Publ. I.R.M.A. Strasbourg, 1993, 460/S–23
Actes 23^e Séminaire Lotharingien, p. 51–66

ENUMERATION OF SKEW FERRERS DIAGRAMS

BY

MARIE-PIERRE DELEST (*) and JEAN-MARC FEDOU (*)

ABSTRACT. — In this paper we show that the generating function for the skew Ferrers diagrams according to the paremeters width and area is the quotient of new basic Bessel functions.

Introduction.

Ferrers diagrams, related to the well known partitions of an integer have been extensively studied, see for instance the works of Andrews [3]. A partition of an integer n is an increasing sequence of integers, $n_1,n_2,\ldots,n_k$ such that $n_1+n_2+\ldots+n_k = n$. The geometric figure formed by the k columns having respectively $n_1,n_2,\ldots,n_k$ cells (see figure 1) is called Ferrers diagram associated to the partition $(n_1,n_2,\ldots,n_k)$ of n. Filling Ferrers diagrams with numbers gives plane partitions which are related with representations of the symmetric group [13]. The Young tableaux are examples of such plane partitions and are of a great interest for the computation of the Schur functions. The literature on these subjects is plentiful.

The difference between two Ferrers diagrams is called a skew Ferrers diagram. Thus a skew Ferrers diagram is defined by two increasing sequences of integers, $n_1,n_2,\ldots,n_k$ and $p_1,p_2,\ldots,p_k$ such that, for every $1 \leq i \leq k$, $n_i \leq p_i$ (see figure 2). If the skew Ferrers diagrams have no cut point and are connected then they are a particular case of polyominoes, the so-called parallelogram polyominoes.

Unit squares with vertices at integer points in the cartesian plane are called *cells*. A polyomino is a finite connected union of cells such that the interior is also connected. Polyominoes are defined up to translation. The perimeter of a polyomino is the length of its border, its area is the number of cells which compound it. For example, the skew Ferrers diagram showed in figure 2 is defined by the two sequences (2,4,4,4,4) and (1,1,3). It is also a parallelogram polyomino having perimeter 18 and area 13.

(*) This work is partially supported by the PRC mathématique et Informatique.

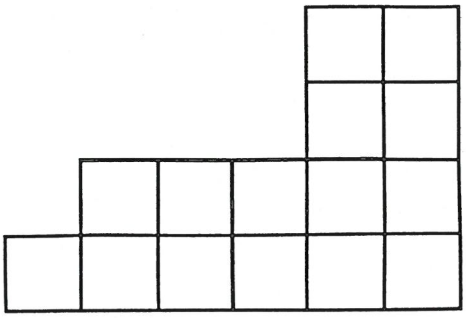

Figure 1. Ferrers diagram corresponding to the sequence (1,2,2,2,4,4)

Counting polyominoes according to the area or perimeter is a major unsolved problem in combinatorics. See for review [18], [20]. The problem is also well known in statistical physics. Usually, Physicits consider animals instead of polyominoes, an equivalent object obtained by taking the center of each elementary cell. They attempt to find some relations for the number a_n of animals having an area or a perimeter n. For results on this subjects the reader would see [23].

A column (resp. a row) is the intersection of the polyomino with an infinite vertical (resp.horizontal) unit strip. A polyomino is said to be convex when all its columns and rows are connex. Recently, convex polyominoes have been enumerated according to the perimeter [8]. The enumeration according to the area is still an open problem.

A parallelogram polyomino is a convex polyomino bordered by two nonintersecting paths having only North and East steps (see figure 2). Parallelogram polyominoes are well known in Combinatorics (see Polya [18], Gessel [14]). The number p_{2n+2} of such polyominoes having perimeter 2n+2 is the Catalan number C_n,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The enumeration of the polyomino parallelograms according to the area has been studied by Gessel in [14], as an application of a q-analog of the Lagrange inversion formula but no explicit formula is given.

