The Classification of Quasi-Regular Polyhedra of Genus 2

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Abstract. The method of chamber systems is used to provide a complete list of all possible tessellations of the closed, orientable surface of genus 2 by (topological) n-gons und m-gons ($n, m > 2$) satisfying a certain local symmetry condition. Using a computer program it is shown that (up to homeomorphism) there are precisely 379 such tessellations. S. Bilinski constructed the first tessellation of the considered type for each of the 17 possible combinations of m- and n-gons using geometrical methods. It is the intention of the authors to demonstrate the usefulness and suitability of chamber systems in dealing with problems of the above type.

1. QUASI-REGULAR POLYHEDRA

A *(topological) polyhedron $\mathcal{P}$ (of genus p)* is a compact, orientable 2-manifold (in E^3) of genus p divided into simply-connected open regions by a finite number of arcs (and simple closed curves) called *edges*. Such a region, together with its boundary, is called a *face*. Edges meet only at their endpoints called *vertices* and each vertex is incident to at least 3 edges (where *loops* are counted twice)(cf. [GS]).

As in [B1], we call a (topological) polyhedron $\mathcal{P}$ *(locally) quasi-regular* if each of its vertex-cycles has the form

$$(m, \underset{1}{n}, m, \underset{2}{n}, \ldots, m, \underset{s}{n}) \tag{1}$$

with $n > m > 2$ and $s > 1$. In other words, when "going around" a vertex, one alternately encounters m-edged and n-edged faces (exactly s of each). We call $\mathcal{P}$ *globally quasi-regular*, if its automorphism group acts transitively on its set of edges.

For a quasi-regular polyhedron $\mathcal{P}$ of genus p, with α_0 vertices, α_1 edges and α_2 faces (q_m of which have m edges and q_n of which have n edges), the following system of Diophantine equations holds:

$$\begin{aligned} s\alpha_0 &= mq_m = nq_n = \alpha_1 \\ \alpha_2 &= q_m + q_n \quad \text{and} \\ 2(1-p) &= \alpha_0 - \alpha_1 + \alpha_2 \quad \text{(Euler formula)} \end{aligned} \tag{2}$$

with m, q_m, n, q_n, and s as above. Note that the numbers α_0, α_1 and α_2 are completely determined by the parameters m, q_m, n, q_n and s.

For $p = 2$ there are exactly 17 choices of positive integers m, q_m, n, q_n and s that satisfy (2)[1]. In [B1] S. Bilinski shows that for each of these 17 solutions, a corresponding polyhedron $\mathcal{P}$ exists[2].

In [DF1] quasi-regular polyhedra are discussed in terms of their associated "chamber systems" and these in terms of systems $(\mathcal{D}; \alpha, \beta)$ where $\mathcal{D}$ is a set of cardinality α_1 and α and β are permutations of $\mathcal{D}$ such that each α-orbit has cardinality n, each $\alpha\beta$-orbit has cardinality m, and each β-orbit has cardinality s. In the case $q_m = 4, m = 3, q_n = 3, n = 4, s = 4$ it is shown how chamber systems can be used to determine all the corresponding homeomorphism classes of (oriented) quasi-regular polyhedra.[3] We will give a complete classification of quasi-regular polyhedra of genus 2 using an algorithm based on the approach discussed in [DF1].

2. The Chamber System of a Polyhedron

To define the *chamber system* of a polyhedron $\mathcal{P}$ of genus p, we first consider the 1-skeleton $\mathcal{P}'$ of an arbitrarily chosen barycentric subdivision of (the decomposition defined on) $\mathcal{P}$. We then associate with each edge e' of $\mathcal{P}'$ adjacent to the barycenter v' of a face of $\mathcal{P}$ the *color* 0 if it joins v' and the barycenter w' of an edge e of $\mathcal{P}$ with $e \subset \overline{f}$, and the *color* 1 if it joins v' and a vertex v of $\mathcal{P}$ with $v \in \overline{f}$, otherwise e' receives the *color* 2. Now the chamber system $G(\mathcal{P}) = (\mathcal{X}, \mathcal{U})$ can be defined as the dual of $\mathcal{P}'$ together with the 3-coloring "inherited" from $\mathcal{P}'$, that is, $G(\mathcal{P})$ is an edge-colored graph with vertex set $\mathcal{X}$ the set of "triangular", 2-dimensional faces (or "chambers") of $\mathcal{P}'$ and with edge set $\mathcal{U}$ the set of edges of $\mathcal{P}'$ with their coloring 0,1,2 as explained above. The end points of any such edge are the two triangles to which it is adjacent. To distinguish the vertices and edges in $G(\mathcal{P})$ from those of $\mathcal{P}$ or $\mathcal{P}'$ they will be called "nodes" and "arcs". Note that different barycentric subdivisions of $\mathcal{P}$ yield isomorphic chamber systems.

