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ON THE PERMANENT OF CERTAIN SUBMATRICES OF CIRCULANT (0,1)-MATRICES

BY

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Summary - Let $A = I_n + P^h + P^k$, where P represents the permutation $(12\ldots n)$ and $1 \le h < k \le n-1$. We prove that the submatrix of A obtained by deleting the rows and the columns intersecting at three non-zero entries belonging to I, P^h, P^k has positive permanent, except in certain cases that are completely determined.

1 Introduction

A matrix of order n is called *circulant* if it is of the form $\sum_{i=1}^{n} s_i P^i$, where P is the $n \times n$-matrix representing the permutation $(12\ldots n)$. Recall that the permanent of an $n-$square matrix $A = [a_{ij}]$ is defined by

$$perA = \sum a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \quad ,$$

where the summation extends over all permutations σ of the symmetric group S_n.

Let $A = I_n + P^h + P^k$, where $1 \le h < k \le n-1$. Denote by a_i, b_i, c_i $(1 \le i \le n)$ the entries of A corresponding to I, P^h, P^k ,call them the first, second, and third diagonals of A , respectively.

Any three non-zero entries of A that belong to distinct rows and columns are said to be *independent.*

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If e_1, e_2, e_3 are three independent entries of A, the submatrix obtained by deleting the lines intersecting at e_1, e_2, e_3 is said to *correspond* to e_1, e_2, e_3.

Let e be an entry of A ; we denote by (e) and $[e]$ respectively the row and the column containing e.

In this paper we consider the permanent of the submatrices of A corresponding to three independent entries. In particular, propositions 2.3 and 2.4 determine the values of n, h, k for which there are submatrices corresponding to three independent entries belonging to distinct diagonals with a zero line and, therefore, zero permanent. Moreover, theorem 3.1 proves that every submatrix R corresponding to three independent entries does not contain a zero submatrix of type (r, s) with $r + s = n - 2$ and $r, s > 1$.

In this way we prove that the submatrix correponding to three independent entries belonging to distinct diagonals has positive permanent, except in certain cases that are completely determined.

2 Submatrices with a zero line

Definition 2.1 *Let x, y be two entries of a row r' of an $n \times n$ circulant $(0,1)$-matrix. We define the* r-distance *between x and y, denoted by* r-dist(x, y), *the number of elements of r' between x and y by moving from left to right cyclically.*
If x, y belong to a column c', then c-dist(x, y) *is the number of elements of c' between x and y by moving from top to bottom cyclically.*

In this way r-dist (x, y) + r-dist$(y, x) = n$.

Moreover, if $A = I + P^h + P^k$, let $\lambda = h, \mu = k - h, \nu = n - k$; we then have r-dist$(a_i, b_i) = \lambda$, r-dist$(b_i, c_i) = \mu$, r-dist$(c_i, a_i) = \nu$, $(1 \le i \le n)$.

Thus, if a', b', c' are the non-zero entries of a column, we have c-dist$(a', c') = \nu$, c-dist$(c', b') = \mu$, c-dist$(b', a') = \lambda$.

Proposition 2.2 *Let $A = I_n + P^h + P^k$, where $1 \le h < k \le n-1$ and where the integers $\lambda = h$, $\mu = k - h$, $\nu = n - k$ are distinct. For every non-zero entry e of A there are submatrices R_1, R_2, C_1, C_2 (where $R_1 \neq R_2$ and $C_1 \neq C_2$) with a zero line, corresponding to e and to two other independent entries belonging to distinct diagonals, .*

Proof. Without loss of generality we can suppose $e = a_1$. Let b, c and a', c' the non-zero entries of $[a_1]$ and (b) , respectively. The second diagonal intersects $[c']$ at the element b'' , while the third diagonal intersects $[a']$ at the element c'' . The entries b'' and c'' belong to different rows. In fact, because c-dist$(c', b'') = \mu$ and c-dist$(a', c'') = \nu$, the contrary would imply $\mu = \nu$, contrary to our assumption. Moreover, $b'' \neq b_1$, because otherwise r-dist$(a_1, b_1) = \lambda$ would coincide with r-dist$(b, c') = \mu$; in a similar way we obtain $c'' \neq c_1$. Thus the entries a_1, b'', c'' are independent, and the submatrix R_1 , obtained by deleting the lines which the preceding elements belong to, contains a zero line, i.e. the line obtained from (b) .

