. The third property turns out to be immediate from our definition.

1.1 Multispecies totally asymmetric zero range process

Fix a partition $\lambda = (\lambda_1, \dots \lambda_k)$ and a positive integer n. Let $m_r = \#\{i : \lambda_i = r\}$ and note $\sum_r m_r = k$. We consider a particle system, denoted by mTAZRP(λ, n), which has n sites (labelled $1, 2, \dots, n$), and k particles of which m_r have type r. Each site may be empty or may contain one or more particles. Particles of the same type are indistinguishable.

We may identify the states of the chain as multiset compositions of type λ with n parts, i.e. a composition $\tau = (\tau_1, \dots, \tau_n)$, each part of which is a (possibly empty) multiset, and such that the union of all the parts has the same content as λ.

The system evolves as a continuous-time Markov chain. Any jump of the system consists of a single particle jumping from site j to site $j-1$, for some $j \in \{1, \dots, n\}$ (sites are considered cyclically mod n, so that a particle jumping out of site 1 enters site n). The rates are governed by a global parameter t and site-dependent parameters $x_1, \dots, x_n$.

For each $j \in \{1, \dots, n\}$ and each $a \geq 1$, a bell of level a rings at site j at rate $x_j^{-1} t^{a-1}$. When such a bell rings: if site j contains at least a particles, then the a'th highest-numbered of them jumps to site $j-1$. If j contains fewer than a particles, nothing changes. (This means that if the number of particles of type r at site j is c_r, and the number of particles of type greater than r at site j is d_r, then the total rate of jumps of particles of type r from site j is $x_j^{-1} t^{d_r} \sum_{i=0}^{c_r - 1} t^i$.)

Example 1.3. mTAZRP$((2, 1, 1), 3)$ has 18 states. Examples of its transitions are:

- The jumps from $(), (), (2, 1, 1)$ are to $(), (2), (1, 1)$ with rate x_3^{-1}, and to $(), (1), (2, 1)$ with rate $x_3^{-1}(t + t^2)$;
- The jumps from $(2, 1), (), (1)$ are to $(2, 1), (1), ()$ with rate x_3^{-1}, to $(1), (), (2, 1)$ with rate x_1^{-1}, and to $(2), (), (1, 1)$ with rate $x_1^{-1} t$.

1.2 Polyqueue tableaux

To state our main result formally, we will need some background. The main combinatorial objects will be polyqueue tableaux, which were originally introduced by the second author with Corteel, Haglund, Mason, and Williams [6]. We provide the definition here.

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition. The *diagram of type* λ, denoted dg(λ), consists of k bottom-justified columns, where the i'th column from left to right has λ_i boxes. See Figure 1 for an illustration of dg$((3, 3, 2, 2, 2, 2, 1, 1))$. The cell (r, j) corresponds to the cell in the j'th column of the r'th row of dg(λ), where rows are labeled from bottom to top. A *polyqueue tableau* or *filling* of type (λ, n) is a filling $\sigma : \text{dg}(\lambda) \to [n]$ of the cells of dg(λ).

For a filling σ, let $\sigma(r, i)$ be the entry in the cell (r, i) of σ. Define $South(x)$ to be the cell $(r-1, i)$ directly below cell x in the same column. Denote by PQT(λ, n) the set of fillings of type (λ, n), and by PQT(λ) all fillings $\sigma : \text{dg}(\lambda) \to \mathbb{Z}^+$. We now define the *reading order* and the three main statistics associated to PQT(λ).

Definition 1.4. Define the *reading order* on $\mathrm{PQT}(\lambda)$ to be along the rows from right to left, which are read from top to bottom.

Definition 1.5. Let $\sigma \in \mathrm{PQT}(\lambda)$. Define the *leg* of a cell (r, i) to be the number of cells in column i above row r, i.e. $\mathrm{leg}((r,i)) = \lambda_i - r$. Define the *major index* of the filling σ to be:

$$\mathrm{maj}(\sigma) = \sum_{x\,:\,\sigma(x)>\sigma(South(x))} (\mathrm{leg}(x)+1).$$

Definition 1.6. Given a diagram $\mathrm{dg}(\lambda)$, a *triple* consists of either

- three cells $(r+1, i)$, (r, i) and (r, j) with $i < j$ with content a, b, and c respectively, as

$$\begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array} \cdots \begin{array}{|c|}\hline c \\ \hline \end{array}\,, \quad \text{or}$$

- two cells (r, i) and (r, j) with content b and c respectively if $\lambda_i = r$, in which case they are said to form a *degenerate triple*.

