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Fundamental expansion of quasisymmetric Macdonald polynomials

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Abstract. The quasisymmetic Macdonald polynomials $G_\gamma(X;q,t)$ were recently introduced by the first and second authors with Haglund, Mason, and Williams to refine the symmetric Macdonald polynomials $P_\lambda(X;q,t)$. We derive an expansion for $G_\gamma(X;q,t)$ in the fundamental basis of quasisymmetric functions.

Keywords: quasisymmetric, Macdonald polynomials, fundamental basis

1 Introduction

The symmetric *Macdonald polynomials* $P_\lambda(X;q,t)$ [10] are a family of functions in $X = \{x_1, x_2, \dots\}$ indexed by partitions, whose coefficients depend on two parameters q and t. The related *nonsymmetric Macdonald polynomials* $E_\mu(X;q,t)$ were introduced shortly after as a tool to study Macdonald polynomials, in a series of papers by Cherednik [2], Macdonald [11], and Opdam [12]. The polynomials $E_\mu(X;q,t)$ are indexed by weak compositions and form a basis for the full polynomial ring $\mathbb{Q}[X](q,t)$. Ferreira [5] and later Alexandersson [1] studied the extension of these to the more general *permuted basement* nonsymmetric Macdonald polynomials $E_\mu^\sigma(X;q,t)$, where $X = \{x_1, \dots, x_n\}$, $\sigma \in S_n$, and the length of μ is n.

The combinatorics of Macdonald polynomials has been actively studied for decades. In [7], Haglund, Haiman, and Loehr gave a combinatorial formula for the *modified Macdonald polynomials*, $\widetilde{H}_\lambda(X;q,t)$, and the *integral form*, $J_\lambda(X;q,t)$. In [8] they subsequently provided a formula for the nonsymmetric Macdonald polynomials $E_\mu(X;q,t)$, which was then broadened to the more general polynomials $E_\mu^\sigma(X;q,t)$ in [1, 5].

In [3], the first and second authors with Haglund, Mason, and Williams introduced a new family of quasisymmetric functions $G_\gamma(X;q,t)$ they named *quasisymmetric Macdonald polynomials*. They showed that $G_\gamma(X;q,t)$ is indeed a quasisymmetric function, and gave a combinatorial formula for $G_\gamma(X;q,t)$ refining the compact formula for P_λ from

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[4]. The Macdonald polynomial $P_\lambda(X;q,t)$ is a sum of these quasisymmetric Macdonald polynomials, and at $q=t=0$, $G_\gamma(X;q,t)$ specializes to the *quasisymmetric Schur functions* $\mathrm{QS}_\gamma(X)$ introduced by Haglund, Luoto, Mason, and van Willigenburg in [9].

The goal of this article is to write an expansion of the polynomials $G_\gamma(X;q,t)$ in the fundamental basis. This basis was introduced by Gessel in [6] and is one of the most common bases of the vector space of quasisymmetric functions. Our main results are the following Theorems, see Section 2 for the relevant definitions.

Theorem 1.1. *Let γ be a strong composition. Then*

$$G_\gamma(X;q,t) = \sum_{\tau \in \mathrm{ST}(\gamma)} t^{\mathrm{coinv}(\tau)} q^{\mathrm{maj}(\tau)} \left(\prod_{\substack{u \in \widehat{\mathrm{dg}}(\gamma) \\ u \notin W(\tau)}} \frac{1-t}{1-q^{\mathrm{leg}(u)+1} t^{\mathrm{arm}(u)+1}} \right)$$
$$\times \sum_{U \subseteq W(\tau)} (-t)^{|U|} \left(\prod_{u \in U} \frac{1-q^{\mathrm{leg}(u)+1} t^{\mathrm{arm}(u)}}{1-q^{\mathrm{leg}(u)+1} t^{\mathrm{arm}(u)+1}} \right) F_{V(\tau)\cup U}.$$

Theorem 1.2. *Let γ be a strong composition. Then*

$$G_\gamma(X;0,t) = \sum_{\tau \in \mathrm{ST}_1(\gamma)} (1-t)^{\omega(\tau)} (-t)^{|\mathrm{Des}(\tau)|} t^{\mathrm{coinv}(\tau) - \mathrm{coinv}(\mathrm{Des}(\tau))} F_{\widehat{\mathrm{V}}(\tau)}.$$

This article proceeds through a series of purely combinatorial proofs and results using a variety of tableaux enumeration techniques, organized as follows. In Section 2, we provide the relevant background. Section 3 provides a proof for Theorem 1.1. In Section 4 we provide an alternative expansion in the Hall–Littlewood case, yielding Theorem 1.2 and a related result for Jack polynomials.

