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A NEW PROOF FOR A TERMINATING "STRANGE" HYPERGEOMETRIC EVALUATION OF GASPER AND RAHMAN CONJECTURED BY GOSPER

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In a letter to Richard Askey, Bill Gosper (1977) conjectured a number of the "mysterious-looking hypergeometric evaluations". All of these except for one identity with more than one parameter were proved by Gessel and Stanton (1982). The terminating and non-terminating versions of this identity were respectively confirmed through the computer certification by Ekhad (1990) and by Gasper & Rahman (1990), in which the q-analogue of Gosper's conjecture was established either.

As known, inverse series relations are partially responsible for the proliferation of combinatorial identities. For the first time to display the talent of inversion technique, we take the original Gosper conjecture as one example and present a new proof here. As the whole technique consists of only the transformations between hypergeometric series and inverse relations, this approch is simpler, shorter and more accessible.

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As the preliminary statement, recall a pair of inverse series relations

$$f(n) = \sum_{k=o}^{n} (-1)^k \binom{n}{k} (a+\lambda k)_n (b-\lambda k)_n \frac{\lambda^{-1}(a-b)+2k}{(\lambda^{-1}(a-b)+n)_{k+1}} g(k) \tag{1}$$

$$g(n) = \sum_{k=o}^{n} (-1)^k \binom{n}{k} \frac{a+\lambda k+k}{(a+\lambda n)_{k+1}} \frac{b-\lambda k+k}{(b-\lambda n)_{k+1}} (\lambda^{-1}(a-b)+k)_n f(k) \tag{2}$$

which follow from the suitable limiting process in the bibasic inversions due to Bressoud (1988) and Gasper (1989), or directly from the equivalent orthogonality in the special case C=-n of Gasper's (1989) formula

$$ {}_6F_5 \left[\begin{matrix} C, & A, & 1+A/(1+\lambda), & B, & 1+B/(1-\lambda), & -C+(A-B)/\lambda \\ & 1+A-\lambda C, & A/(1+\lambda), & 1+B+\lambda C, & B/(1-\lambda), & 1+(A-B)/\lambda \end{matrix} \right] $$

$$= \frac{\Gamma(1+(A-B)/\lambda)\ \Gamma(1+A-\lambda C)\ \Gamma(1+B+\lambda C)}{\Gamma(1+A)\ \Gamma(1+B)\ \Gamma(1+C)\ \Gamma(1-C+(A-B)/\lambda)}. \tag{3}$$

THEOREM Gosper's conjecture is true. i.e.

$$ {}_7F_6 \left[\begin{matrix} a, 1+a/2, & a+1/2, & b, & 1-b, & n+(2a+1)/3, & -n \\ a/2, & 1/2, & (2a-b+3)/3, & (2a+b+2)/3, & 3n+2a+1, & -3n \end{matrix} \right] $$

$$= \frac{((1+b)/3)_n((2-b)/3)_n((2a+2)/3)_n((2a+3)/3)_n}{(1/3)_n(2/3)_n((3+2a-b)/3)_n((2+2a+b)/3)_n}. \tag{4}$$

Rewrite it in the form

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{4k+2a}{(3n+2a)_{k+1}}\,\frac{-2k-1}{(-3n-1)_{k+1}}\,(k+(2a+1)/3)_n\,\frac{(a)_k\,(a+1/2)_k\,(b)_k\,(1-b)_k}{(3/2)_k((2a-b+3)/3)_k((2a+b+2)/3)_k}$$

$$=\frac{(2a)_{3n}\,(1)_n\,((1+b)/3)_n\,((2-b)/3)_n}{(2)_{3n}\,((3+2a-b)/3)_n\,((2+2a+b)/3)_n}\,, \qquad (5)$$

which could be telescoped in (2) and yields the dual relation from (1)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(2a+3k)_n\,(-1-3k)_n\,\frac{2k+(2a+1)/3}{(n+(2a+1)/3)_{k+1}}\,\frac{(1)_k((1+b)/3)_k((2-b)/3)_k}{((3+2a-b)/3)_k((2+2a+b)/3)_k}\,\frac{(2a)_{3k}}{(2)_{3k}}$$

$$=\frac{(a)_n\,(a+1/2)_n\,(b)_n\,(1-b)_n}{(3/2)_n((2a-b+3)/3)_n((2a+b+2)/3)_n}\,. \qquad (6)$$

Noting that

$$(-1-3k)_n = (-1)^k\,(-1-2n)_{3n-3k}\,(2)_{3k}\,/(2)_{2n}$$

and

$$(2a+3k)_n = (-1)^{n+k}\,(2a)_{4n}\,/\{(2a)_{3k}\,(-1-2a-4n)_{3n-3k}\}\,,$$

we can reformulate (6) as a very well-poised series

$${}_7F_6\left[\begin{matrix}-2n-(2a+1)/3, & -n-(2a-5)/6, & -(1+2n)/3, & -2n/3, & -(2n-1)/3, & -n-(2a-b)/3, & -n-(2a+b-1)/3\\ & -n-(2a+1)/6, & (3-2a-4n)/3, & (2-2a-4n)/3, & (1-2a-4n)/3, & -n+(2-b)/3, & -n+(1+b)/3\end{matrix}\right]$$

$$=\frac{(b)_n\,(1-b)_n\,((1+2a)/3)_{2n}}{((1+b)/3)_n\,((2-b)/3)_n\,(2n+2a)_{2n}}\,, \qquad (7)$$

which is one very special case of the Dougall-Dixon theorem. This confirms the truth of (4). □

One profound fact hidden behind this example is that the inverse series relations could be compared to the "black box". The operation demonstrated above is just like the "input-output" process which should be expected to have the high potential for creating combinatorial identities. The author of the present paper is currently developing a project along this direction which would convince a remarkable fact that almost all terminating hypergeometric identities (e.g. those covered by the works due to Andrews [1979], Gasper & Rahman [1990] and Gessel & Stanton [1982,1983]) are the dual relations of only three hypergeometric formulae named after Chu-Vandermonde-Gauss, Pfaff-Saalschutz and Dougall-Dixon-Kummer, as long as one generalized version of reciprocal pair (1) and (2), and its q-analogue (which contain the inverse pair due to Gould and Hsu [1973] and Carlitz' q-analogue [1973] as special cases) are accepted in advance. Its extensive exhibition will appear in the forthcoming paper.

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