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SPECIALIZATIONS OF GENERALIZED LAGUERRE POLYNOMIALS

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ABSTRACT. Three specializations of a set of orthogonal polynomials with "8 different q's" are given. The polynomials are identified as q-analogues of Laguerre polynomials, and the combinatorial interpretation of the moments give infinitely many new Mahonian statistics on permutations.

1. Introduction.

The Laguerre polynomials $L_n^\alpha(x)$ have been extensively studied, analytically [E] and combinatorially [F-S],[V]. There is also a classical set of q-Laguerre polynomials, [M]. Recently, a set of orthogonal polynomials generalizing the Laguerre polynomials has been studied [Si-St]. These polynomials in some sense have "8 different q's". Various specializations of them give many orthogonal polynomials associated with different types of set partitions. The purpose of this paper is to present the specializations which are true q-analogs of $L_n^0(x)$. By this we mean that the n^{th} moments, instead of being $n!$, are basically $n!_q$.

The combinatorial description of these moments leads to new Mahonian statistics on permutations, in fact infinitely many such statistics. These statistics are given in Theorems 2, 3 and 4.

We shall use the terminology and notation found in [G-R], and let

$$[n]_q = \frac{1-q^n}{1-q}, \quad [n]_{r,s} = \frac{r^n - s^n}{r-s}.$$

2. The polynomials and their moments.

Any set of monic orthogonal polynomials satisfies the three term recurrence relation

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x).$$

For the set of orthogonal polynomials with 8 different "q's" considered in [Si-St], the coefficients are

$$(2.1) \qquad b_n = a[n+1]_{r,s} + b[n]_{t,u}, \quad \lambda_n = ab[n]_{p,q}[n]_{v,w}.$$

The polynomials defined by (1) are the generalized Laguerre polynomials that we will specialize.

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The fundamental combinatorial fact (Theorem 1) that we need here concerns the moments for these polynomials. They are generating functions for permutations according to certain statistics. For the definition of these statistics, it is convenient to represent a permutation σ as a word $\sigma(1)\sigma(2)\cdots\sigma(n)$ consisting of increasing runs, separated by the descents of the permutation. For example, the permutation $\sigma = 26|359|7|148$ has 4 runs and 3 descents. The elements $\sigma(i)$ of σ fall into four classes: the elements which begin runs of length ≥ 2 (openers), the elements which close runs of length ≥ 2 (closers), the elements which form singleton runs (singletons), and the elements which continue runs (continuation). We shall abbreviate these classes of elements "op", "clos", "sing", and "cont" respectively. In the example, the openers are {2,3,1}, the closers are {6,9,8}, the singleton is {7}, and the continuation elements are {5,4}.

Definition 1. *For $\sigma \in S_n$, the statistics $lsg(\sigma)$ and $rsg(\sigma)$ are defined by*

$$lsg(\sigma) = \sum_{i=1}^{n} lsg(i), \qquad rsg(\sigma) = \sum_{i=1}^{n} rsg(i),$$

where $lsg(i)$ = the number of runs of σ strictly to the left of i which contain elements smaller and greater than i, and $rsg(i)$ = the number of runs of σ strictly to the right of i which contain elements smaller and greater than i.

In the example, lsg(7)=1, from the run 359 to the left of 7, and $lsg(\sigma) = 6$. The theorem in [Si-St] computes the joint distribution of the lsg and rsg statistics on the 4 types of elements of σ. We put $lsg(sing)(\sigma)$ equal to the sum of lsg(i) over all i which are singletons. Analogously we define lsg and rsg on the other three classes of points of σ. We also use $run(\sigma)$ for the number of runs of σ.

Theorem 1. *The n^{th} moment μ_n for the polynomials given in (1) is*

$$\mu_n = \sum_{\sigma \in S_n} r^{\mathrm{lsg(sing)}(\sigma)} s^{\mathrm{rsg(sing)}(\sigma)} t^{\mathrm{lsg(cont)}(\sigma)} u^{\mathrm{rsg(cont)}(\sigma)} p^{\mathrm{lsg(op)}(\sigma)} q^{\mathrm{rsg(op)}(\sigma)}$$
$$v^{\mathrm{lsg(clos)}(\sigma)} w^{\mathrm{rsg(clos)}(\sigma)} a^{\mathrm{run}(\sigma)} b^{n-\mathrm{run}(\sigma)}.$$

For the example $\sigma = 26|359|7|148$ the weight is $r^1 s^1 t^3 u^1 p^1 q^2 v^1 w^2 a^4 b^5$. The parameters {r, s, t, u, p, q, v, w} are the so-called "8 different q's."