We call a triple a *quinv triple* if the entries are oriented counterclockwise when read in increasing order, with ties being broken with respect to reading order. Note that if the triple is degenerate with content b, c, it is a quinv triple if and only if $b < c$. This is equivalent to thinking of a degenerate triple as a regular triple with $a = 0$.

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-2} 3 & 2 \\ \cline{1-5} 1 & 3 & 1 & 3 & 3 \\ \hline 1 & 1 & 2 & 1 & 2 & 3 & 3 \\ \hline \end{array}$$

Figure 1: A polyqueue tableau of type $\lambda = (3,3,2,2,2,1,1)$ and $n = 3$. The weight of this filling is $x_1^5 x_2^3 x_3^6 q^5 t^{12}$, and it corresponds to the state $\tau = ((3,3,2),(2,2),(1,1))$.

The *weight* of a filling σ is $\mathrm{wt}(\sigma) = x^\sigma t^{\mathrm{quinv}(\sigma)} q^{\mathrm{maj}(\sigma)}$. See Figure 1. Each $\sigma \in \mathrm{PQT}(\lambda, n)$ is associated to a state τ of the $TAZRP(\lambda, n)$ through its bottom row as follows. For $j = 1, \ldots, n$, set $\tau_j = \{\lambda_i \in \lambda \;:\; \sigma(1, i) = j\}$ to be the multiset of the heights of the columns of T whose bottom-most entry is j. For example, the following are all six tableaux that correspond to the state $\tau = ((),(2,1),(1)) \in TAZRP((2,1,1),3)$:

$$\begin{array}{|c|c|c|}\cline{1-1} 1 \\ \hline 2 & 3 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\cline{1-1} 2 \\ \hline 2 & 3 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\cline{1-1} 3 \\ \hline 2 & 3 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\cline{1-1} 1 \\ \hline 2 & 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\cline{1-1} 2 \\ \hline 2 & 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\cline{1-1} 3 \\ \hline 2 & 2 & 3 \\ \hline \end{array}$$

We denote by $\mathrm{PQT}(\tau)$ all polyqueue tableaux that correspond to the mTAZRP state τ. In the forthcoming article [1], we prove the following by constructing a Markov chain on polyqueue tableaux that projects to the mTAZRP.

Theorem 1.7. *The stationary probability of state $\tau \in TAZRP(\lambda, n)$ equals*

$$\frac{1}{\mathcal{Z}_{(\lambda,n)}} \sum_{\sigma \in \mathrm{PQT}(\tau)} \mathrm{wt}(\sigma).$$

2 Background and preliminaries

The proofs of several of our results will rely on the techniques used in [12].

2.1 The axioms uniquely characterizing $\widetilde{H}_\lambda$

Let $\Lambda \equiv \Lambda(q,t)$ be the algebra of symmetric functions. Recall that $\{p_\mu\}$ is the power sum basis for Λ, and ω is the standard involution. For a formal power series A in indeterminates $a_1, a_2, \cdots$, in our case with coefficients in $\mathbb{Q}(q,t)$, $p_k[A]$ is the formal substitution of a_i^k for each indeterminate a_i. Then for an arbitrary $f \in \Lambda$, the *plethysm* $f[A]$ is defined by expressing f in the power sum basis and substituting $p_k[A]$ for each p_k in the expansion. By convention, we define the *plethystic alphabets* $X = x_1 + x_2 + \cdots$ and $Y = y_1 + y_2 + \cdots$, so that $f[X] = f(X)$, $f[-X] = (-1)^d \omega(f(X))$ if f is homogeneous of degree d, $f[X+Y] = f(X,Y)$, where $f(X,Y)$ represents the concatenation of the alphabets X and Y, and $f[X(1-q)]$ is the image of f under the algebra homomorphism mapping $p_k(X)$ to $(1-q^k)p_k(X)$. See [13, Section 2] for a complete description.