The same considerations hold for (c) .

The new submatrix R_2 cannot coincide with the preceding one, because otherwise the columns $[b'']$ and $[c'']$ would intersect column (c) at the elements $\bar{a}$ and $\bar{b}$. This in turn implies that r-dist$(\bar{a}, \bar{b}) = \lambda$ coincides with r-dist$(c', a') = \nu$, which is impossible by our assumption.

We can repeat the same considerations for the non-zero elements b_1, c_1 of (a_1) ; so we have two other distinct submatrices C_1, C_2 with a zero line respectively obtained from $[b_1]$ and $[c_1]$. □

Proposition 2.3 *Let $A = I + P^h + P^k$, where $1 \leq h < k \leq n-1$, and suppose that the integers $\lambda = h,\ \mu = k-h,\ \nu = n-k$ are distinct. For every entry $e \neq 0$ of A the four submatrices $R_1,\ R_2,\ C_1,\ C_2$ with a zero line (corresponding to e and to two other independent elements belonging to distinct diagonals) are all distinct, except in the following cases:*

$n = 7\mu$ and either $\lambda = 2\mu, \nu = 4\mu$ or $\lambda = 4\mu, \nu = 2\mu$, or

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where exactly two of the submatrices coincide.

Proof. Without loss of generality we can suppose $e = a_1$. If b and c are the non-zero elements of $[a_1]$, let R_1, R_2 be the submatrices with a zero line obtained respectively from the rows (b) and (c) as proved in proposition 2.2.

Similarly, submatrices C_1, C_2 are obtained from (a_1) and the columns $[b_1]$ and $[c_1]$.

We have already proved that $R_1 \neq R_2$ and $C_1 \neq C_2$.

Now we prove that $R_1 \neq C_2$.

Let c, b and b_1, c_1 the non-zero entries of $[a_1]$ and (a_1) , respectively. R_1 is obtained by considering the row (b), while C_2 is obtained by considering the column $[c_1]$. Let c', a' and b'', a'' be the non-zero entries of (b) and $[c_1]$, respectively.

We have two possibilities to consider.

The first is : $[a'']$ precedes $[a']$. In this case, as r-dist$(c_1, a_1) =$ r-dist(c', a'), we see that $[c']$ precedes $[a'']$. Then the element $\bar{b} = [c'] \cap (a'')$ can not belong to the second diagonal. In fact, on the contrary, it satisfies r-dist$(\bar{b}, a'') <$ r-dist(c', a'), that is $\mu + \nu < \nu$, which implies the impossible relation $\mu < 0$.

The second possibility is : $[a'']$ follows $[a']$. In this case $[c']$ precedes $[a']$ and the element $\bar{c} = [a'] \cap (b'')$ does not belong to the third diagonal. In fact, on the contrary, it satisfies r-dist$(\bar{c}, b'') <$ r-dist(a', b), that is $\lambda + \nu < \lambda$, which implies the impossible relation $\nu < 0$. In a similar way we obtain $R_2 \neq C_1$.

Now consider the possibility that $R_1 = C_1$. Let a', c' and a'', c'' the non-zero elements of (b) and $[b_1]$, respectively. Moreover, let $\bar{b} = [c'] \cap (c'')$ and $\bar{c} = [a'] \cap (a'')$; so R_1 is obtained by deleting the lines intersecting at $a_1, \bar{b}, \bar{c}$.

We have two cases to consider:

1) $[c']$ precedes $[c'']$;
2) $[c']$ follows $[c'']$.