We will be concerned with the case when the moments in Theorem 1 are multiples of $n!_q = [n]_q[n-1]_q \cdots [1]_q$. It is clear from (2.1) that the moments are fixed under the interchange of {r, s}, {t, u}, {p, q}, and {v, w}, and also fixed if p and q are interchanged with v and w. Also, the parameter b can be rescaled to 1.

3. The specializations.

In this section we state three different specializations of the polynomials in (1). Each of these three cases will have moments which are basically $n!_q$.

First we choose the parameters so that the polynomials coincide with the monic little q-Jacobi polynomials [G-R, p. 166], $p_n(x(1-q)q, q^\alpha, 0; q)$, which have

$$\text{(3.1)} \qquad b_n = q^{n-1}[n+1+\alpha]_q + q^{n+\alpha-1}[n]_q, \qquad \lambda_n = q^{2n-3+\alpha}[n]_q[n+\alpha]_q.$$

The appropriate specialization occurs only for $\alpha = 0$, $p_n(xq(1-q), 1, 0)$, and is $r = t = p = v = q^2$, $s = u = q = w$, $a = 1/q$, $b = 1$. The measure for $p_n(x; a, b, q)$ is purely discrete, with masses of $\frac{(aq)^i (bq;q)_i (aq;q)_\infty}{(q;q)_i (abq^2;q)_\infty}$ at $x = q^i$. An easy calculation shows that the moments are given by $\mu_n = \frac{(aq;q)_n}{(abq^2;q)_n}$. Thus for $p_n(xq(1-q), 1, 0)$, we have $\mu_n = q^{-n} n!_q$. Based upon these remarks, Theorem 1 becomes the following theorem.

Theorem 2. *For $\sigma \in S_n$, let*

$$s(\sigma) := n - run(\sigma) + 2lsg(\sigma) + rsg(\sigma).$$

Then

$$\sum_{\sigma \in S_n} q^{s(\sigma)} = n!_q.$$

Moreover, we see from the symmetry of (2.1) with respect to the 4 pairs of "q's" that Theorem 2 holds for 16 statistics related to $s(\sigma)$. These 16 are obtained by choosing the coefficients 1 and 2 for lsg and rsg independently for the four types of elements of σ. This means for example that

$$\begin{aligned} s'(\sigma) = n - \mathrm{run}(\sigma) &+ \mathrm{lsg(sing)}+2\mathrm{rsg(sing)}+\mathrm{lsg(op)}+2\mathrm{rsg(op)} \\ &+ 2\mathrm{lsg(cont)}+\mathrm{rsg(cont)}+2\mathrm{lsg(clos)}+\mathrm{rsg(clos)} \end{aligned}$$

also satisfies Theorem 2. We will see later (Proposition 1) that in fact there are infinitely many statistics related to $s(\sigma)$.

We state here the explicit formula for the polynomials, which is just the definition of the little q-Jacobi polynomials:

(3.2), (r=t=q^2, s=u=q, a=1/q, b=1)

$$p_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q [n]_q \cdots [n-k+1]_q (-1)^k x^{n-k} q^{\binom{k-1}{2}-1}.$$

For our second specialization we consider

(3.3) $$b_n = q^{n+1}[n+1]_q + q^{n-1}[n]_q, \quad \lambda_n = q^{2n-1}[n]_q[n]_q.$$

