Block Designs With Prescribed Automorphism Group

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Abstract

Kramer and Mesner [Kra76] showed that the t-designs admitting a given automorphism group A are 0-1-solutions $\vec{x}$ of a system of equations

$$M_{t,k}^{A} \cdot \vec{x} = (\lambda, \ldots, \lambda)^{t}.$$

Based on this approach we present an algorithm for the construction of a system of representatives of designs with given parameters $t-(v,k,\lambda)$ and a given automorphism group A.

Firstly we present a method for computing the incidence matrices $M_{t,k}^{A}$ by means of double cosets.

Solving the above system of equations is an NP-complete problem. We use a heuristic approach and represent the set of all solutions implicitly by a graph. This gives us the possibility either to extract the solutions explicitly, if there are not too many of them, or to compute their numbers.

Finally we can construct the isomorphism types of t-designs with given parameters and given automorphism group A, if we know about the structure of overgroups of A, or, if there are too many designs, we are in many cases still able to give the precise number.

With the help of the complete algorithm we verify many prominent results. To the best of our knowledge our approach for the first time allowed to compute the precise number of isomorphism types or even these designs themselves for substantial numbers; see the examples and tables at the end of this publication.

A longer paper containing proofs and more detailed tables is in preparation.

1 Introduction

Our aim is to present an algorithm for the construction of systems of representatives of designs with given parameters $t-(v,k,\lambda)$ and given automorphism group A. We start with some basic facts leading us to the three parts of our algorithm, which will then be examined step by step.

1.1 Definition: A *simple t-design* with parameters $t-(v,k,\lambda)$ is a tuple $(V,\mathcal{B})$ with

- $|V| = v$, say $V = \{1,\ldots,v\}$,
- $\mathcal{B} \subseteq \binom{V}{k} := \{M \subseteq V \mid |M| = k\}$ — the elements of $\mathcal{B}$ are called *"blocks"* —,
- each $T \in \binom{V}{t}$ is contained in exactly λ blocks.

Often we say *designs* or $t-(v,k,\lambda)$ *designs* instead of *t-designs.* In the case of non simple designs $\mathcal{B}$ is a multiset, i.e. each block can occur more than once.

Designs with $\lambda = 1$ are called *Steiner systems.* •

In this paper we will restrict our attention to *simple* designs although the methods are essentially the same for non simple designs.

Let's think of a matrix $M^v_{t,k}$ the columns of which are labelled by the k-subsets (subsets of cardinality k) and the rows of which are labelled by the t-subsets of V. A matrix entry $m_{i,j}$ is 1 if and only if the set T being label of row i is subset of the set K being label of column j; otherwise it is 0. Then a $t-(v,k,\lambda)$ design can be interpreted as a 0-1-solution of

$$M^v_{t,k} \cdot \vec{x} = (\lambda,\ldots,\lambda)^t.$$

In such a 0-1-solution $\vec{x}$ all components x_i are either 0 or 1, so that they describe a subset of $\binom{V}{k}$, namely the blocks of the design. The fact that all entries of the vector on the right hand side of the equation are λ forces all t-subsets of V to be contained in exactly λ blocks.

A great problem with this approach is that the numbers of rows and columns of this matrix, which are $\binom{v}{t}$ and $\binom{v}{k}$, grow exponentially with v, t and k. This motivates prescribing automorphism groups:

1.2 Definition: We say, a design $(\{1,\dots,v\},\mathcal{B})$ is *admitting automorphism group* A, A being a subgroup of the full symmetric group S_v on $\{1,\dots,v\}$, if and only if

$$a(\mathcal{B})=\mathcal{B}\quad\forall a\in A,$$

where $a(\mathcal{B})=\{a(B)\mid B\in\mathcal{B}\}$ and $a(B)=\{a(b)\mid b\in B\}$.

A is called *full* automorphism group or simply *the* automorphism group of a design $(V,\mathcal{B})$, if and only if $(V,\mathcal{B})$ isn't admitting automorphism group A' for all $A'>A$. •

We can now define an incidence matrix $M_{t,k}^{A}$, such that the matrix $M_{t,k}^{v}$ from above corresponds to the special case where the group A is the identity subgroup of S_v, the full symmetric group on $\{1,\dots,v\}$:

1.3 Definition: For integers v, t, k and a group $A\le S_v$ the matrix $M_{t,k}^{A}$ with the following properties is called *incidence matrix* with parameters t, k and A:

- The labels of the rows of $M_{t,k}^{A}$ are the orbits of the action of A on the t-subsets of V (t-orbits),
- the labels of the columns of $M_{t,k}^{A}$ are the orbits of the action of A on the k-subsets of V (k-orbits),
- an entry $m_{i,j}$ of the matrix is the number of elements of the k-orbit being label of column j which contain a fixed element of the t-orbit being label of row i.