Case 1). Then r-dist(b, c) + r-dist$(\bar{b}, c'')$ = r-dist(a_1, b_1) , that is

$$\lambda = 2\mu \tag{1}$$

Moreover, r-dist(a_1, b_1) + r-dist$(a'', \bar{c})$ = r-dist(b, a') , i.e. $\lambda + \lambda + \mu = \mu + \nu$. So we obtain

$$\nu = 2\lambda \tag{2}$$

From (1) and (2) we obtain $\lambda = 2\mu$, $\nu = 4\mu$ and $n = 7\mu$. (see fig.1)

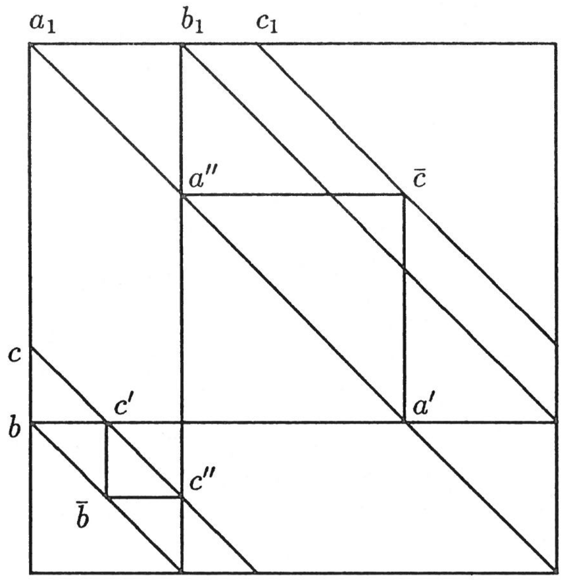

fig. 1

Case 2). Then r-dist(a_1, b_1) + r-dist$(c'', \bar{b})$ = r-dist(b, c') , that is

$$\mu = 2\lambda + \nu \tag{3}$$

Moreover r-dist(a_1, b_1) + r-dist$(a'', \bar{c})$ = r-dist(b, a') , i.e. $\lambda + \lambda + \mu = \mu + \nu$. So we obtain

$$\nu = 2\lambda \tag{4}$$

From (3) and (4) we obtain $\mu = 4\lambda$, $\nu = 2\lambda$ and $n = 7\lambda$. (see fig.2)

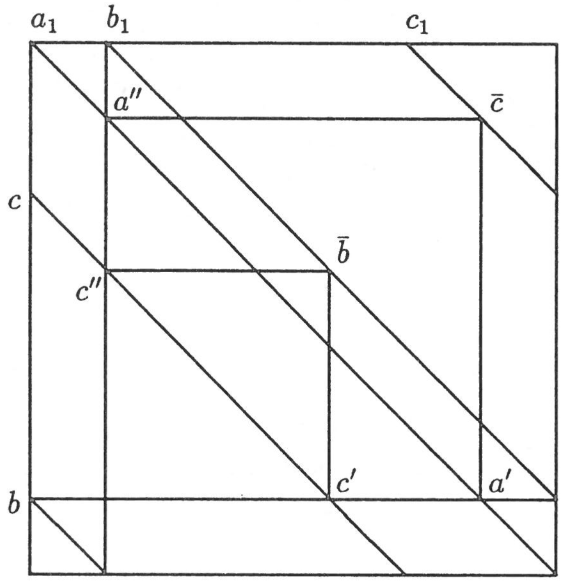

fig. 2

By using a similar procedure, we consider the case $R_2 = C_2$; we obtain either

$n = 7\lambda$, $\mu = 4\lambda$, $\nu = 2\lambda$, or
$n = 7\mu$, $\lambda = 4\mu$, $\nu = 2\mu$

Thus in the preceding cases there are three submatrices of order $n-3$ with a zero line corresponding to a non-zero entry e and to two other independent entries belonging to distinct diagonals ; in every other case the matrices R_1, R_2, C_1, C_2 are distinct. □

Proposition 2.4 *Let $A = I_n + P^h + P^k$, where $1 \le h < k \le n-1$, and suppose that at least two of the integers $\lambda = h$, $\mu = k - h$, $\nu = n - k$ coincide. Then there are no submatrices of order $n-3$, corresponding to three independent entries belonging to distinct diagonals, with a zero line.*

Proof. Suppose $\lambda = \mu$; the other cases can be reduced to this situation by multiplying A by a power of P.

