Séminaire lotharingien de combinatoire 22.Treffen

Hesselberg 28-30 September 1989

GALOIS CORRESPONDENCES IN CATEGORY THEORY

by

Jacques RIGUET

This title has been chosen because of shortness,but the real one is : "Generalised Galois corrrespondences in (generalised) category theory" We shall try to explain in what sense those generalisations have to be understood and what are our motivations for introducing them .At the level of those words of introduction, it is enough to say that a Galois correspondence being a pair of mappings between ordered sets having to satisfy certain conditions, we generalise it by replacing the mappings by difunctional relations and ordered sets by preordered sets or at a later stage by categories or actegories .

1. SOME DEFINITIONS FOR BINARY RELATIONS

If R is a class of pairs ,then $\breve{R}$ denotes its converse , $\rhd R$ its domain , $\lhd R$ ($= \rhd \breve{R}$) its codomain , $\diamond R$ ($= \rhd R \cup \lhd R$) its field $\rhd\ell R = \rhd R \ominus \lhd R$ the class of its minimal elements , $\lhd\ell R = \lhd R \ominus \rhd R$ the class of its maximal elements .

If X is a class , $\mathbb{I}_X = \{ (x,x) \ / \ x \in X \}$ denotes the identical relation on X .

We set $\mathbb{I}_R = \{ (x,x) \ / \ (x,x) \in R \}$ $(= R \cap \mathbb{I}_{\rhd R} = R \cap \mathbb{I}_{\lhd R})$

$R^{\sim} = \{ (x,x') \ / \ \exists\, y \ \ (x,y) \in R \text{ and } (x',y) \in R \}$

R is said to be <u>reflexive on X</u> iff $\mathbb{I}_X \subset R$; to be <u>transitive</u> iff $\exists y$ $(x,y) \in R$ and $(y,z) \in R$ implies $(x,z) \in R$; to be a <u>preorder</u> iff R is reflexive on $\Diamond R$ and transitive ; to be an <u>order</u> iff , moreover , $R \cap \breve{R} \subset \mathbb{I}_{\Diamond R}$.

The <u>section of R by the element x</u> is, by definition : $R(x) = \{ y \,/\, (x,y) \in R \}$. If n is an integer,the <u>class of elements of degree n</u> ((resp. <u>of codegree n)</u> <u>with respect to R</u> is,by definition :

$\rhd_n R = \{ x \,/\, x \in \Diamond R$ and $R(x)$ has exactly n elements $\}$ (resp. $\lhd_n R = \rhd_n \breve{R}$)

Obviously, $\rhd_0 R = \lhd \ell R$ and $\lhd_0 R = \rhd \ell R$.

If J is a set of integers, we denote by $\rhd_J R$ the union of $\{ \rhd_n R \,/\, n \in J \}$

R is an <u>equivalence</u> iff R is a symmetrical preorder (i.e. R=R) ; an <u>equivalence on X</u> iff , moreover, $\Diamond R = X$.

R is <u>functional</u> or <u>univocal</u> iff $\breve{R}^{\sim} \subset \mathbb{I}_{\Diamond R}$ or iff $\breve{R}^{\sim} = \mathbb{I}_{\lhd R}$ or iff $\rhd R = \rhd_1 R$

R is then an equivalence on $\rhd R$.

R is <u>cofunctional</u> iff $\breve{R}$ is functional.

R is <u>biunivocal</u> iff R is functional and cofunctional .

R is called <u>difunctional</u> iff $\exists y$ $\exists y'$ $(x,y) \in R$ and $(y,y') \in \breve{R}$ and $(y',z) \in R$ implies $(x,z) \in R$. $R^{\sim}$ is then an equivalence on $\rhd R$ and $\breve{R}^{\sim}$ an equivalence on $\lhd R$; further, R is the disjoint union of all the rectangles of the form $X \times Y$ which are included in it, X and Y being respectively equivalence classes with respect to $R^{\sim}$ or to $\breve{R}^{\sim}$.

A symmetrical difunctional relation is called an <u>alternance</u>

If X and Y are classes and R a class of couples ,we denote by $R|_X$ the <u>restriction of R to X</u> ,that is to say $\{ (x,y) \,/\, (x,y) \in R$ and $x \in X \}$ and ,similarly ,by $R\rceil_Y$ $(= (\breve{R}|_Y)^{\smile})$ the <u>corestriction of R to Y</u> .

One says that $R|_{\rhd_1 R}$ is the <u>univocal part of R</u> .

An arrow joining some points x and y (x at the beginning ,and y at the end) is said to be a possible <u>sagittal writing</u> of the couple (x,y) .

The sagittal writing is space consuming , but permits a "geometrical grasp" which helps a lot in many cases . For instance the sagittal way of asserting that R is difunctional can be stated by the implication :

$$\left\{ \begin{tikzcd} x \arrow[r] & y \\ y' \arrow[ur] \arrow[r] & z \end{tikzcd} \right\} \subset R \quad \text{implies} \quad (x \longrightarrow z) \in R$$

A <u>word of length n</u> is, by definition ,a functional relation u such that its domain $\triangleright u$ is the interval]n] = {1,2,...,n} .
If $u = \{ \overset{1}{\underset{u_1}{\downarrow}} \cdots\cdots \overset{n}{\underset{u_n}{\downarrow}} \}$ the word u is written $\overleftarrow{u_1, u_2, \dots, u_n}$ or, simply $u_1 u_2 \dots u_n$ providing that no danger of confusion results .
Let $R_1, \dots, R_m$ be classes of pairs . Then $\overrightarrow{R_1, \dots, R_n}$ or $\overrightarrow{R_n, \dots, R_1}$ denotes the result of the <u>composition operation</u> on the word $\overrightarrow{R_1, \dots, R_n}$ = $\overrightarrow{R_n, \dots R_1}$ which, by definition, is the class of all couples (x,z) such that there exists y ,y ...,y ,y such that $y_1 = x$, $y_{n+1} = z$ and $\forall$ i $\in$ n] $(y_i \longrightarrow y_{i+1}) \in R$. In the following pages ,we adopt the traditional shortening which consists of writing ambigously $R_n \dots R_1$ instead of $\overrightarrow{R_n \dots R_1}$. For instance, with this convention, the restriction $R|_X$ can also be written $R\,\mathbb{I}_X$,the corestriction $R\rfloor_Y$ can also be written $\mathbb{I}_Y R$, $R \cap X \times Y$ can also be written $\mathbb{I}_Y R\, \mathbb{I}_X$ and $R^{\sim}$ can also be written $\breve{R}\, R$.
When $R_1 = R_2 = \dots = R_n = R$,then $\overrightarrow{R \dots R}$ is, of course , conventionally written R^n.

