TRANSITIVE ORIENTATIONS, MÖBIUS FUNCTIONS, AND COMPLETE SEMI-THUE SYSTEMS FOR FREE PARTIALLY COMMUTATIVE MONOIDS

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The extended version of the lecture appears in the Proceedings of the 15th International Colloquium on Automata, Languages and Programming (ICALP 88), Lecture Notes of Computer Science, Springer 1988.

We give an abstract of the contents, only:

Free partially commutative monoids were introduced by Cartier/Foata [CaFo69] in order to solve some combinatorial problems. These monoids are given by $M = X^*/\{(ab, ba) \mid (a,b) \in I\}$ where X denotes a finite alphabet and $I \subseteq X \times X$ is a so-called independence relation. One of the basic properties of a free partially commutative monoid concernes the polynomial $\mu_M = \sum(-1)^{|F|}[F]$ in the ring of formal power series $Z\langle\langle M\rangle\rangle$ where the sum is taken over those subsets $F \subseteq X$ which consist of independent letters only and where $[F]$ denotes the trace $\prod_{x \in F} x \in M$. This polynomial has a formal inverse in $Z\langle\langle M\rangle\rangle$ which is the constant function with value one. The polynomial $\mu_M \in Z\langle\langle M\rangle\rangle$ is therefore the so-called Möbius function of M.

If we represent each trace $[F]$ in μ_M by some word $w_F \in X^*$ we obtain a polynomial $\sum(-1)^{|w_F|}w_F$ in the ring of formal power series $Z\langle\langle X^*\rangle\rangle$. There are several choices for these polynomials, and for simplicity each such polynomial is called a Möbius function. Let us call a Möbius function unambiguous if its formal inverse in $Z\langle\langle X^*\rangle\rangle$ is the characteristic function of a set of representatives for M.

It has been conjectured, see [Chof86, Chap II.2] that for every free partially commutative monoid there is at least one unambiguous Möbius function. Further it has been asked how to find the words $w_F \in X^*$ such that $\mu = \sum(-1)^{|w_F|}w_F$ is unambiguous. It turns out that the conjecture is false. But in case of existence, there is a very simple way to compute these words w_F.

This follows since the existence of unambiguous Möbius functions is directly related to the existence of transitive orientations of I. It is known that transitive orientations correspond

to finite complete semi-Thue systems $S \subseteq X^* \times X^*$ with $X^*/S = M$, see [Otto88]. We obtain therefore a close connection between unambiguous Möbius functions and finite complete semi-Thue systems: The formal inverse of an unambiguous Möbius function is exactly the characteristic series over the rational set of irreducible words of a finite complete semi-Thue system which defines M. Moreover, starting with a finite complete semi-Thue system defining M we obtain, by the formal inverse of the characteristic function over the irreducible words, an unambiguous Möbius function for M. Altogether we obtain a canonical bijection between the following sets: transitive orientations of I, unambiguous Möbius functions for M, finite normalized complete semi-Thue systems which define M.

The existence of a transitive orientation is of purely graph-theoretical nature which can be decided in time $O(\#X \cdot d)$ where $d = \max_{a \in X}\{\#\{b \in X \mid (a,b) \in I\}\}$, see [Golum80]. Thus, there is a polynomial time algorithm which decides for a finite alphabet X with independency $I \subseteq X \times X$ whether there is an unambiguous Möbius function in $Z\langle\langle X^* \rangle\rangle$, and which computes such a function in case of existence.

References:

[CaFa69] P. Cartier, D. Foata : Problemes combinatoires de commutation et réarrengements; Lect. Not. in Math., No. 85 , Springer, Berlin 1969

[Chof86] C. Choffrut: Free partially commutative monoids; LITP-report 86/20, Université de Paris 7, 1986

[Golum80] M.C. Golumbic: Algorithmic graph theory and perfect graphs; Academic Press, New York 1986

[Otto88] F. Otto: Finite canonical rewriting systems for congruences generated by concurrency relations; to appear in: Math. Syst. Theory

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