•

1.4 Notation: We denote the set of *orbits* of the operation of a group A on a set Ω by

$$\Omega/A:=\{A(\omega)\mid\omega\in\Omega\}.$$

•

The idea for introducing an automorphism group A was to reduce the dimensions of the incidence matrix from $|\Omega|$ to $|\Omega/A|$. More precisely we have reduced the number of its rows and columns from $\binom{v}{t}$ and $\binom{v}{k}$ to $\left|\binom{V}{t}/A\right|$ and $\left|\binom{V}{k}/A\right|$. Since the set of blocks of a design admitting automorphism group A has to be a union of orbits of A on the k-subsets of V we can now verify the following prominent observation of Kramer and Mesner [Kra76]:

1.5 Theorem: The designs with parameters $t-(v,k,\lambda)$ and admitting automorphism group A are exactly the 0-1-solutions of

$$M_{t,k}^{A}\cdot\vec{x}=(\lambda,\dots,\lambda)^{t},$$

where $M_{t,k}^{A}$ is the incidence matrix. □

1.6 Example: Following an example in [Kre86] we set $t=2$, $k=3$ and prescribe automorphism group

$$A:=\langle(1,4,5)(2,7,6),(2,6)(4,5)\rangle.$$

Firstly we verify the incidence matrix $M_{2,3}^{A}$, where "t_1t_2" and "$k_1k_2k_3$" are abbreviations for $\{t_1,t_2\}$ and $\{k_1,k_2,k_3\}$:

$$
\begin{array}{cl}
\left.\begin{array}{cccccccccc}
123 & 125 & 127 & & & & & & & \\
347 & 147 & 467 & & & & & & & \\
136 & 146 & 167 & & & & & & & \\
356 & 124 & 456 & 126 & 256 & 134 & 137 & & 236 & \\
357 & 457 & 157 & 247 & 257 & 345 & 346 & & 237 & \\
234 & 156 & 245 & 567 & 246 & 135 & 235 & 145 & 367 & 267
\end{array}\right\} & \binom{V}{k}/A \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 1
\end{pmatrix} & \underbrace{\begin{array}{l}
12,47,16,56,57,24 \\
13,34,35 \\
14,45,15 \\
17,46,25 \\
23,37,36 \\
26,27,67
\end{array}}_{\binom{V}{t}/A}
\end{array}
$$

Note that prescribing automorphism group A reduces the dimensions of the matrix from 21×35 to 6×10.

Since $(0,0,0,1,0,0,1,1,0,0)^{t}$ is a solution of

$$M_{2,3}^{A}\cdot\vec{x}=(1,1,1,1,1,1)^{t}$$

columns 4, 7 and 8 yield a Steiner system $2-(7,3,1)$ admitting automorphism group A. Its set of blocks is

$$\mathcal{B}=\Big\{\{1,2,6\},\{2,4,7\},\{5,6,7\},\{1,3,7\},\{3,4,6\},\{2,3,5\},\{1,4,5\}\Big\}.$$

Column 2, 7 and 10 form the other Steiner system with the same parameters and admitting the same automorphism group. ◇

Based on theorem 1.5 we now split our algorithm into three steps:

1.7 The three steps of our algorithm for constructing systems of representatives of t-designs with given parameters $t-(v,k,\lambda)$ and given automorphism group A:

1. Computation of the incidence matrix $M^A_{t,k}$.

2. Looking for all solutions of the integral matrix equation

$$M^A_{t,k} \cdot \vec{x} = (\lambda, \dots, \lambda)^t \text{ with } x_i \in \{0,1\} \; \forall i.$$

3. Check, which designs are non isomorphic and what's their *full* automorphism group, because there is no information on these topics in the theorem of Kramer and Mesner.

2 Computation of Incidence Matrices

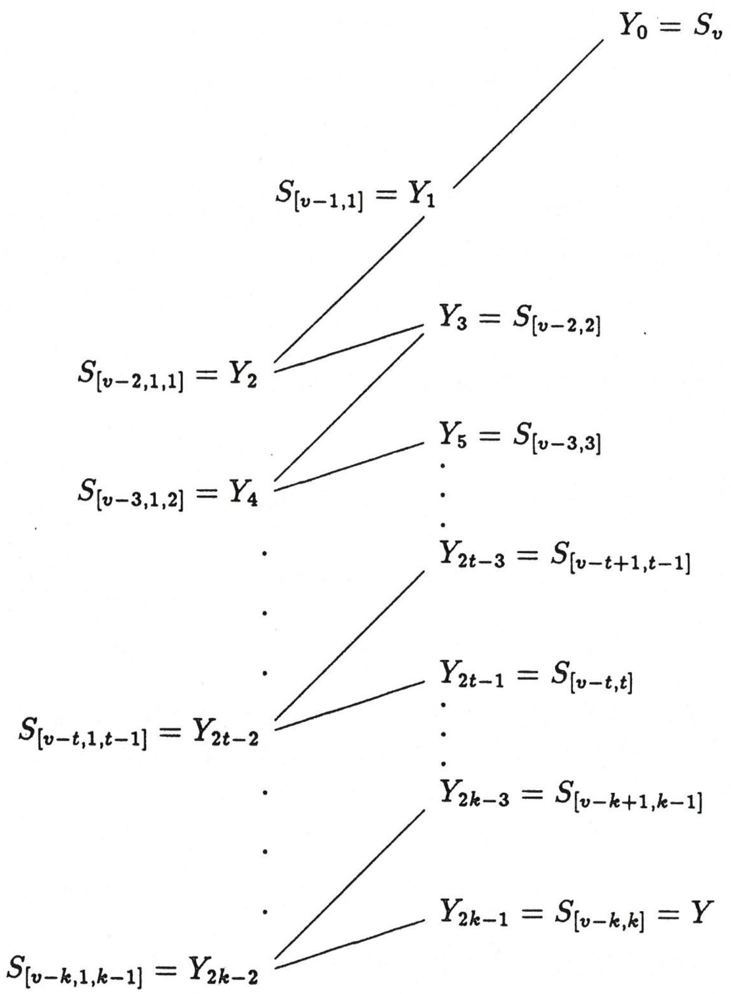
In order to compute the incidence matrices $M^A_{t,k}$ we need the labels