The modified Macdonald polynomials are the basis of Λ with coefficients in $\mathbb{Q}(q,t)$, characterized by the following triangularity and normalization axioms. These axioms are derived from Macdonald's triangularity and orthogonality axioms; see [13, Proposition 2.6] and [14, Section 6.1]. For some coefficients $a_{\mu\lambda}(q,t), b_{\mu\lambda}(q,t) \in \mathbb{Q}(q,t)$:

$$\widetilde{H}_\lambda[X(1-t);q,t] = \sum_{\mu \geq \lambda'} a_{\mu\lambda}(q,t) s_\mu(X), \tag{2.1}$$

$$\widetilde{H}_\lambda[X(1-q);q,t] = \sum_{\mu \geq \lambda} b_{\mu\lambda}(q,t) s_\mu(X), \tag{2.2}$$

$$\left\langle \widetilde{H}_\lambda(X;q,t), s_{(n)}(X) \right\rangle = 1. \tag{2.3}$$

Using elementary properties of plethysm (see [12, Section 5]), we reformulate (2.1) and (2.2) in terms of the monomial basis, for some coefficients $c_{\mu\lambda}(q,t), d_{\mu\lambda}(q,t) \in \mathbb{Q}(q,t)$:

$$\widetilde{H}_\lambda[X(t-1);q,t] = \sum_{\mu \leq \lambda} d_{\mu\lambda}(q,t) m_\mu(X), \tag{2.4}$$

$$\widetilde{H}_\lambda[X(q-1);q,t] = \sum_{\mu \leq \lambda'} c_{\mu\lambda}(q,t) m_\mu(X). \tag{2.5}$$

Along with (2.3), these axioms uniquely characterize $\widetilde{H}_\lambda$.

2.2 Super-alphabets

With PQT, quinv, and maj as defined in Section 1.2, we define:

$$C_\lambda(X;q,t) = \sum_{\sigma \in \mathrm{PQT}(\lambda)} q^{\mathrm{maj}(\sigma)} t^{\mathrm{quinv}(\sigma)} x^\sigma. \tag{2.6}$$

We will show that $C_\lambda(X;q,t)$ satisfies the axioms (2.1), (2.2), and (2.3).

It is easy to see that $C_\lambda(X;q,t)$ satisfies (2.3). The strategy of our proof is to show that $C_\lambda(X;q,t)$ satisfies (2.4) and (2.5) by modifying the proof of the Haglund–Haiman–Loehr formula in [12, Section 4] for the setting of $\mathrm{PQT}(\lambda)$ through the *superization* of C_λ.

We briefly describe the well-known properties of superization in this setting. Define the "super-alphabet" $\mathcal{A} = \mathbb{Z}_+ \cup \mathbb{Z}_- = \{\bar{1}, 1, \bar{2}, 2, \dots\}$ consisting of positive and "negative" letters $i, \bar{i}$ of our original alphabet. One can consider any total ordering on $\mathcal{A}$; in our proofs we will use the two total orderings

$$(\mathcal{A}, <_1) = \{1 < \bar{1} < 2 < \bar{2} < \cdots\},$$
$$(\mathcal{A}, <_2) = \{1 < 2 < 3 < \cdots < \bar{3} < \bar{2} < \bar{1}\}.$$

For $a, b \in \mathcal{A}$ and any fixed total ordering $<$, we will use the notation $I(a,b)$:

$$I(a,b) = \begin{cases} 1, & a > b \text{ or } a = b \in \mathbb{Z}_-, \\ 0, & a < b \text{ or } a = b \in \mathbb{Z}_+. \end{cases}$$

Definition 2.1. The *superization* of a symmetric function $f(X)$ is $\widetilde{f}(X,Y) = \omega_Y f[X+Y]$, where ω_Y acts on $f(x,y)$ considered as a symmetric function of the Y variables.

Definition 2.2. Given a super alphabet $\mathcal{A} = \mathbb{Z}_+ \cup \mathbb{Z}_-$ and a fixed total ordering $<$, a *super filling* of a diagram $\mathrm{dg}(\lambda)$ is a function $\sigma : \mathrm{dg}(\lambda) \to \mathcal{A}$, with the following extension of the definitions of the maj and quinv statistics from Definition 1.5 and Definition 1.6.

