Matroidizing Set Systems

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Assume E is some finite set. Let $< \cdots >: \mathcal{P}(E) \to \mathcal{P}(E)$ denote a closure operator which by definition satisfies

$$F \subseteq < F > = \ll F \gg \text{ for } F \subseteq E$$

and

$$< F_1 > \subseteq < F_2 > \text{ for } F_1 \subseteq F_2 \subseteq E.$$

Then we define an anti chain $\mathcal{B}_{<\cdots>} \subseteq \mathcal{P}(E)$ by

$$\mathcal{B}_{<\cdots>} := \{B \subseteq E \mid < B > = E, < B \setminus \{b\} > \neq E \text{ for all } b \in B\}.$$

Similarly, for any anti chain $\mathcal{B} \subseteq \mathcal{P}(E)$ we define a closure operator $< \cdots >_{\mathcal{B}}: \mathcal{P}(E) \to \mathcal{P}(E)$ by

$$< F >_{\mathcal{B}} := \{e \in E \mid \text{ for } B \in \mathcal{B} \text{ with } e \in B \text{ there exists } f \in F \text{ with } (B \setminus \{e\}) \cup \{f\} \in \mathcal{B}\}.$$

If M denotes some matroid defined on E with $\mathcal{B}$ as its set of bases and $< \cdots >$ as its closure operator, then $\mathcal{B} = \mathcal{B}_{<\cdots>}$ and $< \cdots > = < \cdots >_{\mathcal{B}}$.
More generally we have

Proposition 1:
Assume $\mathcal{B} \subseteq \mathcal{P}(E)$ is some anti chain and $< \cdots >: \mathcal{P}(E) \to \mathcal{P}(E)$ is some closure operator. Then the following statements are equivalent:

(i) $\mathcal{B} = \mathcal{B}_{<\cdots>}$ and $< \cdots > = < \cdots >_{\mathcal{B}}$.

(ii) $\mathcal{B} = \mathcal{B}_{<\cdots>}$ and $< \cdots >$ is the closure operator of some matroid; that means:
For $F \subseteq E$ and $e, f \in E$ we have
$f \in < F \cup \{e\} > \setminus < F >$ iff $e \in < F \cup \{f\} > \setminus < F >$.

(iii) $< \cdots > = < \cdots >_{\mathcal{B}}$ and $\mathcal{B}$ is the set of bases of some matroid; that means:
For $B_1, B_2 \in \mathcal{B}$ and $b \in B_1 \setminus B_2$ there exists some $b' \in B_2 \setminus B_1$ with $(B_1 \setminus \{b\}) \cup \{b'\} \in \mathcal{B}$.

The operations

$$P : \mathcal{B} \mapsto < \cdots >_{\mathcal{B}} \mapsto \mathcal{B}_{<\cdots>_{\mathcal{B}}},$$
$$Q : < \cdots > \mapsto \mathcal{B}_{<\cdots>} \mapsto < \cdots >_{(\mathcal{B}_{<\cdots>})}$$

define maps from the set of anti chains and the set of closure operators into themselves.
By Proposition 1, $P(\mathcal{B}) = \mathcal{B}$ and $Q(< \cdots >) = < \cdots >$ iff $\mathcal{B}$ is the set of bases of some matroid and $< \cdots >$ is the closure operator of some matroid.
Write

$$P^0(\mathcal{B}) := \mathcal{B}, P^{n+1}(\mathcal{B}) := P(P^n(\mathcal{B}))$$

for any anti chain $\mathcal{B} \subseteq \mathcal{P}(E)$ and

$$Q^0(< \cdots >) := < \cdots >, Q^{n+1}(< \cdots >) := Q(Q^n(< \cdots >))$$

for any closure operator $< \cdots >: \mathcal{P}(E) \to \mathcal{P}(E)$ and $n \geq 0$.

