ON THE EXISTENCE OF A FINITE BASE FOR SYSTEMS OF EQUATIONS OF INFINITE WORDS

ALDO DE LUCA

Let A be a finite alphabet; A^* (resp. A^ω) is the set of the finite words (resp.infinite words) over A; moreover we pose $A^\infty = A^* \cup A^\omega$. The elements of A^ω (resp. A^∞) are called *ω-words* (resp. *∞-words*) . The length of a word $v \in A^*$ is denoted by $|v|$. The subsets of A^* (resp. A^ω , A^∞) are called *languages* (resp. *ω-languages*, *∞-languages*).

Let $u \in A^\omega$, if $u = a_1 a_2 \ldots a_n \ldots$, with $a_i \in A$, $i>0$, then we pose $u(i) = a_i$ for $i \geq 1$. Let B a finite or countable alphabet; a *morphism* $f: A^\infty \rightarrow B^\infty$ is any map f from A^∞ to B^∞ such that: (a) the restriction of f to A^* is a morphism from A^* to B^*, (b) for each $u \in A^\omega$, $f(u) = f(u(1))f(u(2))\ldots f(u(n))\ldots$ .The morphism f is determined by the list f(a) for all $a \in A$.

An *equation* with variables in A is a pair $(u, v) \in A^\infty \times A^\infty$, denoted also by u = v. A *solution* of the equation u = v in B^* is a morphism $f: A^\infty \rightarrow B^\infty$ such that f(u) = f(v).

A *system* S of equations with variables in A is a subset of $A^\infty \times A^\infty$. A morphism $f: A^\infty \rightarrow B^\infty$ is a solution of S in B^* if f(u) = f(v) for each $(u, v) \in S$.

Two systems of equations with variables in A, are said to be *equivalent* in B^*, if they have the same set of solutions in B^*.

S' is a *subsystem* of S if $S' \subseteq S$.

In a joint paper with M.Pelagalli and S.Varricchio (cf.[2]) we have proved that any system of equations on ∞- words in a finite number of indeterminates has over a countably generated free monoid a finite equivalent subsystem. In the case of finite words the result has been recently proved by Albert and Lawrence [1] and, independently, by Guba[5-6](proof of the *Ehrenfeucht conjecture*).

Theorem 1 . Any system of equations in a finite number of variables is equivalent in a countably generated free monoid to a finite subsystem.

The proof is based on a suitable injective interpretation of the ∞-words on a finite alphabet B as real numbers in the interval [0,1] . One can associate to each ∞- word-equation $u = v$ of a given system S a formal power series t in 2n indeterminates (n = Card(A)) in such a way that the morphism $f : A^{\infty} \to B^{\infty}$ is a solution of the equation $u = v$ if and only if $t(f_h) = 0$, with $f_h = \{ [f(a)]_h h^{-|f(a)|}, h^{-|f(a)|}, a \in A\}$, where $[\ ]_h$ denotes the *standard interpretation* in base h of a word and h >Card(B)+1. Moreover one has to take into account the fact that the ring C_n of all formal power series of n commutative variables with coefficients in the complex field and having a convergence radius r >0 (*convergent series*) is a Noetherian ring (cf.[3]) , i.e. any ideal of C_n is finitely generated (*Hilbert's basis theorem for convergent power series*).

Let $S =\{ (u^{\alpha}, v^{\alpha})\}_{\alpha \in I}$ be any system of equations and denote by $F = \{(t^{\alpha})\}_{\alpha \in I}$ the corresponding set of formal power series. Since C_{2n} is Noetherian F will admit a finite base $F' = \{t_1,\dots,t_p\}$. Let $S' = \{(u_1, v_1), \dots, (u_p, v_p)\}$ be the finite subsystem of S such that t_i (i=1,..,p) is the formal power series corresponding to the equation (u_i, v_i) . One can prove (cf.[2]) that for any $\alpha \in I$, $t^{\alpha}(f_h) = 0$ if and only if $t_i(f_h) = 0$ (i = 1,...,p) . From this it follows that S is equivalent to S'.

In the case in which $B = \{b_1, b_2, \dots\}$ is a countable alphabet to reach the result it is sufficient to observe that B^{∞} can always be embedded in $\{a,b\}^{\infty}$ by an injective morphism. One can choose, for instance, the morphism $\delta : B^{\infty} \to \{a,b\}^{\infty}$ defined as:

$$\delta(b_i) = a^i b \ , i > 0 .$$

Thus if u=v is an equation then the morphism $f : A^{\infty} \to B^{\infty}$ is a solution i.e. f(u)=f(v), if and only if $\delta(f(u)) = \delta(f(v))$.

Let now f, g be two morphisms from A^{∞} to B^{∞} and L a subset of A^{∞}, we say that f and g *agree* on L if f(u) = g(u) for every u in L. Let L be a language of A^∞; a *finite* subset F of L is called a *test set* for L if for each pair (f, g) of morphisms defined on A^∞, f and g agree on L if and only if they agree on F. As well known[4] the result of Albert and Lawrence is equivalent to the statement that any language over a finite alphabet has a test set.

By theorem 1 one can prove the existence of a test set for any ∞-language over a finite alphabet A. Indeed one can show [2] that this problem is equivalent , for any system S of equations on ∞-words, to the existence of a finite subsystem which is equivalent to S over B^*.

Theorem 2 . Let A be a finite alphabet. Each ∞-language L of A^∞ has a test set for morphisms f,g : $A^\infty \to B^\infty$ if and only if each system of equations of ∞-words has over B^* an equivalent finite subsystem.

REFERENCES

1.M.H.Albert and J.Lawrence , A proof of Ehrenfeucht's Conjecture, Theoret.Comput.Sci. 41(1985), 121-123.

2.A.de Luca, M.Pelagalli and S. Varricchio, Test sets for languages of infinite words, Dept.of Mathematics University of Rome "La Sapienza",preprint March 1988.

3.R.C.Gunning and H.Rossi, Analytic functions of several complex variables, Prentice Hall, Inc. 1985.

4. J. Karhumaki,The Ehrenfeucht conjecture: a compactness claim for finitely generated free monoids,Theoretical Computer Science, 29(1984)285-308.

5.R.McNaughton , A proof of the Ehrenfeucht conjecture, Informal memorandum, Dept.of Computer Science, Rensselaer Polytechnic Insitute, Troy, N.Y. 1985 .

6.D.Perrin, On the solution of Ehrenfeucht's conjecture, EATCS Bull. 27(1985) 68-70.

ALDO DE LUCA
Dipartimento di Matematica,
Università di Roma "La Sapienza"
Roma, Italy

and

Istituto di Cibernetica del C.N.R.
Arco Felice, Napoli,Italy