2 Preliminaries and definitions

For a nonnegative integer n, a *weak composition* $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$ is a list of nonnegative integers called the *parts* of α, summing to n, so that $n = |\alpha| = \sum_{i=1}^k \alpha_i$. Let α^+ denote the composition obtained by collapsing the (weak) composition α by removing the zero-parts from α. We call a composition with no non-zero parts a *strong composition*. If $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$, then α is called a *partition*. We denote by $\mathrm{inc}(\alpha)$ the composition obtained by sorting the parts of α in increasing order. Define $\beta(\alpha)$ to be the permutation of *longest length* such that $\beta(\alpha) \circ \alpha = \mathrm{inc}(\alpha)$, where the length of a permutation is the number of inversions in its word representation.

Example 2.1. For $\alpha = (2,1,0,0,3,0,1)$, we have $\alpha^+ = (2,1,3,1)$, $\mathrm{inc}(\alpha) = (0,0,0,1,1,2,3)$, and $\beta(\alpha) = (6,4,3,7,2,1,5)$.

2.1 Quasisymmetric functions

Similar to the symmetric functions, the vector space of *quasisymmetric functions* has several natural bases consisting of functions of fixed degree. We will focus on the *monomial basis* $\{M_S\}$ and the *fundamental basis* $\{F_S\}$, indexed by subsets $S \subset [n-1]$, for each fixed degree n. The monomial basis functions are defined as

$$M_S := \sum_{i_1<i_2<\cdots<i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k} \tag{2.1}$$

where $k = |S|+1$, and α is the (strong) composition corresponding to the subset S.

The fundamental basis functions are defined as

$$F_S := \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ j \in S \implies i_j \neq i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}. \tag{2.2}$$

For example,

$$M_{\{2,3,6\}} = \sum_{i_1<i_2<i_3<i_4} x_{i_1}^2 x_{i_2}^1 x_{i_3}^3 x_{i_4}^2, \quad \text{and} \quad F_{\{2,3,6\}} = \sum_{i_1 \le i_2 < i_3 < i_4 \le i_5 \le i_6 < i_7 \le i_8} x_{i_1} x_{i_2} \cdots x_{i_8}.$$

Let $S \subseteq [n-1]$. It follows that

$$F_S = \sum_{S \subseteq S'} M_{S'}. \tag{2.3}$$

For example, let $n = 8$ and $S = \{1,4\}$. Then

$$F_{\{1,4\}} = M_{\{1,4\}} + M_{\{1,2,4\}} + M_{\{1,3,4\}} + M_{\{1,2,3,4\}}.$$

The goal of this article is to give an expansion of the quasisymmetric Macdonald polynomial $G_\gamma(X;q,t)$, which we present below, in terms of the fundamental quasisymmetric basis. Let γ be a strong composition. The quasisymmetric Macdonald polynomial is defined by the infinite sum

$$G_\gamma(X;q,t) = \sum_{\alpha:\ \alpha^+ = \gamma} E_{\mathrm{inc}(\alpha)}^{\beta(\alpha)}(X;q,t), \tag{2.4}$$

where $E_\mu^\sigma(X;q,t)$ is the *permuted basement Macdonald polynomial* introduced in [5] and further studied in [1]. We will define G_γ combinatorially in the next section. Note that E_μ^σ is a polynomial in k variables, where k is the number of parts of μ, so we actually mean $E_\mu^\sigma(X;q,t) = E_\mu^\sigma(x_1,\ldots,x_k;q,t)$, and $\sigma \in S_k$.

Remark 2.2. It turns out that $E_{\mathrm{inc}(\alpha)}^{\beta(\alpha)}(X;0,t) = E_\alpha^{id}(X;0,t)$. Thus the quasisymmetric Hall–Littlewood polynomials $\mathcal{L}_\alpha(X;t)$, defined in [9] as

$$\mathcal{L}_\gamma(X;t) = \sum_{\alpha:\alpha^+ = \gamma} E_\alpha^{id}(X;0,t),$$

coincide with $G_\gamma(X;0,t)$.

2.2 Tableaux formula for $E^{\sigma}_{\mu}(X;q,t)$

The polynomial $E^{\sigma}_{\mu}(X;q,t)$ has a combinatorial description in the form of a tableaux formula [7]. We review the relevant statistics for general compositions, though we will primarily focus on the case where the parts of μ are arranged in weakly increasing order.

For any weak composition α, define $\mathrm{dg}(\alpha)$, the diagram of α, to be the composition shape in French notation with α_i boxes in column i from left to right. The rows are labeled from bottom to top starting with row 1, and a cell in row r and column c is denoted by coordinates $(r,c) \in \mathrm{dg}(\alpha)$. Define $\widehat{\mathrm{dg}}(\alpha)$ to be the set of cells in $\mathrm{dg}(\alpha)$ not contained in the bottom row. If T is a filling of $\mathrm{dg}(\alpha)$, the entry in a cell $u \in \mathrm{dg}(\alpha)$ is denoted by $T(u)$. Let $x^T = \prod_{u \in \mathrm{dg}(\alpha)} x_{T(u)}$ be the monomial encoding the content of T.