- maj: If $y = South(x)$ in $\mathrm{dg}(\lambda)$, then $(x,y) \in \mathrm{Des}(\sigma)$ if $I(\sigma(x), \sigma(y)) = 1$. The maj statistic is defined as before.
- quinv: If three cells x, y, z with entries $\sigma(x) = a$, $\sigma(y) = b$, $\sigma(z) = c$ form the triple

$$\begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array} \cdots \begin{array}{|c|}\hline c \\ \hline \end{array}$$

then the triple is a quinv triple if and only if exactly *one* of the following is true:

$$\{I(a,b) = 1,\ I(c,b) = 0,\ I(a,c) = 0\}.$$

It is *not a* quinv *triple* if and only if exactly *two* of the conditions above are true.[1] (Note that in the case of a degenerate triple, $I(a,b) = I(a,c) = 0$ so the triple is a quinv triple if and only if $I(c,b) = 1$.) As before, $\mathrm{quinv}(\sigma)$ is the number of quinv triples in σ.

[1] The reader may check that it is impossible for all or none of the conditions to be true.

We write $|\sigma|$ to denote the regular filling with the positive alphabet, such that $|\sigma|(u) = |\sigma(u)|$ for each $u \in \mathrm{dg}(\lambda)$. It is immediate that when $\sigma = |\sigma|$, the above definitions reduce to those of a polyqueue filling and its statistics as given in Section 1.2. Denote the set of super-fillings $\{\sigma \in \mathrm{dg}(\lambda) \to \mathcal{A}\}$ by $\widetilde{\mathrm{PQT}}(\lambda)$.

Proposition 2.3 ([12, Proposition 4.3])**.** *Let $\lambda \vdash n$. The superization of the polynomial $C_\lambda(x; q, t)$ has the following formula in terms of super-fillings:*

$$\widetilde{C}_\lambda(x, y; q, t) = \sum_{\sigma \in \widetilde{\mathrm{PQT}}(\lambda)} q^{\mathrm{maj}(\sigma)} t^{\mathrm{quinv}(\sigma)} z^\sigma,$$

where $z_i = x_i$ if $i \in \mathbb{Z}_+$ and $z_i = y_i$ if $i \in \mathbb{Z}_-$, and the statistics quinv *and* maj *on super-fillings $\sigma \in \widetilde{\mathrm{PQT}}(\lambda)$ are given in Definition 2.2.*

We use the identies $C_\lambda[X(t-1); q, t] = \widetilde{C}_\lambda(tX, -X; q, t)$ and $C_\lambda[X(q-1); q, t] = \widetilde{C}_\lambda(qX, -X; q, t)$ to obtain

$$C_\lambda[X(t-1); q, t] = \sum_{\sigma \in \widetilde{\mathrm{PQT}}(\lambda)} (-1)^{m(\sigma)} q^{\mathrm{maj}(\sigma)} t^{p(\sigma)+\mathrm{quinv}(\sigma)} x^{|\sigma|}, \tag{2.7}$$

$$C_\lambda[X(q-1); q, t] = \sum_{\sigma \in \widetilde{\mathrm{PQT}}(\lambda)} (-1)^{m(\sigma)} q^{p(\sigma)+\mathrm{maj}(\sigma)} t^{\mathrm{quinv}(\sigma)} x^{|\sigma|}, \tag{2.8}$$

where $p(\sigma) = |\{u : \sigma(u) \in \mathbb{Z}_+\}|$ and $m(\sigma) = |\{u : \sigma(u) \in \mathbb{Z}_-\}|$ are the numbers of positive and negative entries in the super-filling σ, respectively. Note that these formulas are valid for any total ordering chosen on $\mathcal{A}$.

3 Sketch of the proof of Theorem 1.1

LLT polynomials are a well-known family of symmetric polynomials discovered by Lascoux, Leclerc, and Thibon [15], indexed by tuples of skew young diagrams. The following theorem is proved by expanding C_λ in terms of LLT polynomials by summing over PQT with a fixed descent set, applying the strategy of proof from [12, Section 3].

Theorem 3.1. *The polynomial $C_\lambda(X; q, t)$ is symmetric in the variables x_i.*

3.1 Proof that $C_\lambda(X; q, t)$ satisfies (2.4)

In this section we will use the ordering $<_1$ on $\mathcal{A}$ and construct a sign-reversing, weight-preserving involution Ψ on super-fillings $\widetilde{\mathrm{PQT}}(\lambda)$, which will cancel out all terms involving x^μ if $\mu > \lambda$.

We begin by defining the following simple map.

