On the Growth Rate of certain Combinatorial Functions

by

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ABSTRACT.– We establish upper and lower bounds for so called Kanamori– McAloon Functions which were studied in [KM 87] by model theoretic reasoning. The bounds we present fit into the hierarchy concept introduced in [Wa 72].

1. Introduction: Undecidable Properties

Gödel in his famous paper [Gö 31] was the first to prove the existence of certain statements formalizable in Peano arithmetic PA, which are neither povable nor refutable from the axioms of PA but nevertheless true. Especially he was able to show that the formalized version of PA's consistency itself remains undecidable if PA happens to be consistent at all. The proof was completed by Gentzen [Ge 36] in 1936, when he showed the latter condition to be valid if one is only willing to accept transfinite induction up to ε_0 as a meaningfull mathematical statement. From a combinatorial point of view the situation still remained unsatisfactory, because the formalized consistency-predicate involved all the coding of the formal language under consideration. A breakthrough was obtained by Paris and Harrington [PH 77] in 1977, when they came up with a purely combinatorial Ramsey-type statement featuring the nice properties of undecidability as well as validity. They essentially showed this statement to be equivalent to PA's constistency using strong model-theoretic methods. Later on Ketonen and Solovay [KS 81] succeeded in stripping the proof of its proof-theoretic garments by giving recursive lower and upper bounds for the related combinatorial functions, which PA failed to prove total. In defining such functions the afore mentioned results suggest, that one has to refer to ordinals at least up to ε_0 in one way or another.

Although Paris and Harrington were the first to obtain undecidable combinatorial statements the combinatorial proof given by Ketonen and Solovay still was technically involved. Kanamori and Mc Aloon [KM 87] owe the credit of skipping the inessentials and it is mainly the aim of this talk to sketch the construction of lower and upper bounds for the combinatorial functions they invented. Because the more sophisticated proofs are technically involved we restrict ourselves to give the reader an overview of the results in the next section and return to a "sketch of proof" in section 4.

2. Growth rates of certain combinatorial functions

1. In order to present our results we list the functions under consideration first.

(PH) Let $k, r \in \omega \backslash \{0\}$ be given. Then
there exists a least $n = PH(r,k) \in \omega\backslash\{0\}$ such that for all $\Delta : \binom{n}{k} \rightarrow r$ there exists $L \subseteq n$, $minL \leq |L|$, $k < |L|$ satisfying $\Delta \restriction L$ is constant.

(KM) Let $k, m \in \omega\backslash\{0\}$ be given. Then
there exists a least $n = KM(id, m, k) =: KM(m, k) \in \omega\backslash\{0\}$ such that $n \rightarrow (m)^k_{id-reg}$,
with $\rightarrow$ denoting the usual arrow notation, which requires Δ to depend on minimum elements only if restricted to $\binom{M}{k}$ for some $M \subseteq n$ satisfying $|M| = m$ and considering only $\Delta : \binom{n}{k} \rightarrow \omega$ with $\Delta(A) \in id(minA) \cup \{0\}$ (id-regressive colorings). Let $KM(m) := KM(m, 2)$.

(KM*) Same statement as in (KM), where id is replaced by an increasing function $f_k : \omega \rightarrow \omega$ and $KM(id, m, k)$ by $KM(f_k, m, k) =: KM^*(m, k)$. More precisely, if $T_1(n) = n + 1$ and $T_{k+1}(n) = (n + 1)^{T_k(n)}$, f_k is defined for $k \geq 3$ by $f_k := (n + 1)^{k-2}T_{k-2}(n) \cdot T_{k-3}(n) \cdot \ldots \cdot T_1(n)$.

(KS) Let $k, m \in \omega\backslash\{0\}$ and λ be an ordinal. Then
there exists a least $n = KS_\lambda(m, k) \in \omega\backslash\{0\}$ such that for all $\Delta : \binom{n}{2} \rightarrow \lambda\times\omega$ with $\Delta(x, y) \in N(\lambda_k, x)\times x$ there exists $M \in \binom{n}{m}$ such that for all $x < y < z$ in M we have :

1) $\Delta(x, y) = \Delta(x, z)$

2) $\alpha \leq \alpha'$ for $\Delta(x,y) = (\alpha, l)$, $\Delta(y,z) = (\alpha', l')$.