The *reading order* of a diagram is the total order given by reading the entries along the rows from top to bottom, and from left to right within each row. Two cells are said to *attack* each other if they are in the same row, or if they are in adjacent rows where the one above is strictly northeast of the one below. A filling T is considered *non-attacking* if $T(u) \neq T(v)$ for any pair of attacking cells u, v.

For a cell $u \in \mathrm{dg}(\alpha)$, we call $\mathrm{leg}(u)$ the number of cells above u in the same column. We call $\mathrm{arm}(u)$ the number of cells to the right of u in columns whose height does not exceed the height of the column containing u, plus the number of cells to the left of u in columns of height strictly smaller than the height of the column containing u. More precisely, let $u = (r,i)$. Then

$$\mathrm{arm}(u) = |\{(r,j) \in \mathrm{dg}(\alpha) : \; j > i, \alpha_j \leq \alpha_i\}| + |\{(r-1,j) \in \mathrm{dg}(\alpha) : j < i, \alpha_j < \alpha_i\}|$$

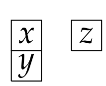
See Figure 2.1. Denote by $\mathrm{South}(u)$ the cell directly below u in the same column. The set of descents of a filling of $\mathrm{dg}(\alpha)$ is

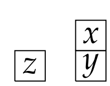
$$\mathrm{Des}(T) = \{u \in \widehat{\mathrm{dg}}(\alpha) \; : \; T(u) > T(\mathrm{South}(u))\},$$

and the *major index* is

$$\mathrm{maj}(T) = \sum_{u \in \mathrm{Des}(T)} \mathrm{leg}(u) + 1.$$

Triples consist of a cell x, the cell $y = \mathrm{South}(x)$ directly below, and a third cell z in the arm of x. If z is in the same row as x, this is called a *type A triple*, and if z is in the same row as y, this is called a *type B triple*, as shown:

Type A: 

Type B: 

Coinversion triples consist of type A triples where the entries are increasing in clockwise orientation, plus type B triples where the entries are increasing in counterclockwise orientation. The $\mathrm{coinv}(T)$ statistic is defined as the total number of all such triples.

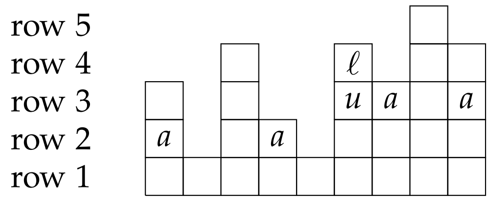
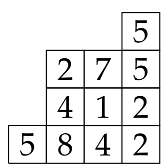


Figure 2.1: The diagram of the composition $(3,1,4,2,1,4,3,5,4)$ and the cells in the leg and the arm of the cell $u = (3,6)$. Here $\mathrm{leg}(u) = 1$ and $\mathrm{arm}(u) = 4$.

Let γ be a strong composition, and let $\sigma = \beta(\gamma)$ be the longest permutation such that $\sigma \circ \gamma = \mathrm{inc}(\gamma)$. Define $\mathrm{NAT}(\gamma)$ to be the set of non-attacking fillings of $\mathrm{dg}(\mathrm{inc}(\gamma))$ such that the entries of the first row are order-equivalent to σ when read in reading order.

Example 2.3. Let $\alpha = (0,4,0,3,1,0,0,3)$. Then $\mathrm{inc}(\alpha) = (0,0,0,0,1,3,3,4)$, $\alpha^+ = (4,3,1,3)$, and $\beta(\alpha) = (7,6,3,1,5,8,4,2)$. The NAT associated to α are fillings of $\mathrm{dg}(\mathrm{inc}(\alpha^+))$ with the bottom row equal to $(5,8,4,2)$: the last ℓ entries of $\beta(\alpha)$, where $\ell = \ell(\alpha^+) = 4$. Notice that $(5,8,4,2)$, is order-equivalent to $(3,4,2,1)$, and $(3,4,2,1) = \beta(\alpha^+)$. Thus in particular, all tableaux associated to α also belong to $\mathrm{NAT}(\alpha^+)$, such as the one below.