Definition 3.2. Let $u \in \mathrm{dg}(\lambda)$. For $\sigma \in \widetilde{\mathrm{PQT}}(\lambda)$, define the map Φ_u by

$$\Phi_u(\sigma(w)) = \begin{cases} \sigma(w), & w \neq u, \\ -\sigma(w), & w = u. \end{cases}$$

Definition 3.3. Let $\sigma \in \widetilde{\mathrm{PQT}(\lambda)}$. We define $\Psi(\sigma)$ as follows. A pair of cells $u = (r, i)$ and $v = (r', j)$ with $i < j$ is *attacking* if $r = r'$ or $r = r' + 1$, i.e. they are either in the same row, or the one to the right is one row below.

- If there is no pair of attacking cells u, v in σ such that $|\sigma(u)| = |\sigma(v)|$, set $\Psi(\sigma) = \sigma$.
- Otherwise let a be minimal such that $|\sigma(x)| = |\sigma(y)| = a$ for some pair x, y of attacking cells in σ. Let v be the last cell in reading order among all such attacking pairs, and let u be the last cell in reading order attacking v, such that $|\sigma(u)| = a$. Set $\Psi(\sigma) = \Phi_u(\sigma)$.

We have defined Ψ such that if σ is not a fixed point, the map flips the sign of the entry at a designated cell u that depends only on $|\sigma|$: thus $\Psi(\Psi(\sigma)) = \sigma$. The following lemma is straightforward to prove by examining all possible cases.

Lemma 3.4. *Let $\sigma \in \widetilde{\mathrm{PQT}}(\lambda)$. Fix the ordering $<_1$ on $\mathcal{A}$. We have*

1. $\mathrm{maj}(\Psi(\sigma)) = \mathrm{maj}(\sigma)$, *and*
2. *if σ is not a fixed point of Ψ, then Ψ increases the number of* quinv *triples by exactly one:*

$$\mathrm{quinv}(\Psi(\sigma)) = \mathrm{quinv}(\sigma) + 1.$$

From this we easily obtain the following using Equation (2.7).

Theorem 3.5. *Let λ be a partition.*

$$C_\lambda[X(t-1); q, t] = \sum_{\substack{\sigma \in \widetilde{\mathrm{PQT}}(\lambda) \\ \sigma : \Psi(\sigma) = \sigma}} (-1)^{m(\sigma)} q^{\mathrm{maj}(\sigma)} t^{p(\sigma) + \mathrm{quinv}(\sigma)} x^{|\sigma|} = \sum_{\mu \leq \lambda} c_{\mu\lambda}(q, t) m_\mu.$$

3.2 Proof that $C_\lambda(X; q, t)$ satisfies (2.5)

We describe sign-reversing, weight-preserving maps on super-fillings $\widetilde{\mathrm{PQT}}$ using the ordering $<_2$ on $\mathcal{A}$, which will cancel out all terms involving x^μ if $\mu > \lambda'$. We follow the strategy of the proof of the corresponding result for HHL tableaux in [12, Section 5.2]. Our proof deviates for the subset of Φ-*degenerate* fillings, which we treat separately.

Definition 3.6. For $\sigma \in \widetilde{\mathrm{PQT}}(\lambda)$, let $a \in \mathbb{Z}_+$ be the smallest positive integer such that there exists a cell $(r, j) \in \mathrm{dg}(\lambda)$ with $|\sigma((r, j))| = a$ with $r > a$. If such a exists, call a the *distinguished label* of σ, and call the first cell in PQT reading order whose absolute value is a the *distinguished cell* of σ. Then σ falls into one of the following three categories:

(a) if no such a exists, we say σ is Φ-*trivial*.

(b) if the distinguished cell does not belong to any degenerate triples (regardless of whether they are quinv triples), we say σ is Φ-*nondegenerate*.

(c) if the distinguished cell is part of any degenerate triple (regardless of whether it is a quinv triple), we say σ is Φ-*degenerate*. If r is the row with the distinguished cell, the set of cells in row r that form degenerate triples is the *degenerate segment*.

See Figure 2 for examples of all three categories.

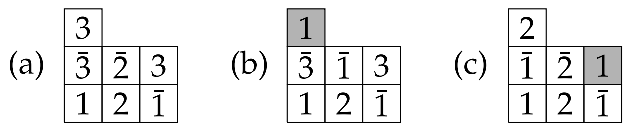

Figure 2: (a) Φ-trivial, (b) Φ-nondegenerate, and (c) Φ-degenerate tableaux. The grey box marks the distinguished cell if such exists. The degenerate segment in (c) is composed of the two cells at the tops of columns 2 and 3 in row 2.