Proposition 2:
Assume $\mathcal{B} \subseteq \mathcal{P}(E)$ is some anti chain with $\mathcal{B} \neq \emptyset$ and $< \cdots >: \mathcal{P}(E) \to \mathcal{P}(E)$ denotes some closure operator.

(i) For every $B' \in \mathcal{B}_{(<\cdots>_\mathcal{B})}$ there exists some $B \in \mathcal{B}$ with $B \subseteq B'$.

(ii) There exists some $m = m(\#E) \leq 2^{(2^{\#E})}$ such that $P^{m+1}(\mathcal{B}) = P^m(\mathcal{B})$ and $Q^{m+1}(< \cdots >) = Q^m(< \cdots >)$.
In particular, $P^m(\mathcal{B})$ is the set of bases of some matroid $M = M(\mathcal{B})$ for any anti chain $\mathcal{B} \subseteq \mathcal{P}(E)$.

(iii) If r denotes the rank of $M(\mathcal{B})$, then

$$r \geq min\{\#B \mid B \in \mathcal{B}\}.$$

Definition:
For some anti chain $\mathcal{B} \subseteq \mathcal{P}(E)$ let $\Gamma_\mathcal{B}$ denote the graph with $\mathcal{B}$ as its set of vertices and

$$\mathcal{K}_\mathcal{B} = \{\{B_1, B_2\} \mid \#B_1 = \#B_2 = \#(B_1 \cap B_2) + 1\}$$

as its set of edges.

Remark:
If $\mathcal{B}$ is the set of bases of some matroid, then $\Gamma_\mathcal{B}$ is the base graph of M .
In general, the number of connected components of $\Gamma_\mathcal{B}$ is not less than $b_\mathcal{B} := \#\{\#B \mid B \in \mathcal{B}\}$.

The next two statements yield some more information about relations between $\mathcal{B}$ and $\mathcal{B}_{(<\cdots>_\mathcal{B})}$.

Proposition 3:
If $min\{\#B \mid B \in \mathcal{B}\} = min\{\#B \mid B \in \mathcal{B}_{(<\cdots>_\mathcal{B})}\}$, then $\Gamma_\mathcal{B}$ is connected. In particular, $min\{\#B \mid B \in \mathcal{B}\} = min\{\#B \mid B \in \mathcal{B}_{(<\cdots>_\mathcal{B})}\}$ implies $b_\mathcal{B} = 1$.

Proposition 4:
Assume $2 \leq n \leq \#E$ and $\mathcal{B} \subseteq \mathcal{P}_n(E)$ with $\mathcal{B} \neq \emptyset$. Put

$$\mathcal{H} := \{H \subseteq E \mid B \not\subseteq H \text{ for all } B \in \mathcal{B}, \text{ but for every } e \in E \backslash H \text{ there exists } B \in \mathcal{B} \text{ with } B \subseteq H \cup \{e\}\}.$$

Furthermore, put $\tilde{E} := E \dot{\cup} \mathcal{H}$ and

$$\mathcal{A} := \mathcal{P}_n(E) \dot{\cup} \{\{e_1, \ldots, e_{n-1}, H\} \mid H \in \mathcal{H}, \{e_1, \ldots, e_{n-1}\} \in \mathcal{P}_{n-1}(E) \setminus \mathcal{P}_{n-1}(H)\}.$$

Then $\mathcal{A} \subseteq \mathcal{P}_n(\tilde{E})$ is some anti chain with $\mathcal{B}_{(<\cdots>_\mathcal{A})} = \mathcal{B}$.

Corollary:
For every $n \in \mathbb{N}$ with $n \geq 2$ and every $m \in \mathbb{N}$ there exists some finite set E and some anti chain $\mathcal{B} \subseteq \mathcal{P}_n(E)$ with $P^{k+1}(\mathcal{B}) \subsetneqq P^k(\mathcal{B})$ for $0 \leq k \leq m$.

Conjecture: $m(\#E) = \mathcal{O}(\#E)$.