We would like to hint at the difference in the first two statements. The concept of relative largeness ($\min L \leq |L|$) in (PH) is replaced by minimum homogeneity in (KM) and id– regressive colorings are considered rather than finite colorings.

Some comments are also necessary in order to understand the meaning of the last statement. A fixed initial segment $\lambda + 1$ of the second number class has been chosen together with an assignment $\alpha[.] \ : \omega \to \alpha$, $\alpha[n] < \alpha$ of a fundamental sequence for each $\alpha \in \lambda + 1$ satisfying:

$\alpha[n] < \alpha[n+1]$; $\lim_{n\to\infty} \alpha[n] = \alpha$, for α a limit; $(\alpha+1)[n] = \alpha$; $0[n] = 0$ for each n.

For $k \in \omega$ define $\lambda_k := \lambda[k]$.

The set of predecessors of $\alpha, N(\alpha, x), (\alpha \in \lambda + 1, x \in \omega)$ is defined by

$$N(\alpha, x) := \{\alpha[x]\underbrace{[x]\dots[x]}_{i-times} | i \in \omega\}$$

At first sight the definition of $KS_\lambda(k,m)$ seems to be slightly artificial but it proves to be useful in order to discard ordinal notions occuring in lower bound estimations. Because of 1) the second condition in (KS) simply requires Δ to be increasing with respect to its first coordinate on M.

2. The sample functions we want to compare with rely on the chosen initial segment as well as the fundamental sequence assignment and occur already in similar form in [Wa 72]:

$$F_0(n) = n + 1$$

$$F_{\alpha+1}(n) = F_\alpha^{(n)}(n), \text{ the n-th iterate of } F_\alpha$$

$$F_\alpha(n) = F_{\alpha[n]}(n), \alpha \text{ a limit ordinal.}$$

Some further requirements of the assignment $\alpha[.]$ as the Bachmann property [Ba 67] should be met to assure certain monotonicity properties of the functions F_α.

The growth rate of these functions is beyond comprehension. $F_3(n)$ for example dominates the n-two's tower $2^{2^{2^{\cdot^{\cdot^{2^n}}}}}$, $F_{\omega+1}$ dwarfs the Ackermann-function and, as Spencer remarked, in the naming-large-number-game $F_{\varepsilon_0+9}(9)$ should win against any non-logician. (The indices always indicate, that a certain assignment of fundamental sequences has been chosen, where in the case of the ordinal ε_0 use is made of the Cantor Normal Form theorem:

$$\omega_1^{\alpha_1} + \cdots + \omega_n^{\alpha_n}[k] = \begin{cases} \omega_1^{\alpha_1} + \cdots + \omega_n^{\alpha_n - 1} \cdot k \text{ if } \alpha_n \text{ is a successor ordinal} \\ \omega_1^{\alpha_1} + \cdots + \omega_n^{\alpha_n[k]}, \text{ if } \alpha_n \text{ is a limit ordinal} \end{cases}$$

$$\omega_k := \varepsilon_0[k] := \underbrace{\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}}_{k-times}$$

3. Presentation of results

3.1 Lower bounds

Let us consider the function $KM(m)$ first. It turns out that $KM(m)$ fails to be primitive recursive, which was already observed by [KM 87] and can also be seen from our result in [PTV 89]:

$$KM(Ram(2, m+3, k)) \geq F_k(m), \text{ where}$$

$$Ram(l, m, r) := \text{ least } n : n \to (m)_r^l,$$

because $Ram(l, m, r)$ happens to be primitive recursive. Generally it can be shown that

$$KM^*(Ram(2+m, 2m+3, 2^m), 2+m) \geq F_{\varepsilon_0}(m)$$

(for details see [PTV 89]). By relating the functions $KM(m, k)$ and $KM^*(m, k)$ similar bounds may be obtained for $KM(m, k)$ thereby establishing the above mentioned undecidability results.

One can see from the proof given by Ketonen and Solovay [KS 81], that $KM^*(m, m)$ serves also as a lower bound for $PH(m, k)$:

$$PH(312 + 24m, 3 + 2 \cdot m) \geq KM^*(m, m)$$

As already mentioned the function KS plays a crucial role in the lower bound estimations and therefore deserves special attention. Under certain assignment conditions for fundamental-sequences (i.e. Bachmann's property) we obtain for the general case:

$$KS_\lambda(m+3,k) \geq F_{\lambda_k}(m), \quad \text{which implies}$$

$$KS_\lambda(m+3,m) \geq F_\lambda(m).$$

We want to conclude this section with the remark that by going beyond ε_0 (F_{Γ_0} or even higher) KS grows so fast that even Feferman's system of predicative analysis fails to prove its totality (for details see again [PTV 89]).