$\in \mathrm{NAT}((4,3,1,3))$

By comparing with [1], we obtain the combinatorial formula for $E^{\beta(\alpha)}_{\mathrm{inc}(\alpha)}(X;q,t)$, where α is a weak composition:

$$E^{\beta(\alpha)}_{\mathrm{inc}(\alpha)}(X;q,t) = \sum_{\substack{T \in \mathrm{NAT}(\alpha^+) \\ T \text{ has bottom row } \pi}} \mathrm{wt}(T) x^T, \tag{2.5}$$

where π is the last ℓ entries of $\beta(\alpha)$, for $\ell = \ell(\alpha^+)$. Here, the weight of a (nonstandard) filling T is

$$\mathrm{wt}(T) = q^{\mathrm{maj}(T)} t^{\mathrm{coinv}(T)} \prod_{\substack{u \in \widehat{\mathrm{dg}}(\alpha^+) \\ T(u) \neq T(\mathrm{South}(u))}} \frac{(1-t)}{(1 - q^{\mathrm{leg}(u)+1} t^{\mathrm{arm}(u)+1})} \tag{2.6}$$

Remark 2.4. We have given the tableaux formula for E^{σ}_{μ} where the parts of μ are weakly increasing. A general formula exists (see [1] for details) for an arbitrary composition μ and a permutation σ by keeping track of the "basement" of a filling. Comparing definitions, it follows that for any composition α, the basement of a filling of $\mathrm{dg}(\mathrm{inc}(\alpha))$ can be recovered uniquely from the bottom row of the filling.

2.3 Standard, packed, and non attacking fillings

A *packed* filling is one that uses every integer from the set $\{1, \ldots, m\}$ for some m. Any filling compresses to a packed filling by shifting the alphabet of values in the filling down as necessary: given a set $\{s_1, \ldots, s_k\}$ with $s_1 < \cdots < s_k$, the entries s_i become i.

It is convenient to work with packed fillings in the context of quasisymmetric functions. We consider every packed filling T to be the representative of the family of fillings which compress to T.

Lemma 2.5. *Suppose* $T' \in \mathrm{NAT}(\gamma)$ *compresses to a packed filling* $T \in \mathrm{NAT}(\gamma)$. *Then* $\mathrm{coinv}(T') = \mathrm{coinv}(T)$ *and* $\mathrm{maj}(T') = \mathrm{maj}(T)$.

The proof of the above lemma follows from the fact that the relative order of entries is preserved by compression. Moreover,

$$\sum_{T'} x^{T'} = M_T,$$

the sum being over all fillings T' that compress to the packed filling T, and M_T is the monomial quasisymmetric function corresponding to the content of T. Thus the q,t-generating function of the family of fillings that compress to the packed representative T is the weight of T times M_T. Hence, we may work with the finite set of packed fillings to represent all possible fillings.

From (2.4) and (2.3), we thus obtain

$$G_\gamma(X;q,t) = \sum_{\substack{T \in \mathrm{NAT}(\gamma) \\ T \text{ packed}}} q^{\mathrm{maj}(T)} t^{\mathrm{coinv}(T)} M_T \prod_{\substack{u \in \widehat{\mathrm{dg}}(\gamma) \\ T(u) \neq T(\mathrm{South}(u))}} \frac{(1-t)}{(1 - q^{\mathrm{leg}(u)+1} t^{\mathrm{arm}(u)+1})}. \tag{2.7}$$

Example 2.6. For $\gamma = (1,2)$, all the packed nonattacking fillings in $\mathrm{NAT}(\gamma)$ are shown below with their weights, to obtain

$$G_{(1,2)} = M_{\{1\}} + \frac{(1-t)(1+t+qt)}{1-qt^2} M_{\{1,1\}}.$$

$$\gamma = (1,2): \quad \begin{array}{|c|c|} \hline & 2 \\ \hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline & 3 \\ \hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline & 2 \\ \hline 1 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline & 1 \\ \hline 2 & 3 \\ \hline \end{array}$$

$$M_{\{1\}} \qquad \tfrac{qt(1-t)}{1-qt^2} M_{\{1,1\}} \qquad \tfrac{(1-t)}{1-qt^2} M_{\{1,1\}} \qquad \tfrac{t(1-t)}{1-qt^2} M_{\{1,1\}}$$

Standard fillings (or standard tableaux), denoted by $\mathrm{ST}(\gamma)$, are fillings of $\mathrm{dg}(\mathrm{inc}(\gamma))$ such that every element in the set $\{1, \ldots, n\}$ appears exactly once, where $n = |\gamma|$. Thus there is a bijection $\tau : \mathrm{dg}(\mathrm{inc}(\gamma)) \to \{1, \ldots, n\}$ between cells of $\mathrm{dg}(\mathrm{inc}(\gamma))$ and the entries $\{1, \ldots, n\}$, and so we can slightly abuse notation and refer to both a cell and its entry when we work with standard tableaux.