The union of { R^n / n positive integer } is called the <u>transitive closure of R</u> and denoted by $\overline{R}$

The <u>cyclic part of R</u> is , by definition , $\mathring{R} = R \cap \breve{\overline{R}}$.

The <u>acyclic part of R</u> is ,by definition , $\check{R} = R \ominus \breve{\overline{R}} = R \ominus \mathring{R}$.

The <u>connex closure of R</u> is ,by definition ,the transitive closure of $R \cup \breve{R}$ .It is an equivalence on $\Diamond R$, its equivalence classes are called <u>connected components of R</u> .

The <u>difunctional closure of R</u> is, by definition $\dddot{R} = R\ \overline{R^{\sim}} = \overline{\check{R}^{\sim}}\ R$.

We say that <u>R is a Ferrers relation</u> iff $\exists y\ \exists y'$ $(x,y) \in R$ and $(y, y') \in \breve{R}^{\dagger}$ and $(y',z) \in R$ implies $(x,z) \in R$; where,by definition, $R^{\dagger} = \overline{R \cup \check{R}} \ominus R$.

2. GRACTS

We have introduced this neologism in order to express , just by saying that R is a <u>gract</u> ,that R is a class of couples for which each second element is itself a couple .

Let us consider a couple $(x,(y,z))$ of the form just described . Writing it sagittaly , as explained in §1,we obtain: $x \longrightarrow (y,z)$ or $y \xrightarrow{\overset{x}{\downarrow}} z$.But,if we use this last way of writing ,we can remark that we do not loose anything but gain in conciseness if we suppress the arrow going from x to $y \longrightarrow z$, obtaining thus $y \xrightarrow{x} z$. This (simplified) sagittal writing of $(x,(y,z))$ justify that one calls x the label, y the source and z the target of $(x,(y,z))$.

The <u>dual of a gract R</u> is, by definition the gract :

$$\text{dual } R = \{ (z \xrightarrow{x} y) \ / \ (y \xrightarrow{x} z) \in R \}$$

A gract which is such that for each possible label $x \in \triangleright R$ there is only one possible source and one possible target is said to be a <u>graph</u> . Equivalently ,a gract R is a graph iff R is functional .

If R is a graph, $\triangleright_1 R$ is called the class of its <u>loop-labels</u> and $R|_{\triangleright_1 R}$ its <u>subgract of loops</u> .

To any gract R , one can associate a graph :

$gR = \{((x,(y,z)),(y,z)) \ / \ (x,(y,z)) \in R \}$ called <u>the graph of R</u> .

cell R = $(gR)^{\sim}$ is an equivalence on R which is called <u>the class of cells of R</u> .

cell R is thus the class of all pairs of elements of R of the type
$$\begin{tikzcd} y \arrow[r, "x"] \arrow[d, phantom, "\downarrow"] & z \\ y \arrow[r, "x'"'] & z \end{tikzcd}$$
,that we write down usually $\begin{tikzcd} y \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & z \end{tikzcd}$, making use of the <u>"cellular writing"</u> of those pairs that we adopt almost exclusively in what follows .

It is useful also to introduce the <u>class of cocells of R</u> denoted cocell R and defined as the class of all pairs of elements of the type
$$\begin{tikzcd} y \arrow[r, "x"] \arrow[d, phantom, "\downarrow"] & z \\ y & z \arrow[l] \end{tikzcd}$$
,that we write down "cellularly" $\begin{tikzcd} y \arrow[r, bend left] \arrow[r, phantom, "\Downarrow"] & z \arrow[l, bend left] \end{tikzcd}$.

It is very useful to introduce the following terminology :

p being some property that can be satisfied by a binary relation , and R being a gract,one says that $x \in \triangleright R$ satisfies the property p in R iff the section R(x) satisfies this property .If some adjective is used to express this property p,then, one qualifies x with this adjective .

For instance , $x \in \triangleright R$ is said to be functional in R , or to be R-functional iff R(x) is functional i.e. $(y \xrightarrow{x} z)$ R and $(y \xrightarrow{x} z')$ R implies z=z'.(One says also that x is a deterministic label) .

3. <u>SOME FUNDAMENTAL GRACTS</u>

The first fundamental gract to be considered is probably <u>the gract of binary relations</u> .It is defined as the class :

$$\underset{\sim}{br} = \{ (x \xrightarrow{R} y) \ / \ R \text{ is a set of pairs and } (x,y) \in R \} .$$

As the section of $\underset{\sim}{br}$ by R is R itself, then ,according to the adjectival terminology introduced at the end of §2, R satisfies the property p iff R satisfies the property p in $\underset{\sim}{br}$.

When $(x \xrightarrow{R} y) \in \underset{\sim}{br}$ and when R is functionnal ,we write frequently Rx instead of y ,as y is uniquely defined from x and R .

One has also to consider as fondamentals the gract of correspondences defined as the class :

$$\text{cor} = \{ \ (X \xrightarrow{R} Y) \ / \ X \text{ and } Y \text{ are sets and } R \subset X \times Y \ \} .$$

and the followings subgracts of it :

$$\text{ucor} = \{ \ (X \xrightarrow{R} Y) \ / \ (X \xrightarrow{R} Y) \in \text{cor} \text{ and } R \text{ is functional} \ \} ,$$

$$\check{\text{u}}\text{cor} = \{ \ (X \xrightarrow{R} Y) \ / \ (X \xrightarrow{R} Y) \in \text{cor} \text{ and } R \text{ is cofunctional} \ \} ,$$

$$\text{vcor} = \{ \ (X \xrightarrow{R} Y) \ / \ (X \xrightarrow{R} Y) \in \text{cor} \text{ and } \rhd R = X \ \} ,$$

$$\check{\text{v}}\text{cor} = \{ \ (X \xrightarrow{R} Y) \ / \ (X \xrightarrow{R} Y) \in \text{cor} \text{ and } \lhd R = Y \ \} ,$$

and their various intersections denoted by $\text{u}\check{\text{u}}\text{cor}$, uvcor , etc.

uvcor is the gract of mappings , $\text{u}\check{\text{u}}\text{vcor}$ is the gract of injections , $\text{uv}\check{\text{v}}\text{cor}$ is the gract of surjections , $\text{u}\check{\text{u}}\text{v}\check{\text{v}}\text{cor}$ the gract of bijections

For later use we need to introduce the following definitions :

If Γ is a gract, the class :

$\text{cell}\,\Gamma = \{ \ \left(\left(y \underset{x'}{\overset{x}{\rightrightarrows}} z\right),(y,z)\right) \ / \ \left(y \underset{x'}{\overset{x}{\rightrightarrows}} z\right) \in \text{cell}\,\Gamma \ \}$ is called the gract of the cells of Γ . cocell Γ is defined in a similar way .