3.2 Upper bounds

While J. Ketonen and R. Solovay come up with estimations for Paris– Harrington functions, most of the investigations concerning upper bounds for Kanamori–McAloon functions is based on the work in [Th 89]), where a thorough analysis of the above mentioned functions as well as regressive colorings of singletons is given. The easiest case was already treated in [PV 89], where it is shown that:

$$KM(m) < F_{m-1}(3) < F_\omega(m)$$

Generally one obtains for k-element subsets:

$$KM^*(m,k) \leq F_{\varepsilon_0[k]}(g(m-3)),$$

where the primitive recursive function g depends on the function f_k introduced earlier. A similar bound can be given for the function $KM(m,k)$ itself.

If we replace the requirement of minimum homogeneity occuring implicitely in the statement (KM) by the stronger condition of minimum-ascendency

($min(A) \leq min(B) \Rightarrow \Delta(A) \leq \Delta(B)$) and reserve the notation $KM_*(m) := KM_*(m,2)$ for this case we obtain:

$$KM_*(m) \leq F_{\omega^2}(m),$$

which shows that increasing the ordinal to ω^ω (as follows from the proof given by Kanamori and McAloon (see [KM 87])) in order to satisfy the ascending condition is not really necessary.

As far as KS_λ is concerned it is possible for the cases $\lambda = \omega$ and $\lambda = \varepsilon_0$ to obtain upper bounds as well (see [Th 89]) e.g.:

$$F_\omega(k^2 \cdot m) \geq KS_\omega(m, k),$$

which proves the lower bounds to be not too bad.

If $\lambda = \varepsilon_0$ we obtain generally

$$KS_{\varepsilon_0}(m, k) \leq F_{\omega^2_{k+1}}(g(m))$$

for a certain primitive recursive function g.

In order to state our results for regressive colorings of singletons let us introduce the following notation:

Let $KS(\lambda, m)$ be the smallest $n \in \omega \setminus \{0\}$ such that for all $\Delta : [1, n] \to \lambda, \Delta(x) \in N(\lambda, x)$ there exists $M \subseteq [1, n]$, $|M| = m$ such that $\Delta \restriction M$ is increasing. We then have

$$KS(\omega_k, m) \leq F_{\omega^2_k}(g(m)).$$

The general case is also investigated in [Th 89], where upper bounds could be obtained by λ- recursion. We conclude with the remark that sharp upper bounds are available only in case $\lambda = \omega$ at the time being.

4. Selection of Proofs

This section is devoted to a sketch of those ideas mainly responsible for the results we obtained.

4.1. The Tree Argument

We need some notation first. Unless otherwise stated we adopt the following conventions:

Natural numbers as well as ordinals are always identified with the set of their predecessors and denoted by latin and greek letters respectively. Let ω be the first infinite ordinal and $\mathbb{N} = \omega \setminus \{0\}$. For any set A and $k \in \omega$ let $\binom{A}{k}$ denote the set of all k-element subsets of A and $[k] := \{1, \ldots, k\}$. Depending on context 2^A will either stand for the set $\{f \,|\, f : A \longrightarrow 2\}$ or for the powerset $\mathcal{P}(A)$. Sometimes $\binom{n}{k}$ and 2^n will also be used to denote the natural numbers themselves but the meaning should always be clear from the context. Sets will be listed in ascending order and for $P \neq \emptyset$, where $|P| < \infty$ let $P^0 := P \setminus \{maxP\}$ and $P_0 := P \setminus \{minP\}$. Given $A \subseteq \omega$ and $f, g : A \longrightarrow A$ we define $f^{(n)} : A \longrightarrow A$ for $n \in \omega$ recursively by

$$f^{(0)} := id\,,\; f^{(n+1)} := f^{(n)} \circ f, \text{ where } (f \circ g)(x) := f(g(x)).$$

Although we prefer the latter notation we sometimes feel free to use fgx instead of $f(g(x))$ if formulas become more readable and no confusion is to be expected. Regarding n as the set of its predecessors will never have us $f(n)$ to mean $\{f(x) \,|\, x \in n\}$. A function $F : \omega \to \omega$ is called <u>increasing</u> whenever $x \leq y$ implies $F(x) \leq F(y)$ for all $x, y \in \omega$.