Define the standardization map $\mathrm{std}\colon \mathrm{NAT}(\gamma) \to \mathrm{ST}(\gamma)$ as follows. For $T \in \mathrm{NAT}(\gamma)$, let $\tau = \mathrm{std}(T)$ be the unique standard filling in $\mathrm{ST}(\gamma)$ that preserves the relative order of the original tableau, and where the reading order is used to break ties. It is straightforward to check that if τ is the standardization of T, then $\mathrm{coinv}(T) = \mathrm{coinv}(\tau)$ and $\mathrm{maj}(T) = \mathrm{maj}(\tau)$. See Example 2.7 for the standardization $\mathrm{std}(T)$ of $T \in \mathrm{NAT}((1,4,3))$.

Let $T \in \mathrm{NAT}(\gamma)$ with standardization $\tau = \mathrm{std}(T)$, and $n = |\gamma|$. Define the *reading word* of T to be the sequence of entries of T listed in reading order, denoted by $\mathrm{rw}(T)$. The reading word of τ is thus a permutation of $\{1, \ldots, n\}$. Define $\mathrm{ID}(\tau)$ to be the *inverse descent set*, where $i \in \mathrm{ID}(\tau)$ if $i+1$ precedes i in $\mathrm{rw}(\tau)$.

Example 2.7. We show $T \in \mathrm{NAT}((1,4,3))$ and the corresponding standardization $\tau = \mathrm{std}(T)$. We have $\mathrm{rw}(\tau) = (3,5,1,8,6,2,7,4)$, and $\mathrm{ID}(\tau) = \{2,4,7\}$.

$$T = \begin{array}{|c|c|c|} \hline & & 2 \\ \hline & 3 & 1 \\ \hline & 5 & 3 \\ \hline 1 & 4 & 2 \\ \hline \end{array} \qquad \tau = \mathrm{std}(T) = \begin{array}{|c|c|c|} \hline & & 3 \\ \hline & 5 & 1 \\ \hline & 8 & 6 \\ \hline 2 & 7 & 4 \\ \hline \end{array}$$

For $\tau \in \mathrm{ST}(\gamma)$, define $V(\tau) \subseteq [n-1]$ to be the set of entries such that $i \in V(\tau)$ if $i \in \mathrm{ID}(\tau)$ or if i and $i+1$ are in cells that attack each other. Define $W(\tau) = \{i \in \tau \ : \ \mathrm{South}(i) = i+1\}$ to be the set of entries i with $i+1$ directly below. Note that $V(\tau) \cap W(\tau) = \emptyset$.

Given a standard filling τ with n cells, the cells labelled from 1 to $n-1$ are partitioned into three blocks:

- The cells with entries in $V(\tau)$, namely those cells where $i \in \mathrm{ID}(\tau)$ **OR** i and $i+1$ are in attacking cells.
- The cells with entries in $W(\tau)$, namely those cells where $i+1$ is directly below i.
- The rest of the cells with entries in $[n-1] \backslash (V(\tau) \cup W(\tau))$.

Let $\gamma \models n$. We consider the pre-image in $\mathrm{NAT}(\gamma)$ of standard fillings $\tau \in \mathrm{ST}(\gamma)$. For a choice of $V(\tau) \subseteq S \subseteq [n-1]$, define a *destandardization map* $\delta_S(\tau) \ : \ \mathrm{dg}(\gamma) \to \mathbb{Z}$ as follows. Let α be the (strong) composition corresponding to the set S. Let w be the word containing the content associated to α in weakly decreasing order, given by $w = (1^{\alpha_1}, 2^{\alpha_2}, \ldots, k^{\alpha_k})$ where α has k parts. Define $\delta_S(\tau) := w \circ \tau$ to be the unique filling of $\mathrm{dg}(\gamma)$ with content α that standardizes to τ.

Example 2.8. Consider the standard tableau τ from Example 2.7. $\mathrm{ID}(\tau) = \{2,4,7\}$, and the set of indices i such that i and $i+1$ are in cells that attack each other is $\{6\}$, so $V(\tau) = \{2,4,6,7\}$. Thus, S can be any subset of $[7]$ containing $V(\tau)$. We show some examples of δ_S for various choices of S:

$$\delta_{\left\{\substack{1,2,3,\\4,5,6,7}\right\}}(\tau) = \begin{array}{ccc} & & 3 \\ & 5 & 1 \\ & 8 & 6 \\ 2 & 7 & 4 \end{array} \qquad \delta_{\left\{\substack{1,2,4,\\5,6,7}\right\}}(\tau) = \begin{array}{ccc} & & 3 \\ & 4 & 1 \\ & 7 & 5 \\ 2 & 6 & 3 \end{array} \qquad \delta_{\left\{\substack{2,4,\\6,7}\right\}}(\tau) = \begin{array}{ccc} & & 2 \\ & 3 & 1 \\ & 5 & 3 \\ 1 & 4 & 2 \end{array}$$

The following lemma gives the weight of a destandardized filling in terms of its standardization.