4 . TRAJECTORIES AND PATHS

If Γ is a gract and n a positive integer,we say that

$$w_n\Gamma = \left\{ \begin{array}{c|l} x \xrightarrow{u} y & u = \overline{u\,,\ldots,u} \quad \text{and } \exists\, a_2 ,\ldots, \exists\, a_n \\ & (x \xrightarrow{u_1} a_2) \quad ,\ldots, \quad (a_n \xrightarrow{u_n} y) \end{array} \right\}$$

is the gract of trajectories of Γ of length n, that

$w_0\Gamma = \{ (x \xrightarrow{\emptyset} x) \ / \ x \in \lhd\!\rhd \lhd \Gamma \}$ is the gract of empty trajectories of Γ and that the union $w\Gamma$ of $\{ \ w_n\Gamma \ / \ n \text{ integer} \ \}$ is the gract of trajectories of Γ (or the gract of Γtrajectories) .

$wg\Gamma$ is called the graph of the paths of Γ or the graph of Γpaths .

5. GENERALISED CATEGORIES : ACTEGORIES

An actegory is,by definition, a couple $\mathscr{C} = (\ \Gamma , \ H \ )$, where Γ is a

gract and H = om$\mathcal{C}$ a preconguence of wgΓ. That means that om$\mathcal{C}$ is a preorder contained in the equivalence : cell wgΓ = (gwgΓ)$^{\sim}$, this preorder being ,further , compatible with the concatenation of paths".
For more details cf.[Rig 89] .

om$\mathcal{C}$ is called <u>the class of precommutative path-cells of $\mathcal{C}$</u> , and the congruence com$\mathcal{C}$ = om$\mathcal{C}$ $\cap$ (om$\mathcal{C}$)$^{\cup}$ is called <u>the class of commutative path-cells of $\mathcal{C}$</u> .

As explained in the paper just referred ,in order to express that ,for the actegory $\mathcal{C}$ we have :

$$\left(\begin{array}{c} x \xrightarrow{(x\xrightarrow{u_1}a_2)\cdots\cdots(a_n\xrightarrow{u_n}y)} y \\ \downarrow \\ x \xrightarrow{(x\xrightarrow{v_1}b_2)\cdots\cdots(b_m\xrightarrow{v_m}y)} y \end{array}\right) \in \mathrm{om}\,\mathcal{C} \quad (\text{resp. } \mathrm{com}\,\mathcal{C}) \text{ , we write :}$$

$$\begin{tikzcd} & a_2 \arrow[r, dotted, no head] & a_n \arrow[dr, "u_n"] & \\ x \arrow[ur, "u_1"] \arrow[dr, "v_1"'] & & & y \\ & b_2 \arrow[r, dotted, no head] & b_m \arrow[ur, "v_m"'] & \end{tikzcd} \in \mathrm{om}\,\mathcal{C} \quad (\text{resp. } (x \begin{array}{c} \xrightarrow{u_1}a_2\cdots a_n\xrightarrow{u_n} \\ \xrightarrow{v_1}b_2\cdots b_m\xrightarrow{v_m} \end{array} y) \in \mathrm{com}\,\mathcal{C})$$

In general, it is tacitly supposed that an actegory $\mathcal{C}$ has units.
That means that $\mathcal{C}$ has a subgract of loops id$\mathcal{C}$ such that
$\forall (y \longrightarrow z) \in \mathcal{C}$, $\forall (y \longrightarrow y) \in$ id$\mathcal{C}$ and $\forall (z \longrightarrow z) \in$ id$\mathcal{C}$, one has
$(y \begin{array}{c} \nearrow^{y}\searrow \\ \xrightarrow[x]{} \end{array} z) \in$ com$\mathcal{C}$ and $(y \begin{array}{c} \nearrow^{z}\searrow \\ \xrightarrow[x]{} \end{array} z) \in$ com$\mathcal{C}$.

A subgract Γ' of an actegory $\mathcal{C}$ is said to be commutative iff
cellwg$\Gamma' \subset$ com$\mathcal{C}$

By abuse of language , and in order to be in accordance with the henceforth traditionnal terminology of category theory , we say that Γ' is a commutative diagramm .

The <u>dual of an actegory $\mathcal{C}$</u> is defined as the actegory <u>dual$\mathcal{C}$</u> the gract of which is the dual of the underlying gract of $\mathcal{C}$, om dual $\mathcal{C}$ being om $\mathcal{C}$ dualised in an obvious way

A subgract Γ' of an actegory $\mathcal{C}$ is said to be subactegory-inducing iff om $\mathcal{C} \cap$ cellwg Γ' is a precongruence of Γ'.
Then (Γ', om $\mathcal{C} \cap$ cellwg Γ') is said to be the actegory induced by Γ'.
id $\mathcal{C}$ is actegory-inducing, the actegory induced by it is denoted id $\mathcal{C}$.

The actegory of laxcorrespondences laxcor is defined by the gract cor and the precongruence om laxcor defined as the class :

$$\left\{ \begin{tikzcd} & A_2 \arrow[r, phantom, "\cdots"] & A_n \arrow[dr, "u_n"] & \\ X \arrow[ur, "u_1"] \arrow[dr, "v_1"'] & & & Y \\ & B_2 \arrow[r, phantom, "\cdots"] & B_m \arrow[ur, "v_m"'] & \end{tikzcd} \middle/ \overleftarrow{u_1,\ldots,u_n} \subset \overleftarrow{v_1,\ldots,v_m} \right\}$$

The actegory cor is defined by the gract cor , by the congruence om cor (= com cor) =com laxcor and by id cor $=\{(X \xrightarrow{\mathrm{I}_X} X) /$ X is a set$\}$

Translating the binary relation calculus into the language of the actegory laxcor permits to visualize the demonstrations as a pasting game of (pre)commutative diagramms . In this respect, it is useful to introduce some special properties of diagramms . For instance , the basic statement derived from Dedekind formula which asserts that if $(X \xrightarrow{r} Y)$, $(Y \xrightarrow{s} Z)$, $(Z \xrightarrow{t} X)$ are elements of cor , then :
$sr \cap \breve{t} = \emptyset \rightleftarrows ts \cap \breve{r} = \emptyset \rightleftarrows rt \cap \breve{s} = \emptyset$ can be handled more easily if one introduces the class of cycles :

$$\text{dedk cor} = \left\{ \begin{tikzcd} & Y \arrow[dr, "s"] & \\ X \arrow[ur, "r"] & & Z \arrow[ll, "t"] \end{tikzcd} \;\; \begin{array}{c} \exists x \in X \\ \exists y \in Y \quad \exists z \in Z \end{array} \;\; \begin{tikzcd} & y \arrow[dr, "s"] & \\ x \arrow[ur, "r"] & & z \arrow[ll, "t"] \end{tikzcd} \; \text{is a subgract of br} \right\}$$

The same definition can obviously be extended to any cycle of length larger than 3 of elements of cor .