The appropriate values are $r = t = p = v = q^2$, $s = u = q = w$, $a = q$, $b = 1$. The polynomials turn out to be a sum of two little q-Jacobi polynomials,

$$\begin{aligned} p_n(x) = n!_q q^{\binom{n}{2}} (-1)^n \Bigg[{}_2\phi_1 &\left(\begin{matrix} q^{-n} & 0; \\ & q \end{matrix} \; q, \; xq(1-q) \right) \\ -(1-q^n) {}_2\phi_1 &\left(\begin{matrix} q^{1-n} & 0; \\ & q^2 \end{matrix} \; q, \; xq(1-q) \right) \Bigg], \end{aligned}$$

or equivalently

(3.4), (r=t=q^2, s=u=q, a=q, b=1)

$$p_n(x) = x^n + \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q [n]_q \cdots [n-k+2]_q ([n-k]_q + q^n)(-1)^k x^{n-k} q^{\binom{k}{2}} + (-1)^n q^{\binom{n+1}{2}} n!_q.$$

We omit the proof of these formulas. It is a verification of the recurrence relation (1) from the recurrence relation for the little q-Jacobi polynomials.

Since these polynomials do not explicitly appear in the literature, we cannot compute the moments by quoting the relevant facts about their measure. Nonetheless, the moments and measure are easily determined.

Proposition 1. *The moments for the polynomials in (3) are $\mu_0 = 1$, and $\mu_n = q\,n!_q, n > 0$. The measure is purely discrete, with masses of $q^i(q;q)_\infty/(q;q)_{i-1}$ at $q^{i-1}/(1-q)$, $i \geq 1$, and a mass of $1-q$ at 0.*

Proof. The q-binomial theorem clearly shows that the total mass $\mu_0 = (1-q)+q = 1$. It also implies

$$\begin{aligned}\mu_n &= \sum_{i=1}^{\infty} \frac{q^{in}q^i(q;q)_\infty}{(1-q)^n(q;q)_{i-1}} + (1-q)\delta_{n,0} \\ &= qn!_q + (1-q)\delta_{n,0}.\end{aligned}$$

Thus the stated measure has the right moments. To show that the polynomials are orthogonal with respect to this measure, we show that the linear functional defined by the measure annihilates $p_1, p_2, \cdots$. Then the three term recurrence shows that the polynomials are orthogonal. However, given the explicit formula (3) for the polynomials, it is also easy to check that the moments do annihilate $p_1, p_2, \cdots$. □

We then get a companion theorem to Theorem 2. It also has 16 equivalent versions.

Theorem 3. *For $\sigma \in S_n$, let*

$$s(\sigma) := run(\sigma) - 1 + 2lsg(\sigma) + rsg(\sigma).$$

Then

$$\sum_{\sigma \in S_n} q^{s(\sigma)} = n!_q.$$

We remark that Theorems 2 and 3 are valid for an infinite number of variations of the statistic $s(\sigma)$. It is easy to verify that for each $\sigma \in S_n$

$$lsg(op)(\sigma) + rsg(op)(\sigma) = lsg(clos)(\sigma) + rsg(clos)(\sigma). \tag{3.5}$$

(In fact there is a specialization of $\{r, s, t, u, p, q, v, w\}$ with one free parameter giving (2) that corresponds to (4).) Therefore in the definition of $s(\sigma)$ the pairs of coefficients $\{1, 2\}$ and $\{1, 2\}$ on the openers and closers can be replaced with $\{1 + c, 2 + c\}$ and $\{1 - c, 2 - c\}$. This provides a variation of Theorems 2 and 3 for each choice of the real parameter c. For example, $c = 1$ gives the unusual choice of coefficients $\{2, 3\}$ and $\{0, 1\}$.

Our third choice for specialization is the set of the classical q-Laguerre polynomials $L_n^\alpha(x(1-q); q)$ [M], [G-R, p. 194], whose monic form has

$$b_n = q^{-2n-\alpha}[n]_q + q^{-2n-1-\alpha}[n+1+\alpha]_q, \quad \lambda_n = q^{1-4n-2\alpha}[n]_q[n+\alpha]_q. \tag{3.6}$$

The appropriate values are $r = t = p = v = q^{-2} = b$, $s = u = q = w := q^{-1} = a$ for $L_n^0(x(1-q), q)$. Again a measure of these polynomials is explicitly known [M, Th.1], and the moments for $L_n^\alpha(x(1-q); q)$ can be found as

$$\mu_n = (q^{\alpha+1}; q)_n q^{-n\alpha - \binom{n+1}{2}}/(1-q)^n.$$

For $\alpha = 0$ this is $q^{-\binom{n+1}{2}} n!_q$. However the combinatorial version of this theorem is equivalent to Theorem 2, if q is replaced $1/q$. Thus no new combinatorial theorem results. We record here the explicit formula for the monic form of $L_n^\alpha(x(1-q);q)$,

$$(3.7) \qquad p_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q [n+\alpha]_q \cdots [n+\alpha-k+1]_q (-1)^k x^{n-k} q^{k(k-\alpha-2n)}.$$