Definition 4.1: Let $F : \omega \longrightarrow \omega$ be an increasing function . Call a coloring

$$\Delta : \binom{\omega}{k} \longrightarrow \omega$$

<u>F-regressive</u> if $\Delta(A) \in F(minA) \cup \{0\}$ for all $A \in \binom{\omega}{k}$. Δ is called <u>min-homogeneous</u> on $M \subseteq \omega$ if for all $A, B \in \binom{M}{k}$:

$$\min A = \min B \text{ implies } \Delta(A) = \Delta(B).$$

Δ is called <u>min-ascending</u> on $M \subseteq \omega$ if for all $A, B \in \binom{M}{k}$:

$$\min A \leq \min B \text{ implies } \Delta(A) \leq \Delta(B).$$

Obviously the second condition implies the first.

For the following digression on the Erdös-Rado canonization theorem (see [ER 50]) we imlore the advanced readers foregiveness but think it helpful in order to understand how compactness arguments work in general.

A coloring $\Delta : \binom{S}{k} \longrightarrow \omega$ is called <u>canonical</u> for some $S \subseteq \omega$ and $k \in \omega$ whenever there exists $V \subseteq k$ such that for all $a_i, b_i \in S$, where $i \in k$ we have

$$\Delta(\{a_0, \ldots, a_{k-1}\}) = \Delta(\{b_0, \ldots, b_{k-1}\}) \text{ if and only if}$$

$$a_i = b_i \text{ for all } i \in V.$$

If V happens to be empty the second condition becomes vacuously true and the coloring has to be constant; otherwise it depends only on the pattern induced by V on k. The important thing to note is the equivalence in the characterization of being canonical.

We are now ready to state the canonization theorem in two different forms:

Theorem: i) For every $k \in \omega$ and all colorings $\Delta : \binom{\omega}{k} \longrightarrow \omega$ there exists $R \subseteq \omega$ satisfying $|R| = \infty$, which is canonically colored under Δ. □

ii) For every $k, m \in \omega$ there exists $n \in \omega$ such that for every coloring $\Delta : \binom{n}{k} \longrightarrow \omega$ there exists $R \subseteq n$ satisfying $|R| = m$, which is canonically colored under Δ.

Proof: We do not intend to prove i), but show how i) implies ii) because this is exactly the point where compactness comes into play. Given $k, m \in \omega$ we call a coloring $\Delta : \binom{n}{k} \longrightarrow \omega$ bad if it does not satisfy the requirements imposed under ii). W.l.o.g. we may assume that $\Delta(A) < |\binom{n}{k}|$ for otherwise we could easily produce such a coloring having the same pattern. Arguing by contradiction allows us to to fix $k, m \in \omega$ and to obtain at least one bad coloring $\Delta : \binom{n}{k} \longrightarrow \omega$ for each $n \in \omega$. So the set of all bad colorings is certainly infinite. We identify a coloring with its set of pairs and order the family of all bad colorings by inclusion. The order is generating an infinite tree T, which is finetely branching, because the number of bad colorings $\Delta : \binom{n}{k} \longrightarrow \omega$, where n is fixed, is finite. Königs lemma assures us about the existence of an infinite path

$$\Delta_m \subsetneq \Delta_{m+1} \subsetneq \ldots$$

of bad colorings $\Delta_i : \binom{i}{k} \longrightarrow \omega$, where $i \geq m$ and we may consider $\Delta := \bigcup_{i \geq m} \Delta_i$ in order to contradict the nonexistance of bad $k-$ element colorings of ω : If $R = \{r_1, \ldots, r_m, \ldots\}$ is given as in i) the coloring Δ_{r_m} would certainly have failed to be bad in the first place. □

Using the Erdös-Rado canonization theorem we get:

Theorem 4.1:

i) For all $k \in \omega$, all mappings $F : \omega \longrightarrow \omega$ and all colorings $\Delta : \binom{\omega}{k} \underset{F-reg}{\longrightarrow} \omega$

there exists an infinite $M \subseteq \omega$ satisfying $\Delta \restriction \binom{M}{k}$ is min-homogeneous.

ii) For all $m, k > 0$, and all mappings $F : \omega \longrightarrow \omega$ there exists $n \in \omega$ such that

for all $\Delta : \binom{n}{k} \underset{F-reg}{\longrightarrow} \omega$ there exists $M \in \binom{n}{m}$ satisfying

$\Delta \restriction \binom{M}{k}$ is min-homogeneous.