In the actegory <u>cor</u> ,the subgracts $\underset{\sim}{\mathrm{ucor}}$, $\underset{\sim}{\mathrm{vcor}}$, $\underset{\sim}{\breve{\mathrm{u}}\mathrm{cor}}$,etc. are actegory-inducing,inducing actegories denoted <u>ucor,</u> <u>vcor</u> , <u>ŭcor</u> ,etc.

<u>The actegory of cells of an actegory $\mathcal{C}$</u> =(Γ , om$\mathcal{C}$) is defined as the actegory <u>cell$\mathcal{C}$</u> ,the gract of which is cellwgΓ , om <u>cell$\mathcal{C}$</u> being the class :

$$\left\{ \begin{matrix} x \overset{u_1}{\underset{u'_1}{\rightrightarrows}} a_2 \cdots\cdots a_n \overset{u_n}{\underset{u'_n}{\rightrightarrows}} y \\ x \rightrightarrows b_2 \cdots\cdots b_m \overset{v_m}{\underset{v'_m}{\rightrightarrows}} y \end{matrix} \;\Big/\; \left(x \begin{matrix} \xrightarrow{u_1}\cdots\xrightarrow{u_n} \\ \Downarrow \\ \xrightarrow[v_1]{}\cdots\xrightarrow[v_m]{} \end{matrix} y\right) \in \mathrm{om}\,\mathcal{C} \text{ and } \left(x \begin{matrix} \xrightarrow{u'_1}\cdots\xrightarrow{u'_n} \\ \xrightarrow[v'_1]{}\cdots\xrightarrow[v'_m]{} \end{matrix} y\right) \in \mathrm{om}\,\mathcal{C} \right\}$$

The actegory <u>cocell$\mathcal{C}$</u> is defined in a siimilar way .

The actegory <u>br</u> is defined by the gract $\underset{\sim}{\mathrm{br}}$ and by :

$$\mathrm{om}\ \underline{\mathrm{br}}\ (=\mathrm{com}\ \underline{\mathrm{br}}\ ) = \left\{ \begin{matrix} x \xrightarrow{u_1} a_2 \cdots\cdots a_n \xrightarrow{u_n} y \\ x \xrightarrow[v_1]{} b_1 \cdots\cdots b_m \xrightarrow[v_m]{} y \end{matrix} \;\Big/\; \left(x \begin{matrix} \xrightarrow{u_1}\cdots\xrightarrow{u_n} \\ \Downarrow \\ \xrightarrow[v_1]{}\cdots\xrightarrow[v_m]{} \end{matrix} y\right) \in \mathrm{cellwg}\,\underset{\sim}{\mathrm{br}} \text{ and } \overleftarrow{u_1\cdots u_n} = \overleftarrow{v_1\cdots v_m} \right\}$$

6. IDEMPOTENTS, JINVERSES

Let $\mathcal{C}$ be an actegory .By definition the <u>subgract of the idempotents</u>

<u>of $\mathcal{C}$</u> is

$$\underset{\sim}{\mathrm{idp}}\,\mathcal{C} = \left\{ (x \xrightarrow{y} x) \;/\; (x \xrightarrow{y} x) \in \mathcal{C},\quad \begin{tikzcd} & x \arrow[dr] & \\ x \arrow[ur, "y"] \arrow[rr, "y"'] & & x \end{tikzcd} \in \mathrm{com}\,\mathcal{C} \right\}$$

<u>the gract of jinverses of $\mathcal{C}$</u> is:

$$\underset{\sim}{\mathrm{jnv}}\,\mathcal{C} = \left\{ \left(x \underset{z_2}{\overset{z_1}{\rightleftarrows}} y\right)_{\Downarrow} \;/\; \left(x \underset{z_2}{\overset{z_1}{\rightleftarrows}} y\right)_{\Downarrow} \in \mathrm{cocell}\,\mathcal{C} \text{ and } \begin{tikzcd} & y \arrow[r, "z_2"] & x \arrow[dr, "z_1"] & \\ x \arrow[ur, "z_1"] \arrow[rrr, "z_1"'] & & & y \end{tikzcd} \in \mathrm{com}\,\mathcal{C} \right\}$$

The gract $\underset{\sim}{\breve{\mathrm{j}}\mathrm{nv}}\,\mathcal{C}$ of ǰinverses of $\mathcal{C}$ is defined in a similar manner by exchanging the roles of z_1 and z_2.

When is the actegory of mappings i. e. $\mathcal{C}$ = uvcor = map , it is known that the cartouche writing of $(E \xrightarrow{\varphi} E) \in$ idp map which consists of representing the equivalence classes of $\varphi^{\sim}$ as pointed cartouches ,the cartouche representing $\varphi^{\sim}(x)$ being filled with the elements of it,and the element x being the distinguished point, gives a full and vivid picture of φ without loss of information .

If $c = (E \underset{\beta}{\overset{\alpha}{\rightleftarrows}} F) \in$ jΥnv map,and if we set $\varphi_c = \beta\alpha$, $\psi_c = \alpha\beta$, $\Sigma_c = \alpha \cap \breve{\beta}$ and $D_c = \breve{\beta}\,(\alpha \cap \breve{\beta}\,)\,\alpha$, then,it is known that $(E \xrightarrow{\varphi_c} E)$ and $(F \xrightarrow{\psi_c} F)$ are idp maps, that $(\triangleright\beta \xrightarrow{\Sigma_c} \triangleleft\alpha)$ is a bijection and that $(E \xrightarrow{D_c} F)$ is a difunctionnal correspondence . Moreover :

$$\varphi_c^{\sim} = D_c^{\sim} = \alpha^{\sim} \;, \quad \psi_c^{\sim} = \breve{D}_c^{\sim} = \beta^{\sim} \quad \text{and} \quad \Sigma_c = D_c \cap (\triangleright\beta \times \triangleleft\alpha)$$

As a consequence of those properties, a full and vivid picture of c , with no loss of information can be given as a cartouche writing, which this time, consists of disposing in pairs joined by arrows labelled α and β going from a cartouche to the pointed element of the other one (Cf. fig 26 p.31 of [Rig 89]) .