It follows from our construction that each node of the chamber system of a polyhedron $\mathcal{P}$ is incident to exactly one i-arc (for all $i \in \{0, 1, 2\}$). Furthermore the graph does not contain any loops (since, as a 2-manifold, $\mathcal{P}$ has no boundary), is bipartite (since $\mathcal{P}$ is orientable) and of genus p. Removing all the arcs of color k ($k \in \{0, 1, 2\}$) yields a spanning (planar) subgraph whose connected components (called *$i-j$-components*) are cycles consisting of alternating i- and j-arcs ($\{i, j, k\} = \{0, 1, 2\}$), called *$i - j$-cycles*. Note that there is a canonical 1–1-correspondence between the $0-1$-, $0-2$-, and $1-2$-cycles and the faces,

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[1] Cf. table 1

[2] In [B1] the case $q_m = 2, m = 3, q_n = 1, n = 6, s = 6$ was overlooked (cf. [B2])

[3] Though not explicitely mentioned there, the ten systems $(\mathcal{D}; \alpha, \beta)$ listed in [DF1] represent all homeomorphism classes of *oriented* quasi-regular polyhedra of the type considered there. Six of the ten oriented polyhedra are homeomorphic to their "mirror image" (by an orientation preserving homeomorphism!), the polyhedra $2a2b2c-1$, $2a2b2c-2$ and the polyhedra $2b2b2d-2$, $2b2b2d-3$ are mirror images of each other. Hence, there are precisely 8 homeomorphism classes of such polyhedra if orientation is neglected.

edges, and vertices of $\mathcal{P}$, respectively. Furthermore we have the following theorem which allows a classification of quasi-regular polyhedra in terms of there associated chamber systems (cf. [DF1]):

Theorem A. *There exists a 1–1-correspondence between the homeomorphism classes of quasi-regular polyhedra of genus p as defined above and the isomorphismen classes of finite, connected, arc-colored, bipartite graphs* $G = (\mathcal{X}, \mathcal{U} \subset \{\{a,b\} \mid a, b \in \mathcal{X}\} \times \{0,1,2\})$ *satisfying the following conditions:*

(P1) *there exists natural numbers* m, n *with* $2 < m < n$ *such that* G *consists of exactly* q_m $0-1$*-cycles of cardinality* $2m$, *and* q_n $0-1$*-cycles of cardinality* $2n$;

(P2) *every* $0-2$*-cycle is of cardinality* 4;

(P3) *all* $1-2$*-cycles have the same cardinality, say* $4s$, *which is larger than* 4;

(P4) *every* $0-2$*-cycle intersects exactly one* $0-1$*-cycle of cardinality* $2n$ *and one of cardinality* $2m$;

(P5) *the parameters* q_m, m, q_n, n *and* s *satisfy the equation:*

$$2(1-p) = \frac{mq_m}{s} - mq_m + q_m + q_n.$$

Remark: Since the numerical invariants m, q_m, n, q_n and s of the quasi-regular polyhedron $\mathcal{P}$ coincide with the numerical invariants of the associated chamber system $G(\mathcal{P})$ defined in (P1), (P2) and (P3) they are denoted by the same symbols.

The correspondence can be defined in terms of barycentric subdivisions and topological realizations; compare [DF1] for details.

3. The Algorithm POLYHEDRA

We will confine ourselves to the case $p = 2$ for the rest of this paper. Using theorem A we can provide a complete list of all (homeomorphism classes of) quasi-regular polyhedra by enumerating all (isomorphism classes of) connected arc-colored bipartite graphs $G = (\mathcal{X}, \mathcal{U} \subset \{\{a,b\} \mid a, b \in \mathcal{X}\} \times \{0,1,2\})$ satisfying the conditions (P1)-(P5). We will now formulate a simple algorithm which generates a "standard representative" for every isomorphism class of chamber systems corresponding to a given set of parameters m, q_m, n, q_n, s ($n > m > 2$, $s > 1$) satisfying (2) . In this chapter we start with a preliminary version of the algorithm based on the "brute force" method of generating permutations (cf. [S]), which obviously is not polynomial and only applicable for small n and q_n. Then in the following chapter, we discuss how to speed up the algorithm. The final version of the algorithm has been successfully implemented and it has been used to compute the 379 chamber systems.