Lemma 2.9. *Let $\gamma \models n$, $\tau \in \mathrm{ST}(\gamma)$, and S such that $V(\tau) \subseteq S \subseteq [n-1]$. Then*

$$\mathrm{wt}(\delta_S(\tau)) = t^{\mathrm{coinv}(\tau)} q^{\mathrm{maj}(\tau)} \prod_{\substack{u \in \widehat{\mathrm{dg}}(\gamma) \\ u \notin W(\tau)}} \frac{1-t}{1-q^{\mathrm{leg}(u)+1} t^{\mathrm{arm}(u)+1}} \prod_{u \in S \cap W(\tau)} \frac{1-t}{1-q^{\mathrm{leg}(u)+1} t^{\mathrm{arm}(u)+1}}.$$

For a strong composition $\gamma \models n$ where $\ell(\gamma)$ is the number of parts, define $h(\gamma) = n - \ell(\gamma)$ to be the number of cells in $\mathrm{dg}(\gamma)$ without its bottom row. Note that $h(\gamma)$ is the number of cells in $\widehat{\mathrm{dg}}(\gamma)$.

3 Proof of Theorem 1.1

We will start with a proof for the $q = 0$ specialization of Theorem 1.1 to develop the main ideas of the proof. The proof for the general q case follows the same strategy.

3.1 The $q = 0$ case

We assume $q = 0$ throughout this section. Let γ be a strong composition. When we compute compute $G_\gamma(X; 0, t)$, the only surviving tableaux in (2.7) are those with an empty descent set, which means the entries must be non-increasing as we read the columns from bottom to top. We denote the subsets of $\mathrm{NAT}(\gamma)$ and $\mathrm{ST}(\gamma)$ that have nonzero weight at $q = 0$ by $\mathrm{NAT}_0(\gamma)$ and $\mathrm{ST}_0(\gamma)$, respectively. We will prove the following.

$$G_\gamma(X; 0, t) = \sum_{\tau \in \mathrm{ST}_0(\gamma)} t^{\mathrm{coinv}(\tau)} (1-t)^{h(\gamma) - |W(\tau)|} \sum_{U \subseteq W(\tau)} (-t)^{|U|} F_{V(\tau) \cup U}. \tag{3.1}$$

Observe the following. The denominator in the product of (2.7) vanishes, and the weight of each $T \in \mathrm{NAT}_0(\gamma)$ becomes

$$\mathrm{wt}(T) = t^{\mathrm{coinv}(T)} (1-t)^{|\{u \in \widehat{\mathrm{dg}}(\lambda) \,:\, T(\mathrm{South}(u)) \neq T(u)\}|}.$$

Moreover, for $\tau \in \mathrm{ST}_0(\gamma)$, Lemma 2.9 specializes to

$$\mathrm{wt}(\delta_S(\tau)) = t^{\mathrm{coinv}(\tau)} (1-t)^{h(\gamma) - |W(\tau) \setminus S|}. \tag{3.2}$$

We are now ready to prove Theorem 1.1 at $q = 0$.

Proof of (3.1). From the definition, we have

$$\begin{aligned}
G_\gamma(X;0,t) &= \sum_{T\in \mathrm{NAT}_0(\gamma)} \mathrm{wt}(T)x^T \\
&= \sum_{\tau\in \mathrm{ST}_0(\gamma)} \sum_{V(\tau)\subseteq S\subseteq [n-1]} \mathrm{wt}(\delta_S(\tau))M_S \\
&= \sum_{\tau\in \mathrm{ST}_0(\gamma)} \sum_{V(\tau)\subseteq S\subseteq [n-1]} t^{\mathrm{coinv}(\tau)}(1-t)^{h(\gamma)-|W(\tau)\setminus S|}M_S \\
&= \sum_{\tau\in \mathrm{ST}_0(\gamma)} t^{\mathrm{coinv}(\tau)}(1-t)^{h(\gamma)-|W(\tau)|} \sum_{V(\tau)\subseteq S\subseteq [n-1]} (1-t)^{|S\cap W(\tau)|}M_S \qquad (3.3)
\end{aligned}$$

where the third line is by (3.2). We then reformulate the second summation:

$$\sum_{V(\tau)\subseteq S\subseteq [n-1]} (1-t)^{|S\cap W(\tau)|}M_S = \sum_{W\subseteq W(\tau)} (1-t)^{|W|} \sum_{\substack{V(\tau)\subseteq S\subseteq [n-1] \\ S\cap W(\tau)=W}} M_S$$