Proof: Following the lines of the preceding proof it is easily seen that the infinite version implies the finite one. We therefore restrict ourselves to the proof of the first part of the theorem which can be reduced to the canonization theorem. Using the latter theorem we obtain an infinite subset R of ω on which our given $F-$ regressive coloring Δ is canonical. If $V = \emptyset$ or $V = 1$ we get what we claim letting $M := R$. Otherwise there exists $i \in V$ with $i \geq 1$. For $R = \{r_0, r_1, r_2, \ldots\}$ let us consider the $F(r_0) + 1$ sets

$$A_m = \{r_0, r_1, \ldots, r_{i-1}, r_{i+m}, r_{i+m+1}, \ldots, r_{m+k-1}\},$$

where $0 \leq m \leq F(r_0)$. Obviously $|A_m| = k$ for $0 \leq m \leq F(r_0)$ and because Δ is $F-$ regressive the pigeon hole principle yields p,q with $0 \leq p < q \leq F(r_0)$ satisfying $\Delta(A_p) = \Delta(A_q)$. But this means that $r_{i+p} = r_{i+q}$, which is absurd. □

If we replace the notion of minimum homogeneity by minimum ascendency in theorem 4.1 the same arguments work noting that there are no infinite strictly descending chains of natural numbers. Therefore the least numbers $n \in \omega$ satisfying the requirements of theorem 4.1 are denoted by

$$KM(F, m, k) \quad \text{and} \quad KM^*(F, m, k) \text{ respectively }.$$

We adopt the convenient notation $n \rightarrow (m)^k_{F-reg}$ following the use in the literature.

The proof using the afore mentioned theorem as well as a compactness argument does not provide us with constructive upper bounds. KM and KM^* obviously are functionals involving F. Nevertheless the function F does not affect the growth rate of KM essentially as long as it behaves primitive recursively. Kanamori and McAloon [KM 87] show how KM^* can be estimated in terms of KM at the price of increasing the exponent k, but one can do slightly better than that. Obviously $KM(id, m, 1) = m$ and in the case $k = 1$ for KM^* sharp upper bounds may be obtained.

We will frequently use a tree-argument invented by Erdös and Rado (see [EHMR 84]) in order to prove Ramsey-type theorems and restate a modified version most suitable for our purposes. Let us consider the following definition first.

Definition 4.2: By a rooted tree T on $A \subseteq \omega$ we mean a tree having A as the set of its nodes, where some element $a \in A$ is chosen as a root. Let a rooted tree T on $A \subseteq \omega$ be given. Directing its edges towards the given root we may identify T with the set of pairs (x, y) in A for which x immediately predecedes y and is nearer to the root. The set of immediate successors of x in T will be denoted by $succ_T(x)$. We omit T if the context allows. A rooted tree T on A is called an <u>A-tree</u> if and only if xTy implies $x < y$ for all $x, y \in A$. Let $f : \omega \rightarrow \omega$ be strictly increasing. An A-tree T is called <u>f(x)-small branching</u> provided $|succ(x)| = |\{y|xTy\}| \leq f(x)$.

Because the following theorem plays an important role in several of our arguments we motivate the tree- argument by showing how $A-$ trees may be used to obtain theorems of Ramsey– type. Suppose we want to show that the set $A_n := \{1, 2, \ldots, 2^{2n-3}\}$ for all colorings $\Delta : \binom{A_n}{2} \longrightarrow 2$ always contains an n– element subset N such that we have $\Delta \restriction \binom{N}{2}$ is constant, i.e.

$$\text{for all n} : 2^{2n-3} \longrightarrow (n)^2_2 .$$

We may achieve this by constructing the following A_n– tree T from the given Δ : Let us put the number 1 at the top of the tree and subdivide the remaining elements into two classes according to the color they obtain under Δ if supplemented by the element 1. Now we continue this process with both classes separately, where the immediate successors of the root 1 are given by the smallest elements of both classes. The following picture shows the final situation we end up with if we restrict ourselves to the case $n = 3$, where the original coloring Δ is indicated on the left hand side:

$\Delta(1;2) = \Delta(1;4) = \Delta(1;5) = 0$

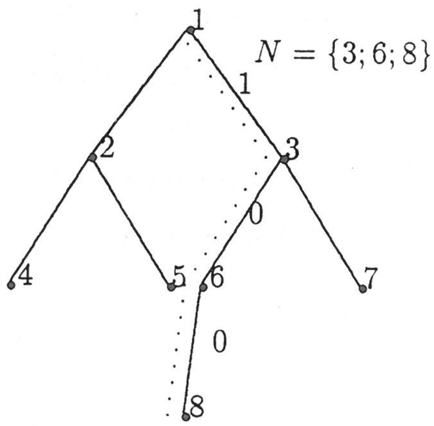
$\Delta(1;3) = \Delta(1;6) = \Delta(1;7) = \Delta(1;8) = 1$

$\Delta(2;4) = 0; \Delta(2;5) = 1$

$\Delta(3;6) = \Delta(3;8) = 0; \Delta(3;7) = 1$

$\Delta(6;8) = 0$

$\vdots$



Picture 1

The binary A_n– tree T has to contain a path P of cardinality $2n - 2$ on which Δ depends only on its minimum elements. Therefore P^0 is of cardinality $2(n-2)+1$ and, by the pigeon hole principle, must contain at least $n - 1$ elements on which Δ is constant by construction. Inserting max P will provide us with the desired n– element set N. The theorem below resembles the reasoning above:

Theorem 4.2: (tree-argument) Let a finite subset $A = \{a_1, \ldots, a_n\}$ of ω be given. Let $f : \omega \to \omega$ be a strictly increasing function and $F : P(A) \to \omega$ be a function satisfying $F(X) < f(maxX^0)$ for all $X \in P(A)$ with $|X| \geq 2$. Then there exists an $f(x)$-small branching A-tree T such that $X^0 = Y^0$ implies $F(X) = F(Y)$ for all paths P in T and all $X, Y \subseteq P$ with $|X| = |Y| \geq 2$.

Proof: The required tree T is defined recursively: Let $T_1 = \emptyset$ be a tree on $\{a_1\}$ and suppose T_i on $\{a_1, \ldots, a_i\}$ where $i < n$ has already been defined. In order to define T_{i+1} consider the following procedure Pr:

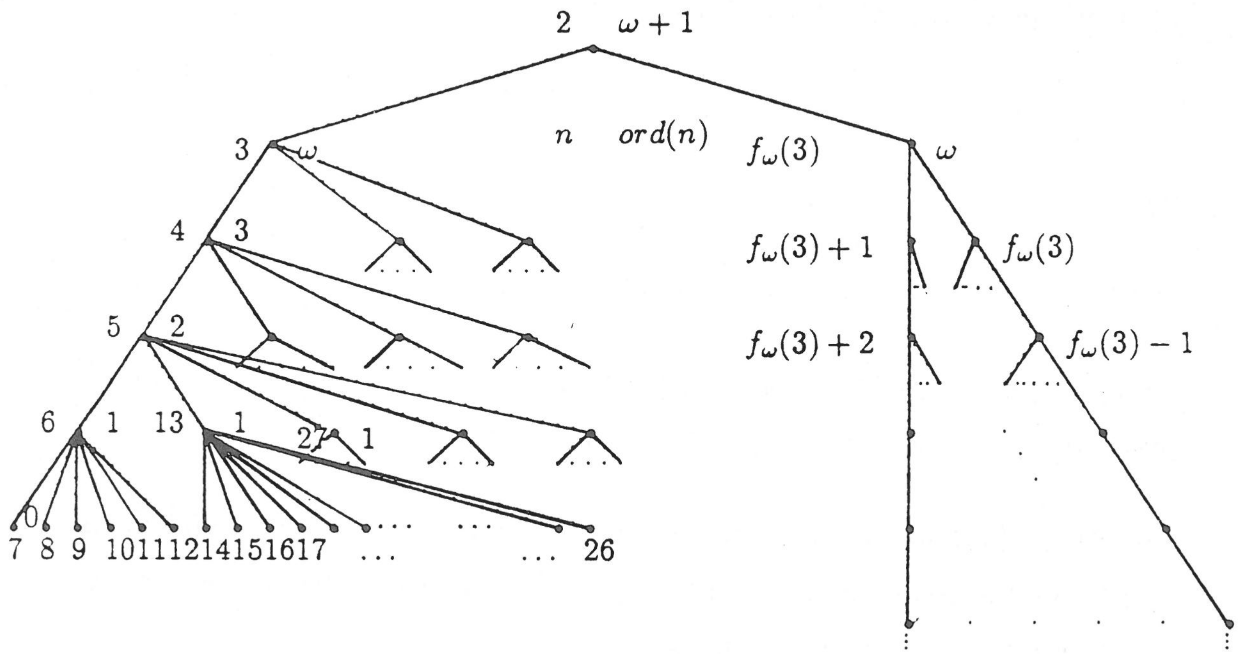
Let $k := 1$.