Here we were interested in the relations between perimeter, number of columns and area in such a polyomino. The reason is that if we want to enumerate convex polyominoes according to the area (which is an open problem) then we must first know some distributions for parallelogram polyominoes (for more explanations see [8]). As shown in section 2, it is easy to give functional equations for their generating function using what we call a q-analog of an algebraic grammar. When A.M. Garsia was visiting in Bordeaux during September 1989, we were stopped at this point. Then we made with him some computations using Macsyma which led us to see some sequences, related to the zeroes of Bessel functions [1], in the *handbook of Integer Sequences* of N.J. Sloane.

We shall show in section 2, that the enumeration of parallelogram polyominoes is intimately connected with the power sums of the zeroes of some q-analogues of Bessel functions. These Bessel functions are different from those defined by Ismail [15] and Jackson [16] and appear to be new. They are introduced in section 3 and some of their properties are given in section 4.

The subject is so rich that it leads us to several combinatorial interpretations for these functions. These can be made in terms of weight enumerators of trees, multichains in Dyck paths, multiwalks in a trees [12]. This not widstanding, further work needs still to be done. Some open questions are given in the conclusion.

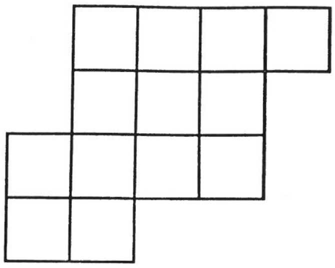

Figure 2. A parallelogram polyomino having area 13 and perimeter 20.

1.DEFINITIONS AND NOTATIONS

A *path* is a sequence of points in the quarter of plane $\mathbb{N}\times\mathbb{N}$. A *step* of a path is a couple of two consecutive points in the path. A *Dyck path* is a path $w = (s_0,s_1,\ldots,s_{2n})$ such that $s_0 = (0,0)$, $s_{2n} = (2n,0)$, having only steps North-East $(s_i=(x,y),s_{i+1}=(x+1,y+1))$ or South-East $(s_i=(x,y),s_{i+1}=(x+1,y-1))$. A *peak* (resp. *trough*) is a point s_i such that the step (s_{i-1},s_i) is North-East (resp. South-East) and the step (s_i,s_{i+1}) is South-East (resp. North-East). The *height* $h(s_i)$ of a point s_i is its ordinate.

A *Dyck word* is a word $w \in \{x, \bar{x}\}^*$ satisfying both conditions:

(i) $|w|_x = |w|_{\bar{x}}$,

(ii) for every factorization $w = uv$, $|u|_x \geq |u|_{\bar{x}}$.

Classically, a Dyck path having length 2n is coded by a Dyck word of length 2n, $w=x_1\ldots x_{2n}$: each North-East (resp. South-East) step (s_{i-1}, s_i) corresponds to the letter $x_i = x$ (resp. $x_i = \bar{x}$). The peaks (resp. troughs) of a Dyck path correspond with the factors $x\,\bar{x}$ (resp $\bar{x}\,x$) of the associated Dyck word. We denote by D_n the set of the Dyck words having length 2n.

Example. The Dyck path showed figure 3 is coded by the following Dyck word from D_8

$$w = x\,x\,\bar{x}\,x\,x\,x\,\bar{x}\,\bar{x}\,x\,\bar{x}\,x\,\bar{x}\;\bar{x}\,\bar{x}\;\;x\;\;\bar{x}.$$

M.P. Delest and X.G.Viennot give in [8] a bijection μ between the parallelogram polyominoes having perimeter 2n+2 and the Dyck words having length 2n. A parallelogram polyomino P can be defined by the two sequences of integers $(a_1,\ldots, a_n)$ and $(b_1,\ldots, b_{n-1})$, where a_i is the number of cells belonging to the i^{th} column and (b_i+1) the number of cells adjacent to the columni and i+1.The Dyck word $\mu(P)$ is the Dyck word having n peaks, whose heights (resp. troughs) are $a_1,\ldots, a_n$ (resp. $b_1,\ldots, b_{n-1}$). They deduce the following