The main result in this section will be the following theorem.

Theorem 3.7.

$$C_\lambda[X(q-1);q,t] = \sum_{\substack{\sigma\in\widetilde{\mathrm{PQT}}(\lambda)\\ \sigma \text{ is } \Phi\text{-trivial}}} (-1)^{m(\sigma)} q^{p(\sigma)+\mathrm{maj}(\sigma)} t^{\mathrm{quinv}(\sigma)} x^{|\sigma|} = \sum_{\mu\le\lambda'} d_{\mu\lambda}(q,t) m_\mu.$$

Proof sketch. First we show that Φ_u (where u is the distinguished cell of σ) is an involution on Φ-nondegenerate fillings. Let $\sigma' = \Phi_u(\sigma)$. Then the contribution to (2.8) from those Φ-nondegenerate fillings cancels:

$$(-1)^{m(\sigma)} q^{p(\sigma)+\mathrm{maj}(\sigma)} t^{\mathrm{quinv}(\sigma)} x^{|\sigma|} + (-1)^{m(\sigma')} q^{p(\sigma')+\mathrm{maj}(\sigma')} t^{\mathrm{quinv}(\sigma')} x^{|\sigma'|} = 0.$$

This mirrors the corresponding involution from [12, Section 5.2]. Next, we prove Theorem 3.8 which shows that the contribution to (2.8) from those fillings cancels as well. Only terms arising from Φ-trivial fillings remain, from which the result easily follows. □

Unfortunately, when $\sigma \in \widetilde{\mathrm{PQT}}(\lambda)$ is Φ-degenerate, a more subtle approach is required. The crux of our proof is the following theorem.

Theorem 3.8. *For a partition λ, the sum over the Φ-degenerate subset of fillings in $\widetilde{\mathrm{PQT}}(\lambda)$ in the right hand side of* (2.8)*, is zero:*

$$\sum_{\substack{\sigma\in\widetilde{\mathrm{PQT}}(\lambda)\\ \sigma \text{ is } \Phi\text{-degenerate}}} (-1)^{m(\sigma)} q^{p(\sigma)+\mathrm{maj}(\sigma)} t^{\mathrm{quinv}(\sigma)} x^{|\sigma|} = 0. \tag{3.1}$$

Proof sketch. Let a be the distinguished label of a Φ-degenerate filling σ. We show in Theorem 4.4 that there exists a quinv-preserving bijection ϕ on words with the same content in absolute value as the content of the degenerate segment of σ (in particular they have the same content in all the letters that are not a or $\bar{a}$), such that $\phi(w)$ has exactly one more or one less $\bar{a}$ as w. We then translate this map to a bijection on ϕ-degenerate fillings in $\widetilde{\mathrm{PQT}}(\lambda)$, which preserves the weight of a filling while reversing the sign, resulting in the cancellation in (3.1). □

Note that in our arguments, we *demonstrate the existence* of this bijection, though we do not succeed in constructing it. It remains an open question to find an explicit bijection.

4 Bijection on words

Let $n \geq 3$ be an integer and let $\alpha = (\alpha_2, \ldots, \alpha_{n-1})$ be a tuple of positive integers. Denote $|\alpha| = \sum_i \alpha_i$ and let $N > |\alpha|$ be an integer. For convenience, let $L = N - |\alpha|$. We consider words of length N in the alphabet $[n] = \{1, \ldots, n\}$ as follows. For $0 \leq k \leq L$, define

$$W_k \equiv W_k^{(\alpha,L)} = \left\{ w = (w_1, \ldots, w_N) \,\middle|\, \begin{array}{l} c_n(w) = k, \; c_1(w) = L - k, \\ \\ c_i(w) = \alpha_i \text{ for } 2 \leq i \leq n-1 \end{array} \right\}, \tag{4.1}$$

where $c_i(w)$ counts the number of times the letter i appears in w.

Example 4.1. For example, with $n = N = 3$, $L = 2$, and $\alpha = (1)$, we have $W_0 = \{112, 121, 211\}$, $W_1 = \{123, 132, 213, 231, 312, 321\}$, and $W_2 = \{233, 323, 332\}$.