If there exists a_j in $succ_{T_i}(a_k)$ for which $F(P_{a_k} \cup \{a_j\}) = F(P_{a_k} \cup \{a_{i+1}\})$ let $k := j$ and continue with the first line of Pr (P_{a_k} denotes the uniquely determined path $a_1 T \ldots T a_k$ in T_i). Otherwise let $T_{i+1} := T_i \cup \{(a_k, a_{i+1})\}$. Finally let $T := T_n$. The required properties are easily checked. □

As will be seen in the following the requirement of the existence of large paths (in order to iterate the argument) will force the set A to be very large. Because the function F does not depend on the last elements if restricted to a path in T we will eventually end up with a subset of A on which F depends only on singletons. If for example $\Delta : \binom{n}{k} \underset{reg}{\longrightarrow} \omega$ is given, we may choose $F(E)$ where $E \subseteq n$ to code the restricted coloring $(\Delta(X))_{X \subseteq E, \max X = \max E}$ yielding $F(E) < f(\max E^0)$, where $f(x) = x^{(2^x)}$. In the case where $k = 2$ it is already sufficient to choose

$$F(E) = \Delta(\{\max E^0, \max E\}) \text{ and } f(x) = x.$$

In order to obtain upper bound estimations like those occuring at the beginning of section 3.2 we have to specify the trees measuring the sizes of the underlying sets. They serve as a link between the combinatorial functions and the numbertheoretic functions we want to compare with. Suffice it to present a typical tree sufficiently large to deal with (KM) statements, which has to replace the binary tree of picture 1. Observe that the number of successors as well as the depth of the tree may no longer be restricted to a given size. Although this fact is mainly responsible for the difficulty in estimating upper bounds, its regular building nevertheless allows us to determine its size by means of the functions F_α :



Picture 2

We conclude this exposition by showing a typical lower bound estimation.

Theorem 4.3: $KS(m+3,k) \geq F_{\omega_k}(m)$.

Proof: Let the mapping $\Delta : \binom{\omega}{2} \longrightarrow \varepsilon_0 \times \omega$ be defined by $\Delta(x,y) = (0,0)$ if $F_{\omega_k}(x) \leq y$ and $\Delta(x,y) = (\alpha, l)$ otherwise, where $\alpha \in N(\omega_k, x)$ and $l \in [1, x-1]$ satisfy

$$F_\alpha^l(x) \leq y < F_\alpha^{l+1}(x).$$

One readily sees that α and l are defined properly. Let $M \in \binom{\omega}{m+3}$ satisfy 1) and 2) of (KS) and let $x < y < z$ be the three largest elements of M. We show that $\Delta(x,y) = (0,0)$ or $\Delta(y,z) = (0,0)$ from which $KS(m+3,k) \geq F_{\omega_k}(m)$ follows.

Assume to the contrary that $\Delta(x,y) = \Delta(x,z) = (\alpha, l)$ and $\Delta(y,z) = (\beta, l')$ with $l, l' > 1$ and $\alpha \leq \beta$. Then

$$F_\alpha^l(x) \leq y < z < F_\alpha^{l+1}(x).$$

We apply F_α to this inequality. The monotonicity of F_α assures that

$$(*) \qquad z < F_\alpha^{l+1}(x) \leq F_\alpha(y).$$

By definition of $\Delta(y,z)$ we know that

$$(**) \qquad F_\beta^{l'}(y) \leq z.$$

But

$$(***) \qquad F_\alpha(y) \leq F_\beta(y) \leq F_\beta^{l'}(y)$$

because $F_\alpha(k) \leq F_\beta(k)$ for all $\alpha \in N(\beta, k)$ and as $l' \geq 1$. Now (*), (**) and (***) produce the obvious contradiction that $z < z$. □

The study of regressive colorings into natural numbers also allows generalizations by considering colorings into transfinite numbers and there is hope for many interesting results coming up in this area.

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