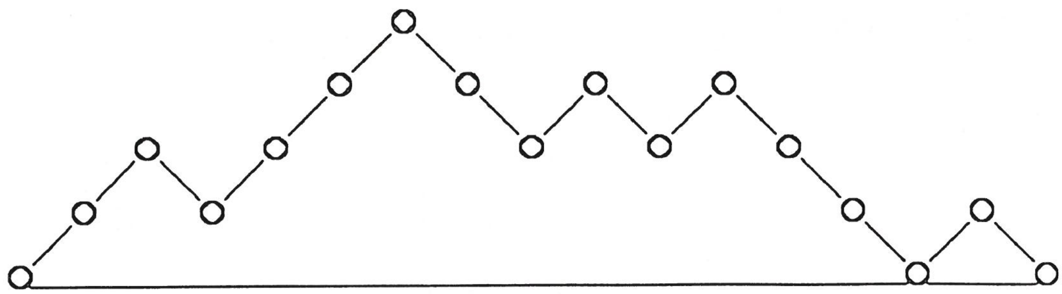

Figure 3. A Dyck path of D_8

Proposition 1. *The map* μ *transforms a parallelogram polyomino having perimeter* 2p+2, n *columns and area* k *into a Dyck word having length* 2p, n *peaks and such that the sum of the height of the peaks is* k.

Example. The parallelogram polyomino showed figure 2 is defined by the two sequences (2,4,3,3,1) and (1,2,2,0) and corresponds to the Dyck path showed figure 3.

On the other hand, Bessel functions are present in Analysis where they are particulary usefull for the resolution of differential equations. There is a lot of works on these functions, see for instance [23] ou [9]. We recall here their classical definition and also a result by Carlitz [6] about the quotient of such functions.

Bessel functions are defined for $\nu>-1$, by

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}x)^{2n+\nu}}{n!\,\Gamma(\nu+n+1)},$$

All the zeroes of $J_\nu(x)$ are real. Let $j_{\nu,k}$ be the k^{th} positive zero of $J_\nu(x)$. The symetric function,

$$\sigma_{2n}(\nu) = \sum_{k=1}^{\infty} (j_{\nu,k})^{-2n},$$

is rational in ν for any positive integer ν.

Rayleigh [19], Airey [1], and others have used this result for the computation of the first zeroes of the Bessel functions. The functions σ_{2n} were known by Jacobi in 1849, as coefficients of the meromorphic functions,

$$\frac{J_{v+1}(x)}{2\,J_v(x)} = \sum_{n=1}^{\infty} \sigma_{2n}(v)\, x^{2n-1}.$$

n	1	2	3	4	5	6	7	8
k								
1	1	1	2	11	38	946	4580	202738
2				5	14	1026	4324	311387
3						362	1316	185430
4						42	132	53752
5								7640
6								429

Figure 4. coefficients $a_k^{(n)}$ for $n \leq 8$

The first values of

$$\sigma_{2n}(v) = 2^{-2n} \frac{\Phi_n(v)}{\pi_n(v)},$$

are given by Carlitz in [6]. Let $\lfloor x \rfloor$ be the integer part of x. Then,

$$\pi_n(v) = \prod_{k=1}^{n} (k+v)^{\left\lfloor \frac{n}{k} \right\rfloor},$$

and$\Phi_n(v) = a_0^{(n)} + a_1^{(n)} v + \ldots + a_d^{(n)} v^d$ is a polynomial of degree

$$d = 1 - n + \sum_{i=2}^{n} \left\lfloor \frac{n}{k} \right\rfloor .$$

The values of the first coefficients are given in figure 4.

Remark. In this array the Catalan numbers appear. An explanation of this fact is given in [12].

2. ENUMERATION OF PARALLELOGRAM POLYOMINOES

In this paragraph, we use the bijection μ between parallelogram polyominoes and Dyck words described in section 1. We apply a method due to Schützenberger [21] in order to get first the generating function of Dyck words according to the parameters length and number of peaks. A particular "reading" of the derivation rules of the Dyck grammar allows us to get the third parameter, sum of the height of the peaks. This method will be described in [7]. We deduce the generating function,

$$^{q}f(t) = \sum_{n,k\geq 1} a_{k,n}\, q^k t^n ,$$

where $a_{k,n}$ is the number of parallelogram polyminoes having n columns and area k, and we show some recurrence on $a_{k,n}$.