We start with some preliminary remarks. Let m, q_m, n, q_n, s be integers with $mq_m = nq_n$, $s \mid mq_m$ and $-2 = \frac{mq_m}{s} - mq_m + q_m + q_n$ and let $G = (\mathcal{X}, \mathcal{U} \subset$

$\{\{a,b\} \mid a,b \in \mathcal{X}\} \times \{0,1,2\})$ be a corresponding connected arc-colored bipartite graphs satisfying the conditions (P1)–(P5). Then we have

$$\#\mathcal{X} = N := 4mq_m \; (= 4nq_n) \tag{3}$$

and the subgraph $G' = (\mathcal{X},\mathcal{U}')$ derived from G by removing all 2-arcs is isomorphic to the graph $G^* = (\mathcal{X}^*,\mathcal{U}^*)$ which is defined as the disjoint union of the two graphs

$$G^{(m)} := (\mathcal{X}^{(m)},\mathcal{U}^{(m)}) \quad \text{and} \quad G^{(n)} := (\mathcal{X}^{(n)},\mathcal{U}^{(n)}) \tag{4}$$

with

$$\mathcal{X}^{(m)} := \{1,\ldots,N/2\}, \quad \mathcal{X}^{(n)} := \{N/2+1,\ldots,N\} \tag{5}$$

and

$$\begin{aligned} \mathcal{U}^{(m)} := \{(\{i,i+1\},0) \mid i \in \mathcal{X}^{(m)},\ i \equiv 1\ (2)\} \\ \cup \{(\{i,i+1\},1) \mid i \in \mathcal{X}^{(m)},\ i \equiv 0\ (2)\} \\ \cup \{(\{i,i-2m+1\},1) \mid i \in \mathcal{X}^{(m)},\ i \equiv 0\ (2m)\}, \end{aligned} \tag{6}$$

and

$$\begin{aligned} \mathcal{U}^{(n)} := \{(\{i,i+1\},0) \mid i \in \mathcal{X}^{(n)},\ i \equiv 1\ (2)\} \\ \cup \{(\{i,i+1\},1) \mid i \in \mathcal{X}^{(n)},\ i \equiv 0\ (2)\} \\ \cup \{(\{i,i-2n+1\},1) \mid i \in \mathcal{X}^{(n)},\ i \equiv 0\ (2n)\}. \end{aligned} \tag{7}$$

Hence it suffices to compute all non-isomorphic extensions of G^* to a chamber system $\tilde{G} = (\mathcal{X},\tilde{\mathcal{U}})$ satisfying (P1)–(P5) by generating all "permissible" sets of 2-arcs for G^* in a systematic manner.

We call an ordered pair $(i,j) \in \mathcal{X} \times \mathcal{X}$ a *bar* (in $G^{(m)}$ or $G^{(n)}$), if $(\{i,j\},0) \in \mathcal{U}^*$ and $i < j$ ($i \in \mathcal{X}^{(m)}$ or $i \in \mathcal{X}^{(n)}$, respectively). Clearly $j = i+1$ in that case. A bar $(i,i+1)$ is *smaller* than a bar $(j,j+1)$ (we write $(i,i+1) < (j,j+1)$) if $i < j$, and is called a ***head*** if it is minimal in the $0-1$-cycle containing the arc $(\{i,i+1\},0)$, i.e. if $i \equiv 1\ (2m)$ for $i \in \mathcal{X}^{(m)}$ and $i \equiv 1\ (2n)$ for $i \in \mathcal{X}^{(n)}$. To *connect two nodes* $i,j \in \mathcal{X}$ means to join them with a 2-arc, and to *connect two bars* $(i,i+1)$ and $(j,j+1)$ means to connect i and j and $i+1$ and $j+1$. Moreover we call a bar *free* if it is not connected to any other bar. Note that connecting a free bar in $G^{(m)}$ to a free bar in $G^{(n)}$ yields a $0-2$-cycle of cardinality 4. Furthermore the resulting graph is bipartite if and only if the initial one was.