7. PREORDERED SETS ,CLOSURE OPERATORS

A pair $E = (X,R)$ is a preordered class iff X is a class and R a class of pairs transitive and reflexive on X .

$\square E = X$ is said the underlying set of E and $\omega_E = R$ its underlying relation . If ω_E is acyclic (i.e. an order relation) E is said to be an ordered class .

The dual of E is defined as $\breve{E} = (\square E, \breve{\omega}_E)$.

If $X \subset \square E$, we say that $E|_X = (X, \omega_E \cap X \times X)$ is the preordered class induced by X .

For any preordered class E , $\underset{\sim}{E} = \underset{\sim}{\mathrm{br}}\big|_{\{\omega_E\}} = \{(x \xrightarrow{\omega_E} y) \;/\; (x,y)\in \omega_E\}$ is a subgract of $\underline{\mathrm{br}}$ which is actegory-inducing .The induced actegory is denoted by $\underline{E}$

If E is a preordered class, we say that a subclass $X \subset \square E$ is convex in E iff $\forall x \in X$, $\forall x' \in X$, $(x,y)\in\omega_E$ and $(y,x')\in\omega_E$ implies $y\in X$.

If R is an equivalence on E , we say that R is convex in E iff $\forall x\in \square E$ $R(x)$ is convex in E .

$\underset{\sim}{\mathrm{orset}} = \{ (E \xrightarrow{f} F) \;/\; (E \xrightarrow{f} F) \in \underset{\sim}{\mathrm{map}}$ and
$$\begin{tikzcd} \square E \arrow[r, "\omega_E"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow"] & \square E \arrow[d, "f"] \\ \square F \arrow[r, "\omega_F"'] & \square F \end{tikzcd} \in \mathrm{om}\ \underline{\mathrm{cor}} \;\}$$
is called the gract of preordered sets .

The actegory $\underline{\mathrm{orset}}$ of preordered sets is defined by

$$\mathrm{om}\ \underline{\mathrm{orset}} = \mathrm{com}\ \underline{\mathrm{orset}} = \left\{ \begin{tikzcd} E \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow"] & F \end{tikzcd} \;\middle/\; \begin{array}{l} \begin{tikzcd} (E \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow"] & F) \end{tikzcd} \in \mathrm{cellwg}\ \underline{\mathrm{orset}} \\ \begin{tikzcd} (\square E \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow"] & \square F) \end{tikzcd} \in \mathrm{com}\ \underline{\mathrm{cor}} \end{array} \right\}$$

The subactegory $\underline{\mathrm{ordset}}$ of ordered sets is defined in an obvious way .

The gract of closure operators is defined as

$\underset{\sim}{\mathrm{cl}}\ \underline{\mathrm{ord}} = \{(E \xrightarrow{\varphi} E) \;/\; (E \xrightarrow{\varphi} E) \in \underset{\sim}{\mathrm{idp}}\ \underline{\mathrm{ord}}$ and $\varphi \subset \omega_E\}$

The gract of coclosure operators is defined as

$\underset{\sim}{\breve{\mathrm{cl}}}\ \underline{\mathrm{ord}} = \{(E \xrightarrow{\psi} E) \;/\; (E \xrightarrow{\psi} E) \in \underset{\sim}{\mathrm{idp}}\ \underline{\mathrm{ord}}$ and $\psi \subset \breve{\omega}_E\}$

Obviously : $(E \xrightarrow{\varphi} E) \in \underset{\sim}{\breve{\mathrm{cl}}}\ \underline{\mathrm{ord}}$ iff $(\breve{E} \xrightarrow{\psi} \breve{E}) \in \underset{\sim}{\mathrm{cl}}\ \underline{\mathrm{ord}}$

If $(E \xrightarrow{\varphi} E) \in \underset{\sim}{\mathrm{cl}}\ \underline{\mathrm{ord}}$, then $(\square E \rightarrow \square E) \in \mathrm{idp}\ \underset{\sim}{\mathrm{map}}$.

The cartouche writing of $(E \xrightarrow{\varphi} E)$ consists of the cartouche writing of $(\square E \xrightarrow{\varphi} \square E)$ enriched by dotted arrows going from x to x' iff $(x,x')\in\omega_E$,those dotted arrows being directed downwards when x and x' are lying in the same cartouche . As $\varphi^{\sim}$ is convex in E, every cartouche is convex in that sense that, when x and x' are inside it and such that $(x,x')\in\omega_E$,then the interval having x and x' for extremities is included in it .

The cartouche writing of a coclosure operation is defined similarly. One can view it as obtained by "mirorring"that one relative to a closure operator and by reversing the direction of the dotted arrows . Thus the dotted arrows joining elements of a same cartouche are this time,directed upwards .

8 ADJUNCTION COCELLS ,GALOIS COCELLS IN ord

The gract of adjunction-cocells is defined by :

$$\text{adj ord} = \left\{ E \underset{\beta}{\overset{\alpha}{\rightleftarrows}} F \;\Big/\; \left(E \underset{\beta}{\overset{\alpha}{\rightleftarrows}} F\right) \in \text{cocell ord} \ \text{ and } \begin{array}{ccc} E & \xrightarrow{\alpha} & F \\ {\scriptstyle\omega_E}\downarrow & & \downarrow{\scriptstyle\omega_F} \\ E & \xrightarrow[\breve{\beta}]{} & F \end{array} \in \text{com cor} \right\}$$

It is an actegory-inducing subgract of cocell ord , which induces in it the subactegory of adjunctions cocells : adj ord

The gract of Galois-cocells is defined by :

$$\text{gal ord} = \left\{ E \underset{\beta}{\overset{\alpha}{\rightleftarrows}} F \;\Big/\; \left(E \underset{\beta}{\overset{\alpha}{\rightleftarrows}} F\right) \in \text{cocell ord} \ \text{ and } \begin{array}{ccc} E & \xrightarrow{\alpha} & F \\ {\scriptstyle\omega_E}\downarrow & & \downarrow{\scriptstyle\breve{\omega}_F} \\ E & \xrightarrow[\breve{\beta}]{} & F \end{array} \in \text{com cor} \right\}$$

It is a subgract of cocell ord ,but is no more actegory-inducing .