Proposition 2. *The number of Dyck words having length* 2n *and* k *peaks, is*

$$\frac{1}{n}\binom{n}{k}\binom{n}{k-1} .$$

This is a classical property related with the Narayana numbers (see for instance [17]).

Proof. Let D' be the set of the words written over the alphabet $\{x, \bar{x}, t\}$, obtained by substituing each factor $xt\bar{x}$ to the factor $x\bar{x}$ in the nonempty Dyck words. We say that we "mark" each peak with the letter t. This language is solution of the following equation,

$$D' = xt\bar{x} + xt\bar{x}\,D' + xD'\bar{x} + xD'\bar{x}D'.$$

Let

$$d(t,x) = \sum_{n,k\geq 0} a_{n,k}\, x^n t^k ,$$

where $a_{n,k}$ is the number of Dyck words having length 2n and k peaks. Commuting the variables in the equation of D' gives the following equation,

$$d(t,x) = xt + xt\, d(t,x) + xd(t,x) + x\, d(t,x)^2,$$

At last, the Lagrange inversion formula proves proposition 2.

Proposition 3. *Let* $^{q}f(t)$ *be the generating function of the Dyck words according to the number of peaks and the sum of the height the peaks . Then* $^{q}f(t)$ *satisties the following functional equation,*

$$^{q}f(t) = qt + qt\, ^{q}f(t) + {}^{q}f(qt) + {}^{q}f(t)\, ^{q}f(qt).$$

Preuve. The method used is described in [7]. It deals with the more general problem of getting the generating function of some combinatorial objects according to two parameters, for instance perimeter and area. More details can be found in [7] and [11]. We just recall here the principle of the method which is divided in four steps.

(1) We code the studied objects by the words of an algebraic language L so that the perimeter can be directly read on the length of the words. This is the classical methodology of Schützenberger [21]. Commuting the variables in the algebraic system, one obtains from a grammar G of L gives the generating function accorting to the perimeter.

(2) For each word w of L, we consider the monomial $\varphi(w) = q^k$ where k is the area of the object coded by w. The idea is to define recursively the function φ from the derivation rules of the grammar G in order to construct the q-analog ${}^qL$ of the language L. It is the set of words (q;w) obtained by applying the recursive definition of φ to w.

(3) We consider the formal series

$$^qS = \sum_{w \in L} (w;q) ,$$

which satisfies a q-analog of the system of algebraic equation satisfied by

$$S = \sum_{w \in L} w .$$

(4) Commuting the variables, we get a functional equation satisfied by the generating function,

$$^q1(t) = \sum_{n,k=0}^{\infty} a_{n,k} \, t^n \, q^k ,$$

where $a_{n,k}$ is the number of studied objects having perimeter n and area q.

Let g be the map which associates to each word from D' the monomial q^k where k is the sum of the height of the peaks of w. The following recursive relations allow us to construct the q-analog ${}^qD'$of the language D' which is the set of the words (w;q) when w describes D'.

$(xt\bar{x};q) = xqt\bar{x}$,

$(xt\bar{x}\, u;q) = xqt\bar{x}\,(u;q)$ for every word u in D',

$(xu\bar{x};q) = x\, q^{|u|_t}\,(u;q)\,\bar{x}$, for every word u in D',

$(xu\bar{x}\, v;q) = x\, q^{|u|_t}(u;q)\,\bar{x}\,(v;q))$, for u and v words in D' (see figure 5).