$$\binom{V}{t}/A \quad \text{and} \quad \binom{V}{k}/A$$

of its rows and columns. Remembering the bijection between the t-subsets of V and the cosets of the Young subgroup $S_{[v-t,t]} \cong S_{v-t} \oplus S_t$ in S_v (and the same for k) we know that these labels are isomorphic to the double cosets

$$S_{[v-t,t]} \backslash S_v / A \quad \text{and} \quad S_{[v-k,k]} \backslash S_v / A.$$

In [Sch90] a double coset algorithm can be found, which is very appropriate doing these calculations. It works with *subgroup ladders*. For the double coset problem $Y \backslash G / A$ this is a sequence $(Y_i)_{1 \le i \le s}$ of groups with $Y_0 = G$ and $Y_s = Y$ and for all i: $Y_{i+1} \ge Y_i \vee Y_{i+1} \le Y_i$. In our special case we use the following ladder:



The algorithm computes step by step the double cosets

$$Y_i \backslash S_v / A$$

out of the

$$Y_j \backslash S_v / A$$

for all $j < i$. So it computes for $t \leq k$

$$S_{[v-k,k]} gA, \ g \in S_v$$

out of

$$S_{[v-t,t]} gA, \ g \in S_v.$$

This is the reason why it not only yields the labels of the rows and columns of our incidence matrix but also its entries. Details can be found in [Sch90].

Using this algorithm we are able to compute matrices with more than thousand rows and columns, and this is much more than our second algorithm, which has to solve the systems of equations, can deal with.

3 Solving Integral Matrix Equations

Slightly generalized our problem is the following one:

3.1 Find *all* integral solutions of the integral system of equations

$$M \cdot \vec{x} = \vec{r}, \text{ with } \nu_i \leq x_i \leq \mu_i \ \forall i.$$

In the special case of simple block designs M is an incidence matrix $M^A_{t,k}$, the components of $\vec{r}$ all are λ and for all i the ν_i are 0 and the μ_i are 1:

$$M^A_{t,k} \cdot \vec{x} = (\lambda, \ldots, \lambda)^t, \text{ with } 0 \leq x_i \leq 1 \ \forall i.$$

This is essentially the problem called "subset sum" in the list of NP-complete problems in the book of Garey and Johnson [GJ79], and so we have:

3.2 Proposition: Problem 3.1 is *NP-complete.* □

But we don't want to give up, even if the numbers of solutions shown in the following table for fairly interesting parameters and automorphism groups aren't encouraging at all.

Table 1: some numbers of solutions

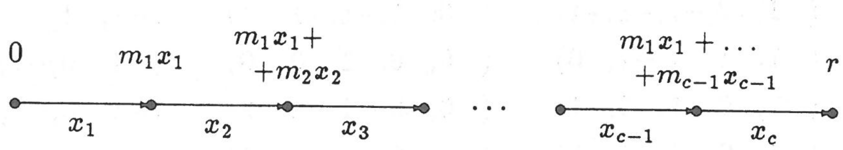
$t-(v,k,\lambda)$	A	size($M^A_{t,k}$)	# solutions
$3-(28,7,6000)$	$\mathrm{P\Gamma L}_2(3^3)$	1×29	4 205 630
$3-(28,6,1000)$	$\mathrm{PGL}_2(3^3)$	1×34	274 688 628
$3-(24,8,1470)$	$\mathrm{PGL}_2(23)$	1×83	1 567 457 262

They have been computed using a data structure, which represents the solutions *implicitely.*

In order to understand it let's first have a look at a one row matrix as in table 1:

$$(m_1, \ldots, m_c) \cdot \vec{x} = r.$$

One solution $(x_1, \ldots, x_c)$ can by visualized by the following directed, labelled and connected graph:



Its vertex labels are the partial sums $m_1x_1 + \ldots + m_ix_i$ for $0 \leq i \leq c$, and its edge labels are the componentes x_i, $1 \leq i \leq c$, of the solution vector $\vec{x}$.