We have:

$\left(E \underset{\beta}{\overset{\alpha}{\rightleftarrows}} F\right) \in$ gal ord iff $\left(E \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \breve{F}\right) \in$ adj ord iff $\left(F \underset{\alpha}{\overset{\beta}{\rightleftarrows}} \breve{E}\right) \in$ adj ord

$\left(E \underset{\beta}{\overset{\alpha}{\rightleftarrows}} F\right) \in$ adj ord iff $\left(\breve{F} \underset{\alpha}{\overset{\beta}{\rightleftarrows}} \breve{E}\right) \in$ adj ord

If $c = \left(E \underset{\beta}{\overset{\alpha}{\rightleftarrows}} F\right) \in$ cocell ord , we define $\square c$ by: $\square c = \left(\square E \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \square F\right)$

It is known that , if $c \in$ adj ord ,or, if $c \in$ gal ord ,then $c \in$ jjnv ord

As a consequence, the cartouche writing of c consists of the cartouche writing of $\square$ c enriched by dotted arrows joining x to x' iff $(x,x') \in \omega_E$

and by dotted arrows (of a different type if it is necessary to avoid confusions) joining y to y' iff $(y,y') \in \omega_F$. Moreover, when x and x' are in the same cartouche the arrows are directed downwards; and when y and y' are in the same cartouche, the arrows are directed downwards in the case of adj or upwards in the case of gal .

(Remark : the dotted arrows are just the arrows of $\underset{\sim}{br}\big|_{\{\omega_E,\omega_F\}} = \underline{E} \cup \underline{F}$)

It is known that, if $c \in \underset{\sim}{adj}\ \underline{ord}$,then

$(E \xrightarrow{\varphi_c} E) \in \underset{\sim}{cl}\ \underline{ord}$, $(F \xrightarrow{\psi_c} F) \in \underset{\sim}{\check{cl}}\ \underline{ord}$, $(E|_{D\beta} \xrightarrow{\Sigma_c} F|_{\Delta\alpha}) \in \underline{ord}$

If Q_c is defined by $Q_c = \omega_F\,\alpha = \breve{\beta}\,\omega_E$ one has

$$Q_c = \breve{\omega}_F D_c \omega_E \quad , \quad D_c = Q_c \cap \breve{\beta}\ Q_c\ \alpha$$

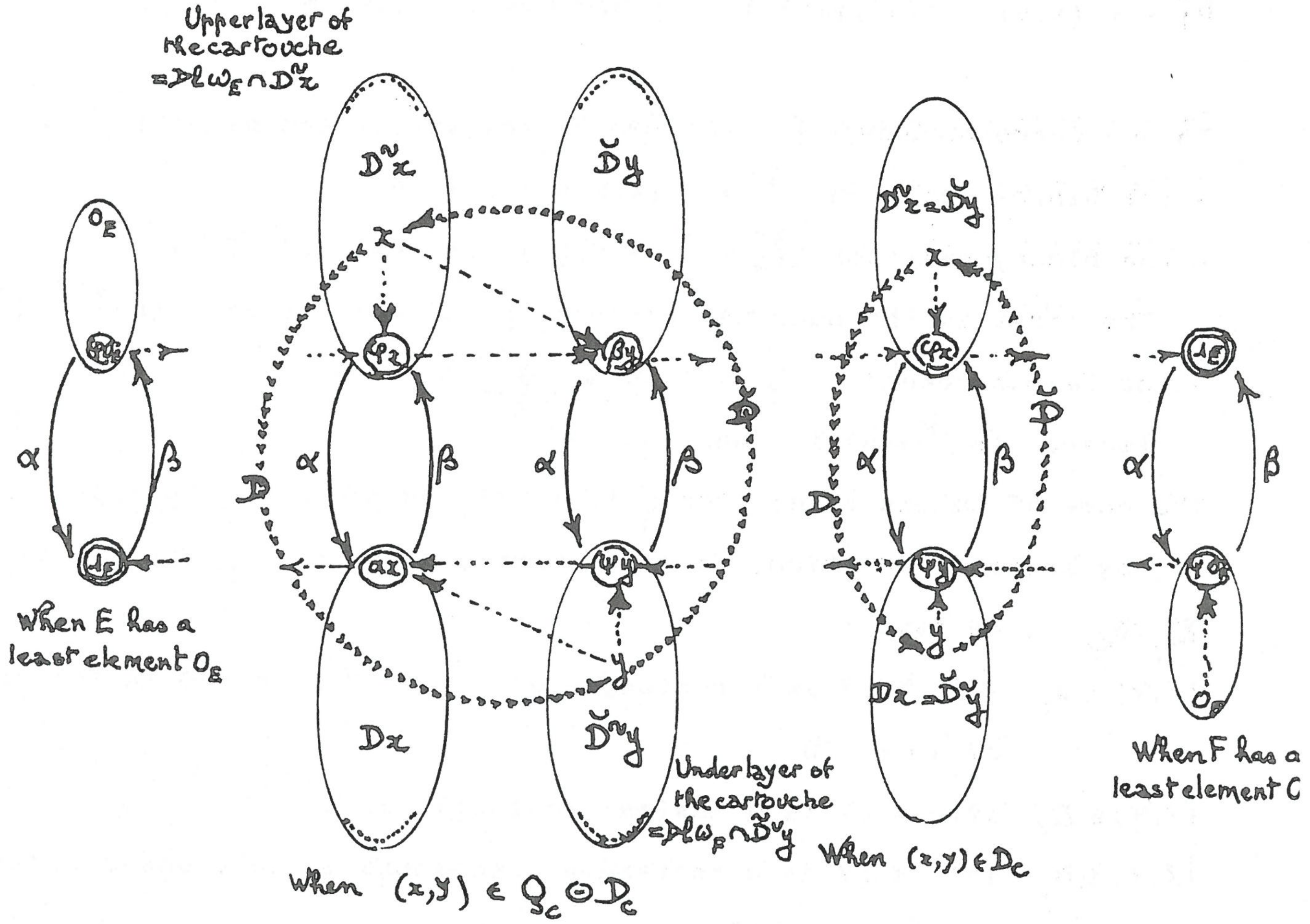

Fig 1 :

Cartouche writing of the Galois-cocell $c = (E \underset{\beta}{\overset{\alpha}{\rightleftarrows}} F) \in$ gal $\underline{ord}$

9. GALOIS COCELL OF A CORRESPONDANCE

In this paragraph, after having recalled that to any correspondance is associated a Galois cocell (generalising the association of an achieved ordered set of Dedekind cuts to any ordered set), we indicate how the "theories d'exactitude" of Van den Bril are related to this concept .