By the binomial theorem, $(1-t)^{|W|} = \sum_{U\subseteq W}(-t)^{|U|}$. Plugging in gives

$$\begin{aligned}
\sum_{W\subseteq W(\tau)} \sum_{U\subseteq W} (-t)^{|U|} \sum_{\substack{V(\tau)\subseteq S\subseteq [n-1] \\ S\cap W(\tau)=W}} M_S &= \sum_{U\subseteq W(\tau)} (-t)^{|U|} \sum_{W\supseteq U} \sum_{\substack{V(\tau)\subseteq S\subseteq [n-1] \\ S\cap W(\tau)=W}} M_S \\
&= \sum_{U\subseteq W(\tau)} (-t)^{|U|} \sum_{V(\tau)\cup U\subseteq S\subseteq [n-1]} M_S \\
&= \sum_{U\subseteq W(\tau)} (-t)^{|U|} F_{V(\tau)\cup U},
\end{aligned}$$

which completes the proof. □

3.2 The general q case

Let γ be a strong composition of n. Recall that the weight of a tableau $T\in \mathrm{NAT}(\gamma)$ is

$$\mathrm{wt}(T) = t^{\mathrm{coinv}(T)}q^{\mathrm{maj}(T)} \prod_{\substack{u\in \widehat{\mathrm{dg}}(\gamma) \\ T(\mathrm{South}(u))\neq T(u)}} \frac{1-t}{1-q^{\mathrm{leg}(u)+1}t^{\mathrm{arm}(u)+1}}.$$

As in the $q=0$ case, we split the set $\mathrm{NAT}(\gamma)$ via the destandardization map δ_S into disjoint components indexed by their representative standard fillings in $\mathrm{ST}(\gamma)$. For $\tau\in \mathrm{ST}(\gamma)$ and any $V(\tau)\subseteq S\subseteq [n-1]$, again, the only cells that will potentially change the weight of the destandardized tableau $\delta_S(\tau)$ are those in $W(\tau)$. This is because when an entry $i\in\tau$ has $i+1$ above it, it is possible for that pair to destandardize to the same value if $\delta_S(\tau)\circ\tau^{-1}(i) = \delta_S(\tau)\circ\tau^{-1}(i+1)$, changing the product in the weight function.

We require the following lemma.

Lemma 3.1. *Let W be any subset of the cells of* $\mathrm{dg}(\lambda)$. *Then*

$$\prod_{u \in W} \frac{1-t}{1-q^{\mathrm{leg}(u)+1}t^{\mathrm{arm}(u)+1}} = \sum_{U \subseteq W} (-t)^{|U|} \left(\prod_{u \in U} \frac{1-q^{\mathrm{leg}(u)+1}t^{\mathrm{arm}(u)}}{1-q^{\mathrm{leg}(u)+1}t^{\mathrm{arm}(u)+1}} \right).$$

We can now prove the main result.

Proof of Theorem 1.1. The proof is now completely analogous to the $q = 0$ case with addition of Lemma 3.1 in place of the binomial theorem. □

4 Further simplifications and specializations

In this section, we further simplify the result for the Hall–Littlewood case ($q = 0$) from Section 3.1. First, we introduce some notation. Let

$$\mathrm{ST}_1(\gamma) = \{\tau \in \mathrm{ST}(\gamma) \colon i \in \mathrm{Des}(\tau) \implies \mathrm{South}(i) = i-1\}.$$

That is, reading down columns, values can decrease by at most 1 per cell. We may send any element of $\tau' \in \mathrm{ST}_1(\gamma)$ to an element of $\tau \in \mathrm{ST}_0(\gamma)$ by sorting entries within their columns to become weakly decreasing from bottom to top. In this case the cells containing descents are sent to some $U \subset W(\tau)$, though the values in the cells may change. To make this sorting function invertible, we need to keep track of U. For any $U \subseteq W(\tau)$, consider the map ι_U that sends $\tau \in \mathrm{ST}_0(\gamma)$ to $\tau' \in \mathrm{ST}_1(\gamma)$ by reversing the order of certain consecutive values in columns of τ. Specifically, for each maximal set $\{i, i+1, \ldots, i+k-1\} \in U$, the values in $[i, i+k]$ are reversed so that $\{i+1, i, \ldots, i+k\}$ is similarly maximal in $\mathrm{Des}(\tau')$. All other values are fixed. Note that since the cells of standard filling τ are identified with the values they contain, we represent U by a subset of $[n]$. Further, since $\tau \in \mathrm{ST}_0(\gamma)$ is the result of sorting the entries within the columns of τ', τ' has a unique preimage.