Algorithm POLYHEDRA

input: the graph $G = (\mathcal{X},\mathcal{U})$, initially equal to $G^* = (\mathcal{X}^*,\mathcal{U}^*)$ as defined in (4)—(7)

output: a representative $G = (\mathcal{X},\mathcal{U})$ of each isomorphism class of graphs that satisfy (P1)—(P5)

```
{
        IF G^(m) contains a free bar THEN {
(*)         LET A be the smallest free bar in G^(m)
            FOR EACH free bar B in G^(n) (in ascending order) DO {
                call CONNECT (A, B)
            }
        }
        ELSE IF the graph G = (X, U) is "OK" THEN print it
}
```

Procedure CONNECT (A, B)

```
{
        connect A and B in U
        call POLYHEDRA
        disconnect A and B in U
}
```

In the "ELSE IF"-line the graph G is "OK" if it has the following properties:

(OK1) every bar has been connected,

(OK2) every $1-2$-cycle is of cardinality $4s$ and

(OK3) the graph is not isomorphic to any graph previously printed out by the algorithm.

4. Speeding up the Algorithm

While "running" the algorithm POLYHEDRA we will indicate the current "level of recursion" by a subscript i. So initially we call POLYHEDRA$_1$ and for the i-th nested call of the algorithm, we write POLYHEDRA$_i$.

Assume that we are running the algorithm POLYHEDRA and that it has already connected the first $i-1 < \frac{N}{4}$ bars in $G^{(m)}$ and that we have just entered POLYHEDRA$_i$. The algorithm POLYHEDRA$_i$ takes the smallest free bar A in $G^{(m)}$ and connects it to the smallest free bar B in $G^{(n)}$. It then goes on to call POLYHEDRA$_{i+1}$ (which in turn may lead to some graphs being printed out). Once this call has been completed, the bars A and B are disconnected. The algorithm POLYHEDRA$_i$ would now go on to select the next free bar $B' > B$ in $G^{(n)}$ and then reenter the main loop to connect A and B' etc. But if A is a head, then repeating the main loop will not produce any graphs not isomorphic to ones already printed during the first execution of the main loop. The reason for this is that, with respect to isomorphism, the bar A and all following bars A' in $G^{(m)}$ "play the same role". Each is a free bar contained in an $\langle\sigma_0, \sigma_1\rangle$-cycle of cardinality $2m$, which consists entirely of free bars. If we were to connect A to B', then in POLYHEDRA$_{i+1}$ some $A' > A$ would be connected to B. But by the previous statement this would not lead to a new isomorphism type of graph. Now consider the algorithm POLYHEDRA$_i$ under the assumption that A is not a head. If at some point the main loop chooses a free bar B' in $G^{(n)}$ that is a head, then the above argument applies similarly to B' and again, after performing POLYHEDRA$_{i+1}$ with A and B' connected, the algorithm POLYHEDRA$_i$ can be aborted.

So we have seen that it suffices if our algorithm considers only graphs G that have the following two properties for any pair of bars A and A' in $G^{(m)}$ connected to bars called B and B' in $G^{(n)}$:

(a) If A is a head and $A \leq A'$ then $B \leq B'$.

(b) If B is a head and $B \leq B'$ then $A \leq A'$.

We can ensure that only such graphs are considered by adding the following line to the bottom of the main loop:

IF A is a head OR B is a head THEN END

where the statement "END" means end the "current incarnation" of POLYHEDRA (but not the whole algorithm).

We shall take a further step to speed up the algorithm. It consists of making the procedure CONNECT ensure that, from the beginning on, the algorithm only considers and produces graphs whose $1-2$-cycles have the correct cardinality. For any intermediate graph produced by the algorithm, let $\mathcal{C}(i)$ denote the $1-2$-component that contains the node i. Any such set $\mathcal{C}(i)$ is either a $1-2$-cycle, or a chain using alternating 1- and 2-arcs (a $1-2$-*chain*). Let $i \in \mathcal{X}^{(m)}$ and $j \in \mathcal{X}^{(n)}$ be two nodes not incident to a 2-arc. Then $\mathcal{C}(i)$ and $\mathcal{C}(j)$ are $1-2$-chains. If $\mathcal{C}(i) \cap \mathcal{C}(j) \neq \emptyset$, then $\mathcal{C}(i) = \mathcal{C}(j)$ and connecting i and j produces a $1-2$-cycle of cardinality $\#\mathcal{C}(i)$. Otherwise, connecting i and j produces a $1-2$-chain of cardinality $\#\mathcal{C}(i) + \#\mathcal{C}(j)$.