If X is a set, eX denotes the set of all subsets of X, $\underset{\sim}{e}X = eX \ominus \{\emptyset\}$ the set of all non empty subsets of X , $e_n X$ the set of all subsets of X having n elements, and the ordered set $oeX = (eX, \omega_{oeX})$ where $\omega_{oeX} = \{ (A,B) \ / \ A \subset B \subset X \}$

If R is a binary relation, we define its punctual restriction $R\upharpoonleft$ by $R\upharpoonleft = \{ (x,y) \ / \ (\{x\},y) \in R \}$. (Obviously $R\upharpoonleft = (R|_{e_1 DR})\upharpoonleft$.)

To any correspondance $\rho = (A \xrightarrow{R} B)$ are associated by definition :

- the binary relation $\rho^{\circ} = \{ (x,R(x)) \ / \ x \ \ A \}$
- the binary relation $[\rho] = \{ (X,R[X]) \ / \ X \in \underset{\sim}{e}A \} \cup \{ (\emptyset,B) \}$

  (The first is the punctual restriction of the second : $[\rho]\upharpoonleft = \rho^{\circ}$)
- the Galois cocell $c_\rho = \Bigl($ \begin{tikzcd} oeA \arrow[r, bend left, "{[\rho]}"] \arrow[r, phantom, "\Downarrow"] & oeB \arrow[l, bend left, "{[\breve{\rho}]}"] \end{tikzcd} $\Bigr)$

  (where $(B \xrightarrow{R} A)$ is denoted by $\breve{\rho}$)

The name of Galois cocell for c_ρ is justified since $c_\rho \in \underset{\sim}{gal}\ \underline{ord}$.

If, by abuse of notation, we write simply Q_ρ , Σ_ρ, D_ρ instead of Q_{c_ρ} $\Sigma_{c_\rho}, D_{c_\rho}$, we have :

$(X,Y) \in Q_\rho$ iff $X \times Y$ is a rectangle of R and X=A or Y=B in the case of $X \times Y = 0$

$(X,Y) \in \Sigma_\rho$ iff $X \times Y$ is a maximal rectangle of R

$(X,Y) \in D_\rho$ iff $X \times Y$ is a rectangle contained in only one maximal rectangle of R

$(x,y) \in D_\rho\upharpoonleft$ iff (x,y) is an element of only one maximal rectangle of R

This last proposition shows that the "relation d'exactitude"of a correspondance $\rho = (A \xrightarrow{R} B)$ introduced by Van den Bril in [VdB 86] coincides with the relation $D_\rho \rceil$.It results from this that the notion of exactitude is a facet of the Galois correspondances.

10.EXACT SQUARES

In this paragraph we indicate how the theory of Guitart's "carres exacts" can be derived from the last lines of the previous paragraph. Let Γ be a gract . We define a class of pairs mesqΓ, that we call the <u>class of medial sections of the Γ squares</u> by

$$\mathrm{mesq}\,\Gamma = \left\{ \left(\begin{pmatrix} x \xrightarrow{a_1} z_1 \\ \downarrow \\ x \xrightarrow[a_2]{} z_2 \end{pmatrix} , \begin{pmatrix} z_1 \xrightarrow{b_1} y \\ \downarrow \\ z_2 \xrightarrow[b_2]{} y \end{pmatrix} \right) \middle/ \begin{array}{l} (x \xrightarrow{a_1} z_1) \in \Gamma ,\ (z_1 \xrightarrow{b_1} y) \in \Gamma \\ (x \xrightarrow{a_2} z_2) \in \Gamma ,\ (z_2 \xrightarrow{b_2} y) \in \Gamma \end{array} \right\}$$

It is almost obvious that mesqΓ is difunctionnal .

Now, if $\mathcal{C}$ is an actegory having as underlying gract, its <u>class of medial sections of the commutative squares</u> is ,by definition :

$$\mathrm{cesq}\,\mathcal{C} = \left\{ \left(\begin{pmatrix} x \xrightarrow{a_1} z_1 \\ \downarrow \\ x \xrightarrow[a_2]{} z_2 \end{pmatrix} , \begin{pmatrix} z_1 \xrightarrow{b_1} y \\ \downarrow \\ z_2 \xrightarrow[b_2]{} y \end{pmatrix} \right) \middle/ \begin{tikzcd}[row sep=small] & z_1 \arrow[dr, "b_1"] \arrow[dd, phantom, "\Downarrow"] & \\ x \arrow[ur, "a_1"] \arrow[dr, "a_2"'] & & y \\ & z_2 \arrow[ur, "b_2"'] & \end{tikzcd} \in \mathrm{com}\,\mathcal{C} \right\}$$

Then,the notion of exact square in Guitart's sense can be expressed by the following :

$$\begin{tikzcd}[row sep=small] & z_1 \arrow[dr] \arrow[dd, phantom, "\Downarrow"] & \\ x \arrow[ur] \arrow[dr] & & y \\ & z_2 \arrow[ur] & \end{tikzcd}$$ is an exact square iff its medial section $\left(\begin{pmatrix} x \xrightarrow{a_1} z_1 \\ \downarrow \\ x \xrightarrow[a_2]{} z_2 \end{pmatrix} , \begin{pmatrix} z_1 \xrightarrow{b_1} y \\ z_2 \xrightarrow[b_2]{} y \end{pmatrix} \right)$ satisfies the"relation d'exactitude"of cesq$\mathcal{C}$.

11. <u>FROM ORD TO OR</u>

A first step towards generalisations mentioned in the introduction consist of making clear (i.e.in algebrzaic terms) how one can define cl <u>or</u> .For this, it is necessary to replace <u>map</u> by a"larger"category the elements of which are of the form : $\left((A,U) \xrightarrow{R} (B,V)\right)$ where $(A \xrightarrow{R} B) \in \underset{\sim}{\mathrm{cor}}$, where R is difunctionnal and where U resp.V are equivalences on A resp.B such that $U \subset R^{\sim}$ and $R^{\smile\sim} \subset V$.

Once this is done, one can easily define what is the convenient definition for adj or and for gal or

A good example to examine in this perspective is the generalised Galois correspondence given by the difunctionnal relations ker and coker in an abelian category .(Cf.for instance [BriPu 69])

12. FURTHER GENERALISATIONS

After having defined cl or , adj or ,gal or ,one can introduce further generalisations by replacing or by the category of categories (or by the actegory of actegories).In this respect,we can in this paper,just give the Fig 2 that shows what can be the cartouche writing of an adjoint situation in category theory .