Let us consider the formal series ${}^qS$,

$$^qS = \sum_{u \in D'} (u;q) .$$

The image of ${}^qS$ by the morphism χ sending t on t, and x,$\bar{x}$ on 1, is the function ${}^qf(t)$,

$$^qf(t) = \sum_{u \in D'} g(u)\ \chi(u) .$$

So the generating function of the Dyck words according to the parameters number of peaks and sum of the height of these peaks, which is also the generating function of the skew Ferrers diagrams or parallelogram polyominoes according to the parameters number of columns and area, is exactly the function ${}^qf(t)$. Applying the recursive definition of the function g gives

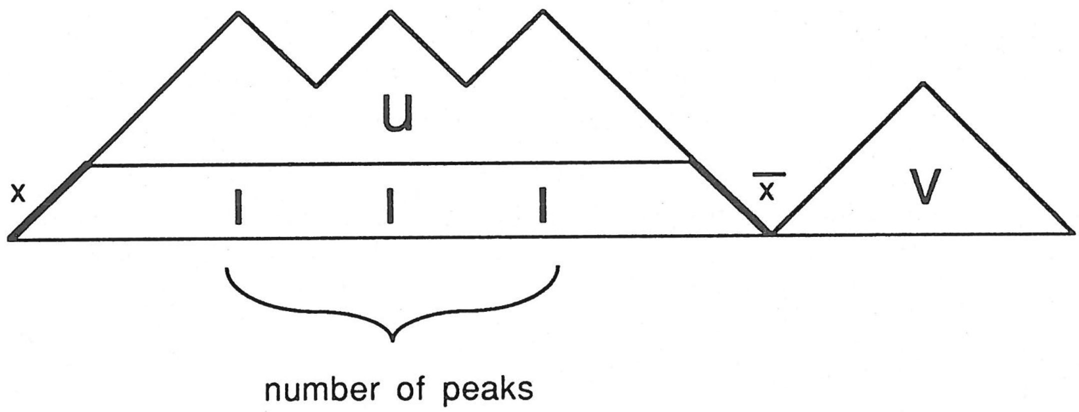

Figure 5. The equality $(x u \bar{x} v;q) = x\, q^{|u|_t} (u;q)\, \bar{x}(v;q)$.

$$^{q}f(t) = qt + qt \sum_{u\in D'} g(u)\,\chi(u) + \sum_{u,v\in D'} q^{|u|_t} g(u)\,(g(v)+1)\chi(u)\chi(v)\ ,$$

that is,

$$^{q}f(t) = qt + qt\, ^{q}f(t) + {}^{q}f(t) + {}^{q}f(t)\, ^{q}f(qt).$$

Let,

$$^{q}f(t) = \sum_{n=1}^{\infty} a_n(q)\, t^n\ ,$$

We denote for short $a_n(q)$ by a_n. The functional equation gives,

$$a_1 = q + q\, a_1,$$

and, if $n > 1$,

$$a_n = q^n a_n + q\, a_{n-1} + \sum_{k=1}^{n-1} a_k\, q^k\, a_{n-k}\ ,$$

thus, setting

$$a_n = \frac{\alpha_n t^n}{(1-q)^{2n-1}}\ ,$$

we have $\alpha_1 = 1$ and for every n, $n>1$,

$$(1-q^n)\,\alpha_n = (1-q)^2\,\alpha_{n-1} + \sum_{k=1}^{n-1} (1-q)q^k\,\alpha_k\,\alpha_{n-k}\ ,$$

$$[n]\,\alpha_n = (1-q)\,\alpha_{n-1} + q\,\alpha_1\alpha_{n-1} + q^{n-1}\,\alpha_1\alpha_{n-1} + \sum_{k=2}^{n-2} q^k\alpha_k\alpha_{n-k}\ ,$$

which is

$$\alpha_2 = \frac{1}{[2]}\ ,$$

and for every n, $n \geq 3$,

$$[n]\, \alpha_n = (1+q^{n-1})\, \alpha_{n-1} + \sum_{k=2}^{n-2} q^k \alpha_k\, \alpha_{n-k} .$$