Replace the entire procedure CONNECT by this new version:

Procedure CONNECT′ (A, B)

```
{
    assume A = (i, i+1) and B = (j, j+1)
    IF C(i) = C(j) AND #C(i) = 4s
    OR C(i) ≠ C(j) AND #C(i) + #C(j) < 4s THEN {
        connect i and j in U
        IF C(i+1) = C(j+1) AND #C(i+1) = 4s
        OR C(i+1) ≠ C(j+1) AND #C(i+1) + #C(j+1) < 4s THEN {
            connect i+1 and j+1 in U
            call POLYHEDRA
            disconnect i+1 and j+1
        }
        disconnect i and j
    }
}
```

Finally, if follows from properties (a) and (b) that adding the lines

IF A is not the smallest bar in $G^{(m)}$ THEN
    IF A is a head AND the smallest free bar in $G^{(n)}$ is a head
        THEN END

to POLYHEDRA directly under the line marked (*) ensures that the graphs considered by the algorithm are connected.

After making all the the above modifications to the algorithm, in the "ELSE IF"-line only the property (OK3) has to be checked, (OK1) and (OK2) are necessarily satisfied. Here is the modified algorithm:

Algorithm POLYHEDRA′

input: the graph $G = (\mathcal{X}, \mathcal{U})$, initially equal to $G^* = (\mathcal{X}^*, \mathcal{U}^*)$ as defined in (4)—(7)

output: a representative $G = (\mathcal{X}, \mathcal{U})$ of each isomorphism class of graphs that satisfy (P1)—(P5)

```
{
        IF G^(m) contains a free bar THEN {
            LET A be the smallest free bar in G^(m)
            IF A is not the smallest bar in G^(m) THEN
                IF A is a head AND the smallest free bar in G^(n) is a head
                    THEN END
            FOR EACH free bar B in G^(n) (in ascending order) DO {
                call CONNECT' (A, B)
                IF A is a head OR B is a head THEN END
            }
        }
        ELSE IF the graph G = (X,U) is not isomorphic to one already
            printed THEN print it
}
```

5. Results and Comments

Implementing the final version of the algorithm POLYHEDRA on a computer yields the following results:

Theorem B. *There are precisely 379 homeomorphism classes of quasi-regular polyhedra of genus 2, 10 of which are globally quasi-regular and 225 of which have orientation reversing automorphisms (cf. table 1, Appendix)*[4]

A remark on the computation time: our implementation of the algorithm POLYHEDRA′ needed approximately 4 000 minutes for the computation of the 90 classes of quasi-regular polyhedra in case 1, where the associated chamber system had $N = 336$ nodes.

Modified versions of the algorithm POLYHEDRA are presently being used to solve classification problems concerning more general polyhedra. Similar algorithms based on the theory of "Delaney symbols" developed by A. W. M.

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[4] Readers interested in in more detailed results should contact the authors. The C-source code of the computer program used for the computation of the results is also available upon receipt of a floppy disk.

Dress and his co-workers (cf. e.g. [D2], [D3], [DS1], [DF2]) have been used successfully to enumerate periodic tilings of the plane (cf. e.g. [DH1/2], [H]). Work is presently being done in developing a computer graphics program to automatically draw the associated structure to a given Delaney symbol or chamber system (cf. [De]). Following Tutte (cf. [T]) recursion formulas counting homeomorphism classes of various types of "pointed"[5] regular polyhedra can also be developed (cf. [D4], [F1], [F2], [DF2], [A1], [A2]).

Case	q_m	m	q_n	n	s	N	Q	Q^+	G
1	28	3	12	7	2	336	90	155	0
2	16	3	6	8	2	192	57	77	1
3	12	3	4	9	2	144	24	37	0
4	10	3	3	10	2	120	15	20	0
5	8	3	2	12	2	96	20	25	0
6	6	3	1	18	2	72	8	9	0
7	10	4	8	5	2	160	51	75	0
8	6	4	4	6	2	96	33	40	2
9	4	4	2	8	2	64	18	19	2
10	3	4	1	12	2	48	6	6	0
11	2	5	1	10	2	40	6	7	1
12	8	3	6	4	3	96	21	28	1
13	5	3	3	5	3	60	7	9	0
14	4	3	2	6	3	48	7	8	0
15	3	3	1	9	3	24	4	4	0
16	4	3	3	4	4	48	8	10	2
17	2	3	1	6	6	24	4	4	1

Table 1. The distribution of the 379 classes of quasi-regular polyhedra. Column 2–6 contains the 17 solutions of the system of Diophantine equations (2) for $p = 2$. N is the cardinality of the node sets, Q the number of homeomorphism classes of the corresponding quasi-regular polyhedra , Q^+ the number of homeomorphism classes of the corresponding oriented quasi-regular polyhedra and G the number of globally quasi-regular polyhedra.

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