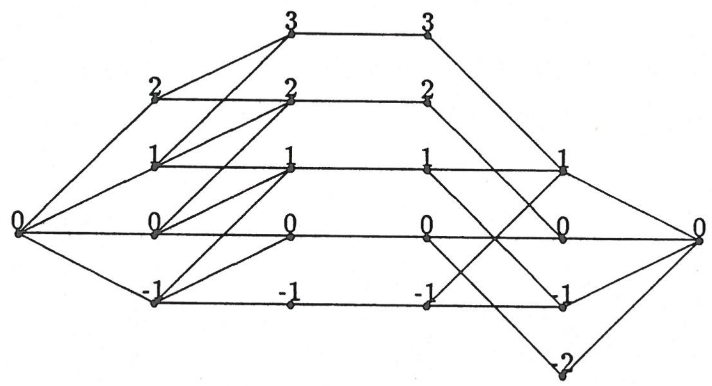
We call the "first" vertex, where the edge labelled x_1 starts, *start vertex*, its label is always 0, and the "last" one, where the edge labelled x_c ends, *stop vertex*; so its label is always the right hand side r of the matrix equation.

This data structure can be generalized, so that the resulting graph — we call it *solution graph* — represents all solutions of the matrix equation. Let's understand this with the help of an

3.3 Example: Consider the one row matrix

$$(1,1,0,2,1)\cdot\vec{x}=0 \quad \text{with} \quad \begin{array}{rcl} -1 \le & x_1 & \le \ \ 2 \\ 0 \le & x_2 & \le \ \ 2 \\ -1 \le & x_3 & \le -1 \\ -1 \le & x_4 & \le \ \ 1 \\ -1 \le & x_5 & \le \ \ 2 \end{array} .$$

Its solution graph is:



The edge labels have been omitted because they can easily be recomputed from the vertex labels with the help of the matrix entries. The direction of the edges always is from left to right.

The solution graph codes all 17 solutions of the matrix equation which are:

(2, 1,−1,−1,−1)	(0, 2,−1,−1, 0)	(−1, 2,−1,−1, 1)
(2, 0,−1,−1, 0)	(0, 1,−1, 0,−1)	(−1, 1,−1, 0, 0)
(1, 2,−1,−1,−1)	(0, 1,−1,−1, 1)	(−1, 1,−1,−1, 2)
(1, 1,−1,−1, 0)	(0, 0,−1, 0, 0)	(−1, 0,−1, 1,−1)
(1, 0,−1, 0,−1)	(0, 0,−1,−1, 2)	(−1, 0,−1, 0, 1)
(1, 0,−1,−1, 1)	(−1, 2,−1, 0,−1)	

◇

The space requirements of this data structure are very small as can be seen in the following table. It is table 1 with an additional column telling the sizes of the corresponding solution graphs in terms of the number of integers the computer needed to code them. This number is a constant plus four times the number of vertices of the graph; so the graph in our last example has about 1400 vertices to code the 1 567 457 262 solutions.

Table 2: some sizes of solution graphs

$t-(v,k,\lambda)$	A	size($M_{t,k}^A$)	# solutions	size(graph)
$3-(28,7,6000)$	$\mathrm{P\Gamma L}_2(3^3)$	1×29	4 205 630	3 066 ints
$3-(28,6,1000)$	$\mathrm{PGL}_2(3^3)$	1×34	274 688 628	5 668 ints
$3-(24,8,1470)$	$\mathrm{PGL}_2(23)$	1×83	1 567 457 262	5 699 ints

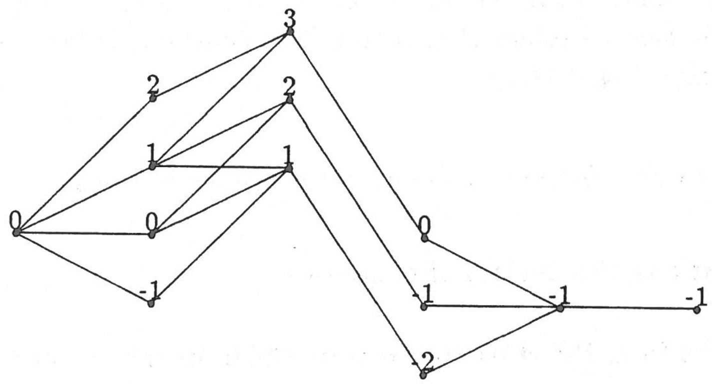
Note that the size of a solution graph does not depend on the number of solutions it codes but on the number of partial sums that can occur.

A second example with another one row matrix leeds us to the next generalization, namely matrices with more than one row.