It was known in the topos theory folklore that a Grothendieck topology can be defined by a "generalised" closure operator. A precise description oi it can be found in [BaW 85] . One can realise that many new ways of studying this concept and various related ones are made possible by the use of the generalisations we have proposed.

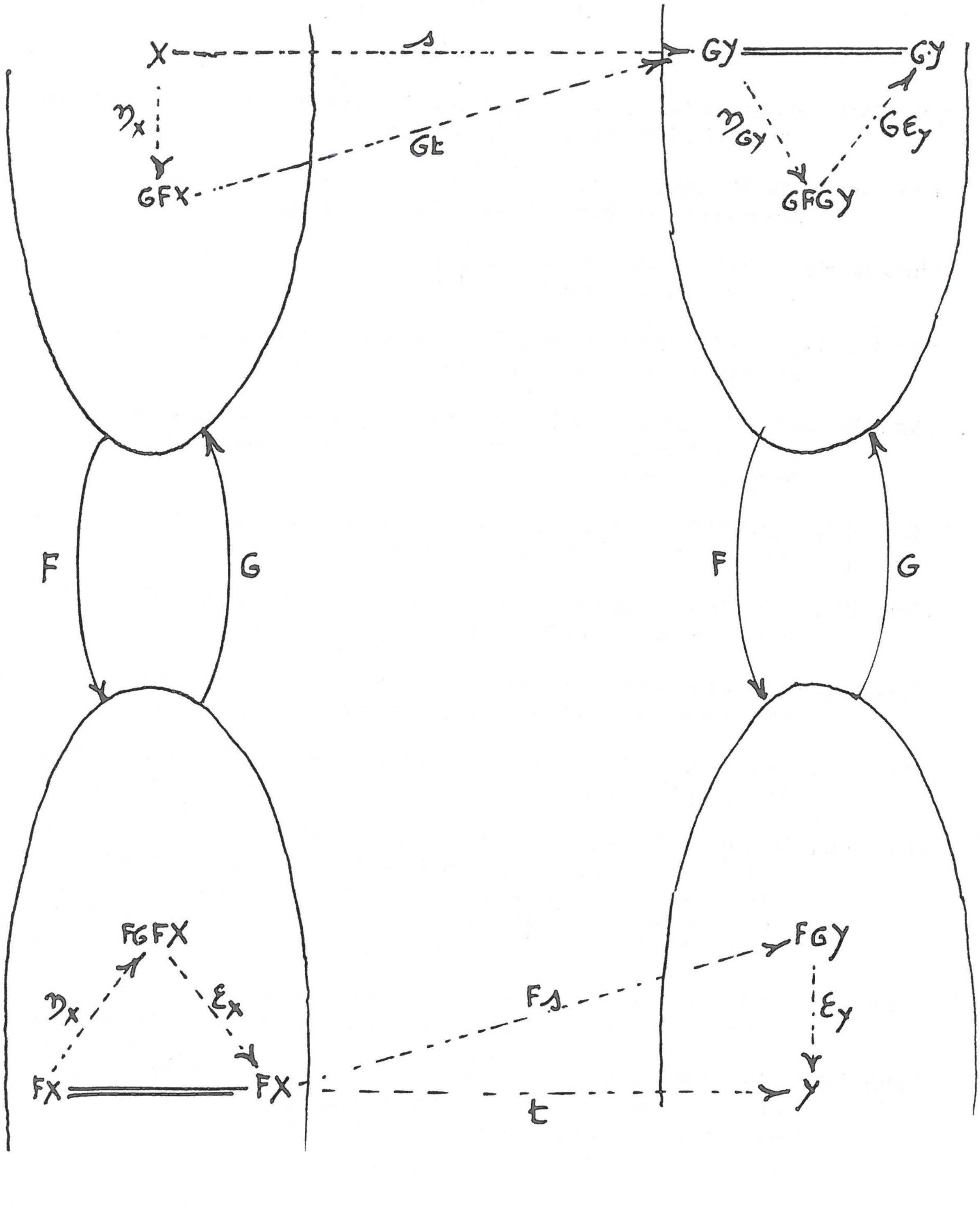

Fig 2 Cartouchewriting of an adjunct situation : $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\Downarrow"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$

$\mathcal{C}$ and $\mathcal{D}$ are categories

F functor from $\mathcal{C}$ to $\mathcal{D}$

G functor from $\mathcal{D}$ to $\mathcal{C}$

η is a unit: $\begin{tikzcd} \mathbb{I}_{\mathcal{C}} \arrow[d, "\eta"'] \\ GF \end{tikzcd}$, ε is a counit: $\begin{tikzcd} FG \arrow[d, "\varepsilon"'] \\ \mathbb{I}_{\mathcal{D}} \end{tikzcd}$

References

[Ba 70] BARR Relational algebras Midwest Category Seminar L.N.math.137 (1970) pp.39-55

[BaW 85] BARR WELLS Toposes Triples and Theories Grundlehren d.m. Wiss.278 Springer 1985

[BriPu 66] BRINKMANN PUPPE Kategorien und Funktoren L.N.math.18 (1966)

[BriPu 69] BRINKMANN PUPPE Abelsche und exakte Kategorien, Korrespondenzen L.N.math.96 (1969)

[CaGiRa 84] CALENKO GISIN RAIKOV Ordered categories with involutions Dissertationes mathematicae 227 Warsawa 1984

[Gui 80] GUITART Relations et carrés exacts Ann.sci.math. Québec 4 (1980) pp.103-125

[Gui 81] GUITART Carrés exacts et carrés déductifs Diagrammes 6 (1981) 17p.

[GuiVdB 81] GUITART VAN DEN BRIL Note sur la détermination des carrés exacts Diagrammes 6 (1981) 7p.

[Hil 66] HILTON Correspondences and exact squares Proc. Conf.cat.algebra. La Jolla Springer 1966

[Rig 89] RIGUET Cartouche writing of Galois and adjunctions pairs and their applications for geometrically depicting consequence relations in logics and in relational data bases Lab.Inf.Theor.Progr Univ. Paris 7 Report 89-4 Janv.1989 57p.

[VdB 80] VAN DEN BRIL Carrés exacts de Hilton dans des contextes non abéliens Ann.soc.math.Québec 4 1980

[VdB 86] VAN DEN BRIL Constructions des théories d' exactitude Cahiers de Topologie et Géométrie différentielle 27 (1986) pp.35-48 .