Valuated Matroids
– A new Look at the Greedy Algorithm

by

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R.G. Bland and M. Las Vergnas have introduced oriented matroids which can be viewed as an abstraction of matroids representable over an ordered field. Analogously, we define valuated matroids as an abstraction of matroids which are representable over some field with a non archimedian valuation.

1. Definition:
A *valuated matroid* M_v of rank m consists of a set E together with a map $v : E^m \to \mathbf{R}^+ \cup \{0\}$ satisfying

(V0) There exist $e_1, \dots, e_m \in E$ with $v(e_1, \dots, e_m) \neq 0$;

(V1) For all $e_1, \dots, e_m \in E$ and all permutations $\pi \in \Sigma_m$ we have $v(e_{\pi(1)}, \dots, e_{\pi(m)}) = v(e_1, \dots, e_m)$.
Moreover, we have $v(e_1, \dots, e_m) = 0$, if $e_i = e_j$ for some i, j with $i \neq j$.

(V2) For $e_0, \dots, e_m, f_2, \dots, f_m \in E$ there exists some i with $1 \leq i \leq m$ and

$$v(e_1, \dots, e_m) \cdot v(e_0, f_2, \dots, f_m) \leq v(e_0, \dots, \hat{e}_i, \dots, e_m) \cdot v(e_i, f_2, \dots, f_m) \quad .$$

$\mathcal{B}_v := \{\{e_1, \dots, e_m\} \mid v(e_1, \dots, e_m) \neq 0\}$ is called the set of *bases* of M_v; v is called a *valuation*.

Remarks:

i) If $v : E^m \to \mathbf{R}^+ \cup \{0\}$ is a valuation, then in view of (V1) we write $v(\{e_1, \dots, e_m\}) = v(e_1, \dots, e_m)$ for pairwise distinct $e_1, \dots, e_m \in E$.

ii) If M is a matroid defined on E with $\mathcal{B}$ as its set of bases, then any valuation $v : E^m \to \mathbf{R}^+ \cup \{0\}$ with $v(B) \neq 0$ if and only if B is a base of M is called a valuation of M.

2. Example:
Assume p is a prime number and consider the p-adic valuation $v_p : \mathbf{Q} \to \mathbf{Q}^+ \cup \{0\}$ given by

$$v_p(0) := 0,$$
$$v_p(\tfrac{l}{k} \cdot p^n) := p^{-n} \text{ for } n \in \mathbf{Z}, l, k \in \mathbf{Z} \setminus p \cdot \mathbf{Z}.$$

If E is a spanning subset of $\mathbf{Q}^m$, then the Grassmann-Plücker identities

$$\sum_{i=0}^{m} (-1)^i \cdot det\,(e_0, \dots, \hat{e}_i, \dots, e_m) \cdot det\,(e_i, f_2, \dots, f_m) = 0 \text{ for } e_0, \dots, e_m, f_2, \dots, f_m \in E$$

imply that $v := v_p \circ det : E^m \to \mathbf{Q}^+ \cup \{0\}$ is a valuation. The underlying combinatorial geometry is induced by linear (in)dependence.

3. Definition and Lemma:

i) Assume $\alpha \in \mathbb{R}^+$ and M is a matroid with $\mathcal{B}$ as its set of bases. Then $v : E^m \to \mathbb{R}^+ \cup \{0\}$ defined by

$$v(e_1, \ldots, e_m) := \begin{cases} \alpha & \text{for } \{e_1, \ldots, e_m\} \in \mathcal{B} \\ 0 & \text{otherwise} \end{cases}$$

is a valuation of M; all these valuations are called *trivial.*

ii) For $\alpha \in \mathbb{R}^+$, a map $\eta : E \to \mathbb{R}^+$ and any map $v : E^m \to \mathbb{R}^+ \cup \{0\}$ we define a new map $w = v[\alpha, \eta] : E^m \to \mathbb{R}^+ \cup \{0\}$ by

$$w(e_1, \ldots, e_m) := \alpha \cdot \prod_{i=1}^{m} \eta(e_i) \cdot v(e_1, \ldots, e_m).$$

If v is a valuation, then w is also a valuation and v and w are called *projectively equivalent.* In case $\alpha = 1$ we write also $v[1, \eta] = v_\eta$.

iii) A valuation $w : E^m \to \mathbb{R}^+ \cup \{0\}$ of a matroid M is called *essentially trivial,* if w is projectively equivalent to some trivial valuation.

iv) A matroid M is called *rigid,* if every valuation of M is essentially trivial.

4. Example:

The uniform matroid $U_{2,4}$ of rank 2 with 4 elements is not rigid.

5. By applying the theory of the Tutte group of a matroid one proves

Theorem:

All binary matroids and all finite projective spaces of dimension at least two are rigid.

6. A Variation of the Greedy Algorithm:

Assume E is a finite set and $v : \binom{E}{m} \to \mathbb{R}^+ \cup \{0\}$ is a function defined on the m-subsets of E with $v(B) > 0$ for some $B \in \binom{E}{m}$. To find some $B \in \binom{E}{m}$ with a large v-value one may proceed according to the following greedy algorithm.

Step 0: Choose some $e_1, \ldots, e_m \in E$ with $v(\{e_1, \ldots, e_m\}) > 0$.

Step $k (1 \le k \le m)$: Assume that $x_1, \ldots, x_{k-1}$ are determined and choose some $x_k \in E$ such that

$$v(\{x_1, \ldots, x_{k-1}, x_k, e_{k+1}, \ldots, e_m\}) \ge v(\{x_1, \ldots, x_{k-1}, x, e_{k+1}, \ldots, e_m\})$$

for all $x \in E \setminus \{x_1, \ldots, x_{k-1}, e_{k+1}, \ldots, e_m\}$.

We say that this algorithm *works* for v, if for all $e_1, \ldots, e_m \in E$ as above and all permitted choices of the $x_1, \ldots, x_m$ one has $v(\{x_1, \ldots, x_m\}) \ge v(B)$ for all $B \in \binom{E}{m}$.

Theorem:

Assume E is finite, $m \le \#E$ and $v : \binom{E}{m} \to \mathbb{R}^+ \cup \{0\}$ is some map with $v(\binom{E}{m}) \ne \{0\}$. Then v defines a valuation $v : E^m \to \mathbb{R}^+ \cup \{0\}$ if and only if the Greedy Algorithm works for v_η for all maps $\eta : E \to \mathbb{R}^+$.

References

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