3.4 Example: The solution graph for the matrix equation

$$(1,1,3,1,0)\cdot\vec{x}=-1$$

with the same conditions as in example 3.3 is the following one. It codes all 32 solutions of the matrix equation:

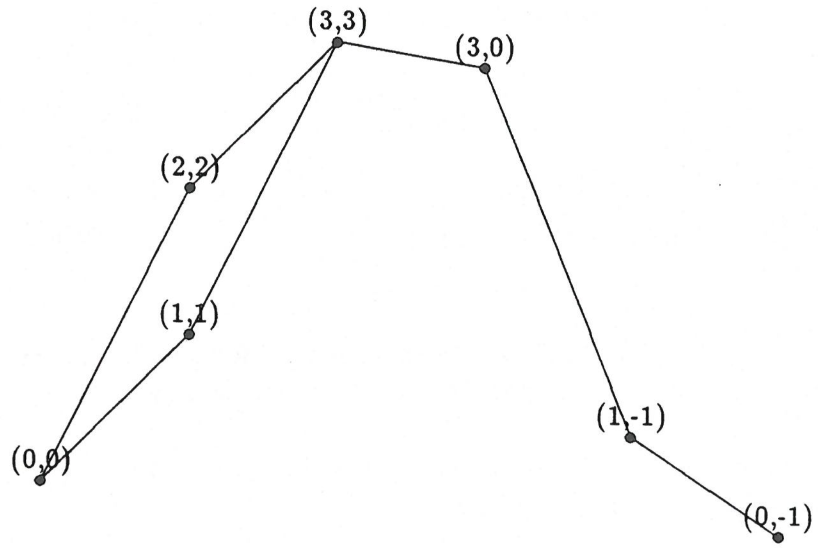


$\diamond$

3.5 Example: We want to solve the following system of equations, the rows of which are the two one row matrices from the other examples. The conditions are still the same.

$$\begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 & 0 \end{pmatrix} \cdot \vec{x} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

The solution graph is:



It codes the two solutions of the system of equations, which are:

$$(2, 1, -1, -1, -1) \quad \text{and} \quad (1, 2, -1, -1, -1).$$

◇

The last example shows that the labels of the vertices of l-row matrices are no longer numbers but l-tuples of numbers — the partial sums are l-tuples. — We won't give the exact definition of solution graphs here, because it is very technical, but we want to stress their main property:

3.6 Proposition: We have a 1-1-correspondence between

- the solutions of a system of equations
- the paths from the start to the stop vertex in the corresponding solution graph.

□

There is an operation on solution graphs, called *intersection*, which allows us to construct the graph of an l-row system of equations out of the solution graphs of systems of equations the rows of which are subsets of the rows of the original l-row system; see examples 3.3, 3.4 and 3.5. The definition again is very technical, but the following proposition is important:

3.7 Proposition: The intersection of two solution graphs

$$\mathcal{G}_1 \quad \text{and} \quad \mathcal{G}_2$$

of

$$M_1 \cdot \vec{x} = \vec{r}_1 \quad \text{and} \quad M_2 \cdot \vec{x} = \vec{r}_2$$

is the solution graph of

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \cdot \vec{x} = \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix}.$$

The conditions $\nu_i \leq x_i \leq \mu_i$ have to be the same for all three systems of equations. □

With the help of the operation intersection we can now construct solution graphs proceeding *step by step*:

1. Computation of the solution graph of the 1^{st} matrix row,
2. computation of the solution graph of the 2^{nd} matrix row,
3. intersection gives the solution graph of the first 2 matrix rows,
4. computation of the solution graph of the 3^{rd} matrix row,
5. intersection gives the solution graph of the first 3 matrix rows,

$$\vdots$$

Proceeding in this way we have split the problem of computing solution graphs of systems of equations with many rows into fairly small problems.

The NP-completeness of the problem, see 3.2, is reflected by the following fact:

3.8 Remark: Intersection of solution graphs reduces the number of solutions, but the size of the graph, i.e. the number of vertices, can grow. □

In practice this means that the solution graphs occuring as intermediate results sometimes get very big. But of course there are many tricks to deal with this problem, for example equation systems and solution graphs can be transformed into "better" ones.

4 Isomorphism Types of Block Designs

4.1 Definition: Two designs $(V, \mathcal{B})$ and $(V, \mathcal{B}')$ with $V = \{1, \ldots, v\}$ are *isomorphic* if and only if

$$\exists \pi \in S_v : \pi(\mathcal{B}) = \mathcal{B}'.$$

•

Our aim is to compute the isomorphism types of $t-(v,k,\lambda)$ designs. First note that these are the *orbits* of the action

$$S_v \times \Omega \longrightarrow \Omega, \qquad (\pi, \mathcal{B}) \longmapsto \pi(\mathcal{B}),$$

where Ω is the set of all $t-(v,k,\lambda)$ designs. Also the solutions of the system of equations

$$M_{t,k}^{A} \cdot \vec{x} = (\lambda, \ldots, \lambda)^t$$

can be described in terms of group theory. They are the *fixed points* Ω_A of A on Ω:

$$\Omega_A = \{(\{1, \ldots, v\}, \mathcal{B}) \in \Omega \mid \forall a \in A : a(\mathcal{B}) = \mathcal{B}\}.$$

This enables us to apply group theoretic theorems, for example Burnside's Lemma. It tells us how to get the number of isomorphism types of block designs with prescribed automorphism group A out of the numbers of fixed points $|\Omega_A|$ with the help of the Burnside matrix or parts of it. See for example [Ker91].