Let us denote by $f_0(t)$ the formal power series

$$f_0(t) = \sum_{n=1}^{\infty} \alpha_n t^n .$$

Finaly, we get

$$f_0\left(\frac{qt}{(1-q)^2}\right) = (1-q)\ {}^{q}f(t) ,$$

which gives the following

Theorem 4. *The generating function* ${}^{q}f(t)$ *of the Dyck words according to the parameters sum of the height of the peaks and number of peaks is* $(1-q)\, f_0(qt\, /\, (1-q)^2)$, *where the coefficients of* f_0 *satisfy*

$$[n]\, \alpha_n = (1+q^{n-1})\, \alpha_{n-1} + \sum_{k=2}^{n-2} q^k \alpha_k\, \alpha_{n-k} .$$

We show in the next section that the function f_0 can be expressed using Bessel functions. For explaning the introduction of the function f_0, we must say that first, we computed (using Macsyma) the first values of a_n, then we read the Sloane and after some others computations, we introduced this function.

3. NEW BASIC BESSEL FUNCTIONS

Let us first recall some technics of q-calculus. The q-analog of an integer n is the polynomial

$$[n] = 1 + q + q^2 + \ldots + q^{n-1},$$

and the q-analog of n factorial is

$$[n]! = \prod_{i=1}^{n} [i] .$$

The q-derivative of a function f(x) is defined by

$$D_q\,(f(x)) = \frac{f(qx)-f(x)}{qx-x} .$$

This q-derivative coïncides with the usual one when $q \to 1$.

Example $D\,(x^n) = [n]\, x^{n-1}$

Classical formulas of derivation are easily extended to the q-derivation. For instance, if u and v are two functions,

$$D_q(u+v) = D_q(u) + D_q(v),$$
$$D_q(uv)(x) = D_q(u)(x).v(x) + u(qx).\ D_q(v)(x),$$
$$D_q(\frac{1}{u})(x) = -\ \frac{D_q(u)(x)}{u(x)u(qx)}.$$

The reader will find in [2], [4], [10] the q-analogs of classical functions and their properties.

Here, we will use a slightly different form of the Bessel functions. This form is close from the one used by some combinatorists (see for instance [5]).

Definition 2. *For any integer* ν, *let* $I_\nu(z)$ *be the function defined by* ,

$$I_\nu = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{n+\nu}}{n!\ (n+\nu)!}$$

Remark. One gets J_ν from I_ν by changing the variable x in z using,

$$J_\nu(z) = (\frac{z}{2})^{-\nu}\ I_\nu(\frac{z^2}{4}),$$

The functions I_ν satisfy a similar property to the Carlitz's one for J_ν, that is

Property 6

$$\frac{I_{\nu+1}(x)}{I_\nu(x)} = \sum_{n=1}^{\infty} \frac{\Phi_n(\nu)}{\pi_n(\nu)} x^n .$$

The usual q-analog of the Bessel function would be

$${}^qI_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\ x^{n+\nu}}{[n]!\ [n+\nu]!},$$

where each occurence of a factorial has been replaced by its q-analog. Here, we need a slightly different definition, which is

Definition 7. *Let* ${}_qI_v(x)$ *be the q-analog of the Bessel function* I_v,

$$ {}_qI_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \, q^{\binom{n+v}{2}} \, x^{n+v}}{[n]! \, [n+v]!} \; . $$

We define $\varphi_v(x)$ *by,*

$$ \varphi_v(x) = \frac{{}_qI_{v+1}(x)}{{}_qI_v(x)} \; , $$

3 PROPERTIES OF THE FUNCTIONS $\varphi_v(x)$

In this paragraph, we first give formulas about q-derivatives of the functions ${}_qI_v(x)$ in order to get a q-differential equation satisfied by $\varphi_0(x)$. Then we show the q-analog of property 6 in the particular case when $v = 0$.