Next, for $\tau' \in \mathrm{ST}_1(\gamma)$, let

$$\mathrm{coinv}(\mathrm{Des}(\tau')) := \sum_{u \in \mathrm{Des}(\tau')} \mathrm{arm}(u),$$
$$\omega(\tau') := h(\gamma) - |\{i \in \tau' \colon i \text{ and } i+1 \text{ share a column}\}|.$$

Lastly, we replace $V(\tau')$ with a new set in the context of $\mathrm{ST}_1(\gamma)$. The *descent group of* i is the maximal connected set of cells in the column of i such that every cell is a decent except the bottom cell. By construction, every cell is contained in a unique descent group. We say *i attacks $i+1$ through descents* if a cell in the descent group of i attacks a cell in the descent group of $i+1$. Notice that if i attacks $i+1$ in $\tau' \in \mathrm{ST}_1(\gamma)$, then i must be at the top of its descent group and $i+1$ must be at the bottom of its descent group. Define $\widehat{\mathrm{V}}(\tau')$ as the set of i in τ' such that $i \in \mathrm{ID}(\tau')$ or i attacks $i+1$ through descents. Since i attacking $i+1$ implies i attacks $i+1$ by descents, it follows that $V(\tau') \subseteq \widehat{\mathrm{V}}(\tau')$.

Example 4.1. Consider $\tau \in \mathrm{ST}_0((1,4,3))$ and $\tau' = \iota_{\{3,4\}}(\tau) \in \mathrm{ST}_1((1,4,3))$.

$$\tau = \begin{array}{ccc} & & \boxed{3} \\ & \boxed{1} & \boxed{4} \\ & \boxed{2} & \boxed{5} \\ \boxed{6} & \boxed{8} & \boxed{7} \end{array} \qquad \tau' = \begin{array}{ccc} & & \boxed{5} \\ & \boxed{1} & \boxed{4} \\ & \boxed{2} & \boxed{3} \\ \boxed{6} & \boxed{8} & \boxed{7} \end{array}$$

Here, $\mathrm{coinv}(\mathrm{Des}(\tau')) = 2$, $\omega(\tau') = 2$, $\mathrm{ID}(\tau') = \{3,4,7\}$, and $\widehat{\mathrm{V}}(\tau') = \{2,3,4,5,6,7\}$.

Theorem 1 (Theorem 1.2)**.** *Let γ be a strong composition. Then*

$$G_\gamma(X;0,t) = \sum_{\tau \in \mathrm{ST}_1(\gamma)} (1-t)^{\omega(\tau)} (-t)^{|\mathrm{Des}(\tau)|} t^{\mathrm{coinv}(\tau) - \mathrm{coinv}(\mathrm{Des}(\tau))} F_{\widehat{\mathrm{V}}(\tau)}.$$

Proof. The proof is a matter of changing the order of summations in (3.1), applying ι_U, tracking the changes to statistics, and combining the sums. Combining the two sums completes the proof. □

4.1 Jack specialization

We also consider the specialization of $G_\gamma(X;q,t)$ to the setting of Jack polynomials, from which we immediately get a new definition of a *quasisymmetric Jack polynomial*. Recall that the Jack polynomial indexed by a partition λ with parameter α is a symmetric polynomial that can be obtained from

$$J_\lambda(X;\alpha) = \lim_{t \to 1^-} \left(\prod_{u \in \mathrm{dg}(\lambda)} \frac{1 - t^{\mathrm{arm}(u)+1} t^{\alpha\, \mathrm{leg}(u)}}{1-t} \right) P_\lambda(X; t^\alpha, t).$$

Thus define the quasisymmetric Jack polynomial indexed by a strong composition γ as

$$G_\gamma(X;\alpha) = \lim_{t \to 1^-} \left(\prod_{u \in \mathrm{dg}(\lambda)} \frac{1 - t^{\mathrm{arm}(u)+1} t^{\alpha\, \mathrm{leg}(u)}}{1-t} \right) G_\gamma(X; t^\alpha, t). \tag{4.1}$$

Using Theorem 1.1 we obtain the following corollary.

Corollary 4.2. *The quasisymmetric Jack polynomial has the following fundamental expansion:*

$$G_\gamma(X;\alpha) = \sum_{\tau \in \mathrm{ST}(\gamma)} \left(\prod_{u \in W(\tau)} (\alpha(\mathrm{leg}(u)+1) + \mathrm{arm}(u)+1) \right) \times \sum_{U \subseteq W(\tau)} (-1)^{|U|} \left(\prod_{u \in U} \frac{\alpha(\mathrm{leg}(u)+1) + \mathrm{arm}(u)}{\alpha(\mathrm{leg}(u)+1) + \mathrm{arm}(u)+1} \right) F_{V(\tau) \cup U}.$$

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