Here, however, we want to *construct* designs, and in order to do this we need one additional notation:

$$\hat{\Omega}_A := \Omega_A - \bigcup_{B>A} \Omega_B,$$

the $t-(v,k,\lambda)$ designs having *full* automorphism group A. Remember, that Ω_A consists of the designs *admitting* automorphism group A.

The following proposition taken from [Lau89] says that computing the isomorphism types of the $t-(v,k,\lambda)$ designs within $\hat{\Omega}_A$ is quite easy, if the normalizer $\mathcal{N}_{S_v}(A)$ of A in S_v is known.

4.2 Proposition: Let a group G act on a set Ω, $A \leq G$, then the operation

$$\mathcal{N}_G(A)/A \times \hat{\Omega}_A \longrightarrow \hat{\Omega}_A, \qquad (nA, \mathcal{B}) \longmapsto n(\mathcal{B})$$

of the factor group $\mathcal{N}_G(A)/A$ on $\hat{\Omega}_A$ is *semi regular*, i.e. each orbit has length

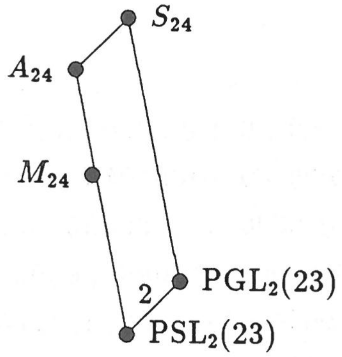
$$|\mathcal{N}_G(A)/A|.$$

□

This theorem will be essential in the following examples.

4.3 Example: $5-(24,6,1)$ designs with full automorphism group $\mathrm{PSL}_2(23)$

From the list of primitive groups we extract the lattice of overgroups of $\mathrm{PSL}_2(23)$ in S_{24}:



Solving

$$M_{5,6}^A \cdot \vec{x} = (1,\ldots,1)^t$$

for all $A \geq \mathrm{PSL}_2(23)$ we get:

A	S_{24}	A_{24}	M_{24}	$\mathrm{PGL}_2(23)$	$\mathrm{PSL}_2(23)$
size($M_{5,6}^A$)	1×1	1×1	1×2	6×22	7×34
$\lvert\Omega_A\rvert$	0	0	0	0	6

The fact that $|\Omega_A| = 0$ for all $A > \mathrm{PSL}_2(23)$ implies $\hat{\Omega}_{\mathrm{PSL}_2(23)} = \Omega_{\mathrm{PSL}_2(23)}$. Since the normalizer of $\mathrm{PSL}_2(23)$ in S_{24} is $\mathrm{PGL}_2(23)$ and the index of these two groups is 2, proposition 4.2 yields:

> There are exactly *three* non isomorphic $5-(24,6,1)$ designs with full automorphism group $\mathrm{PSL}_2(23)$.

Our aim is to explicitely construct these 3 designs. Proposition 4.2 says we have to consider the action of $\mathrm{PGL}_2(23)/\mathrm{PSL}_2(23)$ on $\hat{\Omega}_{\mathrm{PSL}_2(23)}$. Since each design in this set consists of 7084 blocks and there are only 34 orbits $\binom{V}{k}/\mathrm{PSL}_2(23)$, we do all calculations with *orbits of blocks* and not with the blocks themselves.

These 34 orbits are labels of the columns of the incidence matrix $M_{5,6}^{\mathrm{PSL}_2(23)}$. So we have an order on them and can use it to represent the only non trivial element π out of $\mathrm{PGL}_2(23)/\mathrm{PSL}_2(23)$ as a permutation of degree 34:

$$\pi=(2,6)(4,14)(5,13)(8,12)(9,11)(15,17)(18,20)(19,24)(21,22)(26,29)(31,32)(33,34)$$

The six solutions of the system of equations

$$M_{5,6}^{\mathrm{PSL}_2(23)} \cdot \vec{x} = (1,\ldots,1)^t$$

are:

$$\begin{array}{rl}
 & \scriptstyle 2\quad 4\quad 6\quad 8\quad 10\quad 12\quad 14\quad 16\quad 18\quad 20\quad 22\quad 24\quad 26\quad 28\quad 30\quad 32\quad 34 \\
l_1 = & 0000000010000000000100000000000010 \\
l_2 = & 0000000010000000000100000000000001 \\
l_3 = & 0000000001000000000000100000001000 \\
l_4 = & 0000000001000000000000100000000100 \\
l_5 = & 0000000000100000010000000000000010 \\
l_6 = & 0000000000100000010000000000000001
\end{array}$$

We see at once

$$\pi(l_1) = l_6, \qquad \pi(l_2) = l_5, \qquad \pi(l_3) = l_4.$$

and therefore know $\{l_1, l_2, l_3\}$ is a system of representatives of all $5-(24,6,1)$ designs with full automorphism group $\mathrm{PSL}_2(23)$:

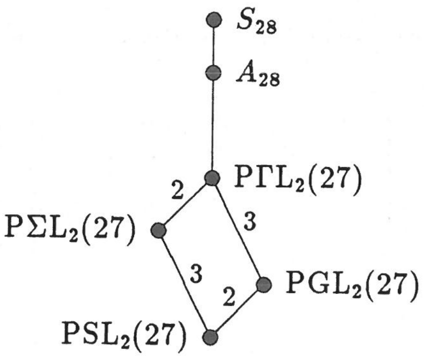
$$\begin{aligned}
l_1 &= \mathrm{PSL}_2(23)(\{11,20,21,22,23,24\}) \cup \mathrm{PSL}_2(23)(\{9,18,21,22,23,24\}) \\
&\cup \mathrm{PSL}_2(23)(\{13,16,20,22,23,24\}), \\
l_2 &= \mathrm{PSL}_2(23)(\{11,20,21,22,23,24\}) \cup \mathrm{PSL}_2(23)(\{9,18,21,22,23,24\}) \\
&\cup \mathrm{PSL}_2(23)(\{\ 2,16,20,22,23,24\}), \\
l_3 &= \mathrm{PSL}_2(23)(\{10,20,21,22,23,24\}) \cup \mathrm{PSL}_2(23)(\{5,18,21,22,23,24\}) \\
&\cup \mathrm{PSL}_2(23)(\{12,19,20,22,23,24\}).
\end{aligned}$$

The only reasonable method to describe these designs is to give representatives of the orbits of the automorphism group, because these orbits are very big: Two of them have cardinality 3036 the other one 1012, what makes 7084 blocks. $\diamond$

The next example shows that the algorithm also works for substantial numbers of isomorphism types of designs.

4.4 Example: Designs with automorphism group $PSL_2(27)$

Again we extract the structure of overgroups from the list of primitive groups:



Firstly we use the zeta matrix of this subgroup lattice:

$$\begin{pmatrix} |\Omega_{PSL_2(27)}| \\ |\Omega_{PGL_2(27)}| \\ |\Omega_{P\Sigma L_2(27)}| \\ |\Omega_{P\Gamma L_2(27)}| \end{pmatrix} = \begin{array}{c} \\ S \\ G \\ \Sigma \\ \Gamma \end{array} \begin{array}{c} \begin{array}{cccc} S & G & \Sigma & \Gamma \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} \end{array} \cdot \begin{pmatrix} |\hat{\Omega}_{PSL_2(27)}| \\ |\hat{\Omega}_{PGL_2(27)}| \\ |\hat{\Omega}_{P\Sigma L_2(27)}| \\ |\hat{\Omega}_{P\Gamma L_2(27)}| \end{pmatrix}.$$

Moebius inversion yields:

$$\begin{pmatrix} |\hat{\Omega}_{PSL_2(27)}| \\ |\hat{\Omega}_{PGL_2(27)}| \\ |\hat{\Omega}_{P\Sigma L_2(27)}| \\ |\hat{\Omega}_{P\Gamma L_2(27)}| \end{pmatrix} = \begin{array}{c} \\ S \\ G \\ \Sigma \\ \Gamma \end{array} \begin{array}{c} \begin{array}{cccc} S & G & \Sigma & \Gamma \end{array} \\ \begin{pmatrix} 1 & -1 & -1 & 1 \\ & 1 & 0 & -1 \\ & & 1 & -1 \\ & & & 1 \end{pmatrix} \end{array} \cdot \begin{pmatrix} |\Omega_{PSL_2(27)}| \\ |\Omega_{PGL_2(27)}| \\ |\Omega_{P\Sigma L_2(27)}| \\ |\Omega_{P\Gamma L_2(27)}| \end{pmatrix}.$$

$P\Gamma L_2(27)$ is the normalizer of the four projective groups and so the lengths of the orbits of

$$Q := P\Gamma L_2(27)/PSL_2(27)$$

on the designs having full automorphism group P are 6, 3, 2 and 1 for P being $PSL_2(27)$, $PGL_2(27)$, $P\Sigma L_2(27)$, $P\Gamma L_2(27)$ respectively. This implies:

$$\begin{pmatrix} |\hat{\Omega}_{PSL_2(27)}/Q| \\ |\hat{\Omega}_{PGL_2(27)}/Q| \\ |\hat{\Omega}_{P\Sigma L_2(27)}/Q| \\ |\hat{\Omega}_{P\Gamma L_2(27)}/Q| \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \\ & \frac{1}{3} & 0 & -\frac{1}{3} \\ & & \frac{1}{2} & -\frac{1}{2} \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} |\Omega_{PSL_2(27)}| \\ |\Omega_{PGL_2(27)}| \\ |\Omega_{P\Sigma L_2(27)}| \\ |\Omega_{P\Gamma L_2(27)}| \end{pmatrix}.$$

Let's apply this result to some special parameters $t-(28,k,\lambda)$:

The numbers of solutions of

$$M^P_{5,6}\cdot\vec{x}=(1,\ldots,1)^t$$

is equal to 1 for all four projective groups P, and since

$$\begin{pmatrix} \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \\ & \frac{1}{3} & 0 & -\frac{1}{3} \\ & & \frac{1}{2} & -\frac{1}{2} \\ & & & 1 \end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

we have:

> There is exactly one $5-(28,7,1)$ design having full automorphism group $\mathrm{P\Gamma L}_2(27)$ and none having one of the other three projective groups as full automorphism group.