4. The "odd' polynomials.

If $r = p$, $s = q$, $t = v$, and $u = w$, then the polynomials defined by (1) are the "even" polynomials for the polynomials defined by (see [C, p.41])

$$b_n = 0, \quad \lambda_{2n} = b[n]_{t,u}, \quad \lambda_{2n+1} = a[n+1]_{r,s}.$$

The odd polynomials have the coefficients

$$(4.1) \qquad b_n = a[n+1]_{r,s} + b[n+1]_{t,u}, \quad \lambda_n = ab[n+1]_{r,s}[n]_{t,u}.$$

The moments for these odd polynomials satisfy $\mu_n(odd) = \mu_{n+1}(even)/\mu_1(even)$. Since all of our specializations in §2 satisfied $r = p$, $s = q$, $t = v$, and $u = w$, these "odd" polynomials also have moments which are multiples of $(n+1)!_q$. There is a version of Theorem 1 for the "odd polynomials" which yields more statistics related to $s(\sigma)$. We do not state this combinatorial theorem here, rather in this section we state what these odd polynomials are, give their moments, and state in Theorem 4 what the statistics related to $s(\sigma)$ are. Clearly the odd polynomials are analogues of the Laguerre polynomials $L_n^1(x)$.

We keep the parameters r, s, t, u. This specialization gives the "even" and "odd" polynomials a combinatorial interpretation as weighted versions of injective maps (see [F-S], $L_n^0(x)$ and $L_n^1(x)$). This family with "4 q's" also contains other families of orthogonal polynomials of combinatorial interest.

We list here the odd polynomials for the three cases in §3, and the respective moments.

(1) little q-Jacobi $p_n(x(1-q)q, q, 0; q)$, $\mu_n = q^{-n}\,(n+1)!_q$,
(2) little q-Jacobi $p_n(x(1-q), q, 0; q)$, $\mu_n = (n+1)!_q$,
(3) q-Laguerre $L_n^1(x(1-q);q)$, $\mu_n = q^{-(n^2+3n)/2}\,(n+1)!_q$.

The combinatorial theorem that results is Theorem 4. We shall need the definition of another statistic $n(\sigma)$. Partition the elements of $\sigma \in S_n$ into three classes: elements to the left of the run with 1, elements in the run with 1, and those to the right of the run with 1. Suppose that the left to right minima of the portion of σ to the right of the run with 1 contains no singletons. Put $n(\sigma) = 0$. Otherwise, let s be the last singleton to the right which is a left to right minima. Also let r be the maximum of the run with 1. Then let $n(\sigma)$ be the number of elements of σ which lie to the left of the run with 1, and lie between s and r.

Theorem 4. *For $\sigma \in S_n$, let*

$$s(\sigma) := run(\sigma) - 1 + 2lsg^*(\sigma) + rsg^*(\sigma) + n(\sigma),$$

where lsg and rsg* are defined by changing the definitions of lsg and rsg: for the closers and singletons, the run containing the element 1 is ignored in the calculation of lsg* and rsg*. For the openers and continuations, the the run containing 1 is always counted in lsg* (if it is to the left), or in rsg* (if it is to the right).*

Theorem 4 also holds if $run(\sigma) - 1$ is replaced by $n - run(\sigma)$. Moreover, the role of closers and openers can be interchanged in Theorem 4, and there are also 16 variations, although complicated ones. A version of Theorem 4 holds for permutations of length $n+1$ with the following condition: the run with 1 also contains $n+1$, and there are no left to right singleton minima after the run with 1. There are $n!$ such permutations in S_{n+1}.

Finally, we remark that these specializations are the only ones we have found for which the moments factor into an analog of $n!$. They are also the only specializations which give the three sets of polynomials that were considered.

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