A *coinversion* of a word w is a pair (i, j), $i < j$, such that $w_i < w_j$, and $\mathrm{coinv}(w)$ as the number of such pairs. In anticipation of our applications in Section 3 where we map $\{1, 2, \ldots, n\}$ to the super-alphabet $\mathcal{A} = \{1, 2, \ldots, \bar{2}, \bar{1}\}$ under total ordering $<_2$, fix ℓ, $2 \leq \ell \leq n$, and we will think of letters $\ell, \ldots, n$ as barred letters. For $w \in W_k$, define

$$\mathrm{quinv}(w) = \mathrm{coinv}(w) + \binom{\alpha_\ell}{2} + \cdots + \binom{\alpha_{n-1}}{2} + \binom{k}{2}. \tag{4.2}$$

This gives the contribution to $\mathrm{quinv}(\sigma)$ from the degenerate word using ordering $<_2$.

For $0 \leq k \leq L$, partition W_k from (4.1) into two subsets. Let $W_k^{\leq}$ (resp. $W_k^{>}$) be those w in W_k whose position of the leftmost n from the left is less than or equal to (resp. greater than) the position of the rightmost 1 from the right ignoring all n's. By convention, take $W_0 = W_0^{>}$ and $W_L = W_L^{\leq}$. Let $W^{\leq} = \bigcup_k W_k^{\leq}$ and $W^{>} = \bigcup_k W_k^{>}$.

For example, with $N = 4$, $L = 3$, and $\alpha = (1)$, we have $W_2^{>} = \{1233, 2133, 2313, 2331\}$, and $W_2^{\leq} = \{1323, 1332, 3123, 3132, 3213, 3231, 3312, 3321\}$.

Recall that the q-analog of an integer n is defined as $[n] := \frac{1-q^n}{1-q}$, the *q-factorial* is $[n]! := [1][2]\cdots[n]$, and the *q-binomial coefficient* is $\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}$. The *$q$-multinomial coefficient* is defined similarly.

Theorem 4.2. *For N, L, α, k and ℓ as above, we have the identities*

$$\sum_{w \in W_k^{>}} q^{\mathrm{quinv}(w)} = q^{\binom{\alpha_\ell}{2}+\cdots+\binom{\alpha_{n-1}}{2}+\binom{k+1}{2}} \begin{bmatrix} N \\ L, \alpha_2, \ldots, \alpha_{n-1} \end{bmatrix} \begin{bmatrix} L-1 \\ k \end{bmatrix}, \tag{4.3}$$

$$\sum_{w \in W_k^{\leq}} q^{\mathrm{quinv}(w)} = q^{\binom{\alpha_\ell}{2}+\cdots+\binom{\alpha_{n-1}}{2}+\binom{k}{2}} \begin{bmatrix} N \\ L, \alpha_2, \ldots, \alpha_{n-1} \end{bmatrix} \begin{bmatrix} L-1 \\ k-1 \end{bmatrix}. \tag{4.4}$$

Example 4.3. As an illustration of Theorem 4.2, consider Example 4.1 and let $\ell = 3$. Then $W_1^{\leq} = \{132, 312, 321\}$ and $W_1^{>} = \{123, 213, 231\}$. One can check that the quinv generating functions of both W_0 and $W_1^{\leq}$ are $1 + q + q^2$, and those of both $W_1^{>}$ and W_2 are $q + q^2 + q^3$. The bijection ϕ claimed in Theorem 4.4 is unique in this case.

We use Theorem 4.2 to show the existence of a quinv-preserving desired involution ϕ on $W \equiv W^{\alpha,L}$.

Theorem 4.4. *There exists a bijection $\phi : W^{>} \to W^{\leq}$ satisfying the following conditions:*

1. *ϕ maps $W_k^{>}$ to $W_{k+1}^{\leq}$ bijectively for $0 \leq k \leq L-1$.*
2. $\mathrm{quinv}(w) = \mathrm{quinv}(\phi(w))$ *for all $w \in W^{\leq}$, and*
3. *The subword of w in the letters $2, \ldots, n-1$ is preserved by ϕ.*

Proof. As a consequence of Theorem 4.2, the quinv generating functions of $W_{k+1}^{>}$ and $W_k^{\leq}$ are equal. The strategy of the proof involves identifying the letters $2, \ldots, n-1$ and is clearly independent of the subword. □

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