Precisely the same constellation of numbers occurs for $4-(28,7,8)$ designs.

More substantial numbers occur when looking at $4-(28,6,72)$ designs:

$$\begin{pmatrix} \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \\ & \frac{1}{3} & 0 & -\frac{1}{3} \\ & & \frac{1}{2} & -\frac{1}{2} \\ & & & 1 \end{pmatrix}\cdot\begin{pmatrix}13\,078\,960\\704\\58\\8\end{pmatrix}=\begin{pmatrix}2\,179\,701\\232\\25\\8\end{pmatrix}.$$

This equation implies:

> There are exactly 2 179 701 non isomorphic $4-(28,6,72)$ designs having full automorphism group $\mathrm{PSL}_2(27)$.
>
> There are exactly 232 non isomorphic $4-(28,6,72)$ designs having full automorphism group $\mathrm{PGL}_2(27)$.
>
> There are exactly 25 non isomorphic $4-(28,6,72)$ designs having full automorphism group $\mathrm{P\Sigma L}_2(27)$.
>
> There are exactly 8 non isomorphic $4-(28,6,72)$ designs having full automorphism group $\mathrm{P\Gamma L}_2(27)$.

We will give two tables to show the efficiency of our algorithm.

Table 3 treats the parameters $t = 5$ and $k = 6$. It has been motivated by a paper of Kreher and Radziszowski [Kre87], in which they showed that each number of fixed points in the column labelled $\mathrm{PSL}_2(27)$ is bigger than zero exept that one belonging to $\lambda = 1$. We can give the precise numbers.

Table 3: $5-(28, 6, \lambda)$-designs with projective automorphism group

λ	$\mathrm{P\Gamma L}_2(27)$	$\mathrm{P\Sigma L}_2(27)$	$\mathrm{PGL}_2(27)$	$\mathrm{PSL}_2(27)$	
1	0	0	0	0	designs
	0	0	0	0	fixed points
2	1	0	1	20	designs
	1	1	4	124	fixed points
3	0	4	0	695	designs
	0	8	0	4 178	fixed points
4	0	0	22	6 132	designs
	0	0	66	36 858	fixed points
5	0	6	0	74 882	designs
	0	12	0	449 304	fixed points
6	0	13	70	370 650	designs
	0	26	210	2 224 136	fixed points
7	0	0	36	1 707 742	designs
	0	0	108	10 246 560	fixed points
8	0	10	261	5 710 925	designs
	0	20	783	34 266 353	fixed points
9	3	14	122	11 496 089	designs
	3	31	369	68 976 931	fixed points
10	0	0	328	22 461 654	designs
	0	0	984	134 770 908	fixed points
11	4	9	274	28 068 645	designs
	4	22	826	168 412 714	fixed points

Of course it is impossible to give such a table for all other possible values of t and k here. Therefore we produced table 4. It gives for all t and k, for which we could solve the systems of equations, the sum of numbers of designs over all interesting values of λ, which are all λ from 1 to $\frac{\sigma_Z}{2}$, σ_Z being the row sum of the incidence matrices.

Table 4: $t-(28,k,\lambda)$-designs with projective automorphism group

t	k	$\mathrm{P\Gamma L_2(27)}$	$\mathrm{P\Sigma L_2(27)}$	$\mathrm{PGL_2(27)}$	$\mathrm{PSL_2(27)}$
3	4	3	2	4	2
3	5	9	14	48	76
4	5	0	0	0	0
3	6	8 191	1 044 480	170 031 544	$\geq 2^{31}$
4	6	23	226	9 929	
5	6	8	56	1 114	69 897 434
3	7	268 435 455	$\geq 2^{31}$		
4	7	207 649	1 885 645 699		
5	7	29 515			
6	7	0	0	0	

◇

We conclude with two results on very prominent problems:

4.5 Example: A result on simple designs with $t \geq 6$:

In 1984 Magliveras and Leavitt [Mag84] found 6 solutions of

$$M_{6,8}^{\mathrm{P\Gamma L_2(32)}} \cdot \vec{x} = (36,\ldots,36)^t;$$

we computed that there are 1179 solutions, what implies:

> There are exactly 1179 non isomorphic $6-(33,8,36)$ designs with full automorphism group $\mathrm{P\Gamma L_2(32)}$.

◇

4.6 Example: A result on simple Steiner systems with $t \geq 5$:

In 1976 Denniston [Den76] published the existence of at least 100 simple 5-(48,6,1) Steiner systems with full automorphism group $PSL_2(47)$. Since there are exactly 918 solutions of the corresponding matrix equation, the normalizer of $PSL_2(47)$ in S_{48} is $PGL_2(47)$, it is not admitted as automorphism group of such a design and there are no other relevant overgroups of $PSL_2(47)$, we have proved:

> There are exactly 459 non isomorphic $5-(48,6,1)$ designs with full automorphism group $PSL_2(47)$.

◇

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