We prove that three independent entries e_1, e_2, e_2 , such that the submatrix obtained by deleting the corresponding lines has a zero line, do not exist. Without loss of generality we can suppose $e_1 = a_1$. Let b, c and a', c' the non-zero entries of $[a_1]$ and (b) respectively; moreover let $\bar{b}, \bar{c}$ the elements of the second and third diagonal belonging to $[c']$ and $[a']$.

As r-dist(b, c') = r-dist(a_1, b_1), this implies $\bar{b} = b_1$; but then $a_1, \bar{b}, \bar{c}$ are not independent.

Consider the row (c) ; let a^*, b^* the non-zero elements of (c) and $\hat{b}, \hat{c}$ the intersections of the second and third diagonal with $[a^*]$ and $[b^*]$. The elements $\hat{b}, \hat{c}$ are in the same row because they are at the same distance of a^*, b^* . Thus also $a_1, \hat{b}, \hat{c}$ are not independent.

In a similar way we can proceed by considering (a_1) and the columns $[b_1], [c_1]$. □

3 Submatrices with positive permanent

Theorem 3.1 *Let R be a submatrix of $A = I_n + P^h + P^k$, $1 \leq h < k \leq n-1$, obtained by deleting the lines intersecting at three independent entries. Then R does not contain a zero submatrix of type (r, s) with $r + s = n - 2$ and $r, s > 1$.*

Proof. Let R be a submatrix corresponding to three independent entries e_1, e_2, e_2 and containing a zero-submatrix H of type (r, s) with $r + s = n - 2$ and $r, s > 1$.

Let $T = \{t_1, t_2, \ldots, t_r\}$ and $U = \{u_1, u_2, \ldots, u_s\}$ denote the rows and the columns of A whose intersection determines H. For every row t_i there are three columns, corresponding to the non-zero entries of t_i , that clearly do not belong to U . Thus there are $3r$ columns v_i $(1 \leq i \leq 3r)$ that do not belong to U. As every column contains three non-zero entries, every element of $V = \{v_1, v_2, \ldots, v_{3r}\}$ can be repeated at most three times in V . Thus V contains at least r distinct elements.

If every element of V is repeated three times, then there are r distinct elements in V , and every non-zero entry of a column of V belongs to a row of T . This implies that the columns $[e_1], [e_2], [e_3]$ do not belong to V , hence A contains $r + s + 3 = n + 1$ distinct columns, a contradiction.

Suppose that not every element of V is repeated three times in V. Then at least $r + i$, $1 \leq i \leq 3$, elements of V are distinct, and there are at least i non-zero entries belonging to elements of V that do not belong to the rows of T. In this case i of the columns $[e_1], [e_2], [e_3]$ can coincide with elements of V ; but then A has again $n + 1$ distinct columns, a contradiction. □

Theorem 3.2 *Let $A = I_n + P^h + P^k$, where $1 \leq h < k \leq n-1$, and suppose that at least two of the integers $\lambda = h$, $\mu = k - h$, $\nu = n - k$ coincide. Then every submatrix of A corresponding to three independent entries belonging to I_n, P^h, P^k has a positive permanent.*

Proof. Let R be a submatrix corresponding to three independent entries belonging to distinct diagonals. Then by the Frobenius-König-Theorem [1] $perR = 0$ if and only if R contains a zero submatrix of type (r, s) such that $r + s = n - 2$. By proposition 2.4 and theorem 3.1 this is impossible. Thus $perR > 0$. □

References

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