Theorem 8. *The function* $\varphi_0(x)$ *satisfies the following q-differential equation,*

$$ D_q(\varphi_0(x)) = 1 + (1-q)\,\varphi_0(x) + \frac{1}{x}\,\varphi_0(x)\,\varphi_0(qx) \; . $$

Proof. This theorem comes from the formulas for the q-derivative of the functions ${}_qI_v(x)$, combined with the formulas of q-derivation. Indeed,

$$ D_q({}_qI_0(x)) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{[n]!\,[n]!} D_q(x^n) \; , $$

$$ = \sum_{n=0}^{\infty} \frac{(-1)^n \, q^{\binom{n}{2}}}{[n-1]!\,[n]!} \, x^{n-1} \; , $$

$$ = - \frac{1}{x} \, {}_qI_1(x) \; . $$

When $v > 0$, we similarly get ,

$$ D_q({}_qI_v(x)) = \sum_{n=0}^{\infty} \frac{(-1)^n \, q^{\binom{n+v-1}{2}}}{[n]!\,[n+v-1]!} \, (qx)^{n+v-1} \; . $$

So we have,

$$ D_q({}_qI_v(x)) = {}_qI_{v-1}(qx) \; , $$

$$ = {}_qI_{v-1}(x) + (qx-x)\, D_q({}_qI_{v-1}(x)) \; . $$

In particular,

$$D_q({}_qI_1(x)) = {}_qI_0(x) - (q-1)\,{}_qI_1(x).$$

Finaly, using q-derivation formulas we get,

$$D_q\left(\frac{{}_qI_1(x)}{{}_qI_0(x)}\right) = \frac{{}_qI_0(x) - (q-1)\,{}_qI_1(x)}{{}_qI_0(x)} + \frac{1}{x}\,\frac{{}_qI_1(qx)\,{}_qI_1(x)}{{}_qI_0(qx)\,{}_qI_0(x)},$$

and theorem 8 follows. We conjecture the following property,

Proposition 9. *The functions* $\varphi_v(x)$ *is given by*

$$\varphi_v(x) = \sum_{n=1}^{\infty} \frac{[\Phi_n](v)}{[\pi_n](v)} x^n ,$$

where $[\pi_n](v)$ *is the natural q-analog of* π_n,

$$[\pi_n](v) = \prod_{k=1}^{n} [k+v]^{\left\lfloor \frac{n}{k} \right\rfloor},$$

and $[\Phi_n](v)$ *is a polynomial in the variables* q and v *and with positive coefficients.*

Definition 10. *We denote by* λ_n *the natural q-analog of* $\pi_n(0)$ *which is the polynomial* $[\pi_n](0)$, *that is,*

$$\lambda_n = \prod_{i=1}^{n} [i]^{\left\lfloor \frac{n}{i} \right\rfloor} .$$

Remark. The polynomials λ_n satisfy the following equalities,

$$\lambda_n = \prod_{j=1}^{n} ([\, n/j\,]!)^{j} ,$$

and,

$$\text{if } n>1,\ \lambda_n = \lambda_{n-1} \prod_{d/n} [d] .$$

The first values of λ_n are 1, [2], [2][3], $[2]^2$ [3][4],...

Definition 11. *For every integers* $n\geq 1$ *and* $i \leq n$, *define the q-binômial of shape* λ *as,*

$$\begin{bmatrix} n \\ i \end{bmatrix}_{\lambda} = \frac{\lambda_n}{\lambda_i \lambda_{n-i}} .$$

It is possible to construct some posets (binomial in the Stanley's sense [22]) such that the number of maximal chains of length n is λ_n. We obtain sets which are too eccentric to be analyzed.

Lemme 12. *For every integers* $n \geq 2$ *and* $1 \leq i \leq n-1$.

$$\frac{1}{[n]} \begin{bmatrix} n \\ i \end{bmatrix}_\lambda ,$$

is a polynomial with integer coefficients.

This lemma is a direct consequence of a basic property of the integer part. The definition of λ_n gives,

$$\frac{1}{[n]} \begin{bmatrix} n \\ i \end{bmatrix}_\lambda = \frac{1}{[n]} \prod_{1 \leq j \leq n} j^{\left\lfloor \frac{n}{j} \right\rfloor - \left\lfloor \frac{i}{j} \right\rfloor - \left\lfloor \frac{n-j}{j} \right\rfloor} .$$

Let n be a strictly positive integer and i, j two integers less than n. Let

$$a = \left\lfloor \frac{n}{j} \right\rfloor , \quad b = \left\lfloor \frac{i}{j} \right\rfloor \text{ and } c = \left\lfloor \frac{n-i}{j} \right\rfloor .$$

We have both inequalities

$$jb \leq i < j(b+1) \text{ and } jc \leq n-i < j(c+1).$$

Thus we have,

$$j(b+c) \leq n < j(b+c) + 2 \text{ and } a-b-c \geq 0.$$

If $j=n$, then $b=c=0$ the equality is trivial. Then we can prove the following

Proposition 13. $f_0(x) = \varphi_0(x)$.

Proof. Let

$$\varphi_0(x) = \sum_{n=1}^{\infty} \alpha_n x^n ,$$

where α_n depends on q. The equality of theorem 4 can be written in,

$$\sum_{n=1}^{\infty} [n] \alpha_n x^{n-1} = 1 - \sum_{n=1}^{\infty} (q-1)\alpha_n x^n + \sum_{i,j=1}^{\infty} \alpha_i \alpha_j q^j x^{i+j-1} .$$

We have $\alpha_1 = 1$ and for every n, $n \geq 1$,

$$[n+1]\alpha_{n+1} = (1-q)\alpha_n + \sum_{k=1}^{n} \alpha_k \alpha_{n-k+1} q^k .$$

$$\beta_1 = 1,$$
$$\beta_2 = 1,$$
$$\beta_3 = 1 + q^2,$$
$$\beta_4 = 1 + q + 2q^2 + 3q^3 + 2q^4 + q^5 + q^6,$$
$$\beta_5 = 1 + q + 3q^2 + 5q^3 + 6q^4 + 6q^5 + 6q^6 + 5q^7 + 3q^8 + q^9 + q^{10}.$$

Figure 6. The polynomials $\beta_n(q)$.

Expanding gives

$$\alpha_2 = \frac{1}{[2]}$$

and for evey n, $n \geq 2$,

$$[n+1]\alpha_{n+1} = (1+q^n)\,\alpha_n + \sum_{k=2}^{n-1} \alpha_k \alpha_{n-k+1} q^k .$$

Using this last equality, we easily conclude.

Remark we have

$$\alpha_1 = 1,$$
$$\alpha_2 = \frac{1}{[2]}$$

and the above equality is enough to define the function φ_0 by recursion.

Theorem 14. *Property 9 holds for* $\nu = 0$.

Proof. The following proof is made using calculus. A more elegant combinatorial proof using valued trees is given in [12]. Let

$$\alpha_n = \frac{\beta_n}{\lambda_n},$$

we have

$$\beta_1 = 1,\ \beta_2 = 1,$$

and for every integer $n \geq 2$,

$$\beta_{n+1} = (1+q^n)\frac{1}{[n+1]}\frac{\lambda_{n+1}}{\lambda_n} + \sum_{k=2}^{n-1} \frac{1}{[n+1]} \begin{bmatrix} n+1 \\ k \end{bmatrix}_\lambda \beta_k \beta_{n-k+1}\, q^k ,$$

Using definition 10 and lema 12, an induction gives the proof of theorem 14.

Conclusions.

(1) The method we used here seems to be a powerfull generalisation of the Schützenberger methodology. In particular, it can be used even when the expected generating function is not algebraic.
(2) We showed that the generating functions $a_n(q)$ of skew Ferrers diagrams having a fixed number n of rows according to the area are rationnal. These functions have others interesting combinatorial interpretations. In [12], it is shown that they are first related to the Ehrhart theory about the enumeration of points with integer coordinates in a convex polytop. This allows to describe these functions by the mean of valued binary trees. On the other hand, these functions appear also in the enumeration of some multichains of the cartesian plane.
(3) The main open problem about this work is to find a combinatorial interpretation of numerators and denominators of the functions a_n.

Aknowledgment. We are very grateful to Adriano Garsia from the University of California at San Diego for initiating us in the q-calculus technics. His help and infectious enthousiasm have been precious all through this work.

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Marie-Pierre DELEST,
Jean-Marc FEDOU,
LaBRI, Département d'informatique,
Université de Bordeaux I,
351, cours de la Libération,
33405 Talence Cedex, France.