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ON A DECOMPOSITION OF SQUARE MATRICES OVER A RING WITH IDENTITY

BY

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Let R be a not necessarily commutative ring with 1 and let P be an (n by n)-matrix over R. Then P is called a *permutation matrix* if, and only if, the following conditions are satisfied:

(1) $P_{ij} \in \{0,1\}$ for all $i, j \in \{0, 1, \ldots, n-1\}$.

(2) Each row of P contains exactly one 1.

(3) Each column of P contains exactly one 1.

Denote by S_n the symmetric group on the set $\{0, 1, \ldots, n-1\}$. If $\pi \in S_n$ then we define $P(\pi)$ by

$$P(\pi)_{ij} := \text{if } \pi(j) = i \text{ then } 1 \text{ else } 0.$$

Then $P(\pi)$ is a permutation matrix and all permutation matrices are ob= tained in this way, as is well-known.

The set $Mat_n(K)$ of all (n by n)-matrices over the field K forms a vector space of dimension n**2 over K and it belongs to the folklore of permutation matrices that

$$\dim(\text{span}(P(\pi) \mid \pi \in S_n)) = (n-1)**2 + 1.$$

Linear Algebra tells us that there exists a basis of the span of permutation matrices consisting entirely of permutation matrices. Sear= ching for such a basis yields a much more general Theorem.

THEOREM 1. Let R be a not necessarily commutative ring with 1 and let n be a positive integer. Consider the set $Mat_{n+1}(R)$ of all ((n + 1) by (n + 1))-matrices over R as a left R-module. Define the submodules V_1, V_2, V_3 of $Mat_{n+1}(R)$ as follows:

1) V_1 consists of all $a \in Mat_{n+1}(R)$ such that $a_{ni} = a_{in} = 0$ for i := 0 to n − 1 and $a_{nn} = \Sigma_{j:=0 \text{ to } n-1}\, a_{n-1,j}$.

2) Let a be an n- and b be an (n − 1)-tuple over R. Define the matrices C(a) and D(b) by

$$
\begin{array}{llllllll}
0 & a_n & \cdots\cdots & & & a_3 & a_2 & a_1 \\
a_1 & 0 & a_n & \cdots\cdots & & & a_3 & a_2 \\
a_2 & a_1 & 0 & a_n & \cdots\cdots & & & a_3 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n-1} & \cdots\cdots & & & a_2 & a_1 & 0 & a_n \\
a_n & \cdots\cdots & & & a_3 & a_2 & a_1 & 0
\end{array}
$$

and

$$
\begin{array}{llllllll}
X & 0 & 0 & \cdots\cdots & & & 0 & 0 \\
0 & 0 & b_{n-1} & \cdots\cdots & & & b_2 & b_1 \\
0 & b_1 & 0 & b_{n-1} & \cdots\cdots & & & b_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & b_{n-1} & \cdots\cdots & & & b_2 & b_1 & 0
\end{array}
$$

respectively where X is the sum of the b_i's. Then V_2 consists of all matrices of the form C(a) + D(b).

3) V_3 is the set of all $a \in Mat_{n+1}(R)$ with $a_{ij} = 0$ for all (i,j) <> (n,0), (n,n).

Then $Mat_{n+1}(R)$ is the direct sum of V_1, V_2, V_3. Moreover, V_1 is , as an R-module, isomorphic to $Mat_n(R)$.

The proof is left as an exercise to the reader.

THEOREM 2. Same assumptions and notations as in Theorem 1. Define the permutations $\alpha, \beta \in S_{n+1}$ by α := (0,1,2,...,n) and β := (1,2,...,n) and set B(i) := P(α**i) for i := 1 to n and B(n + i) := P(β**i) for i := 1 to n − 1. Then {B(i) | i := 1 to 2*n − 1} is a basis for V_2.

Proof. Straightforward.

As a consequence of Theorems 1 and 2 we get

THEOREM 3. Denote by $const_{n+1}(R)$ the set of all $a \in Mat_{n+1}(R)$ such that there exists an $r \in R$ with $\Sigma_{k:=0 \text{ to } n}\, a_{kj} = r = \Sigma_{l:=0 \text{ to } n}\, a_{il}$ for all i and j. Then $const_{n+1}(R)$ is a direct summand of $Mat_{n+1}(R)$ having a basis consisting in n**2 + 1 permutation matrices.

Theorems 1 and 2 give a recursion for a basis of $const_{n+1}(R)$ as well as for a basis of a complement of $const_{n+1}(R)$. As an example, we list the 17 permutations whose permutation matrices form a basis of $const_5(R)$. The /'s indicate the steps in the recursion. Moreover, we list a set of 8 mat= rices forming a basis of a complement of $const_5(R)$.

(0) / (0,1) / (0,1,2), (0,2,1), (1,2) / (0,1,2,3), (0,2)(1,3), (0,3,2,1), (1,2,3), (1,3,2) / (0,1,2,3,4), (0,2,4,1,3), (0,3,1,4,2), (0,4,3,2,1), (1,2,3,4), (1,3)(2,4), (1,4,3,2).

```
0 0 0 0 0     0 0 0 0 0     0 0 0 0 0     0 0 0 0 0
1 0 0 0 0     0 1 0 0 0     0 0 0 0 0     0 0 0 0 0
0 0 1 0 0     0 0 1 0 0     1 0 0 0 0     0 0 1 0 0
0 0 0 1 0     0 0 0 1 0     0 0 0 1 0     0 0 0 1 0
0 0 0 0 1     0 0 0 0 1     0 0 0 0 1     0 0 0 0 1

0 0 0 0 0     0 0 0 0 0     0 0 0 0 0     0 0 0 0 0
0 0 0 0 0     0 0 0 0 0     0 0 0 0 0     0 0 0 0 0
0 0 0 0 0     0 0 0 0 0     0 0 0 0 0     0 0 0 0 0
1 0 0 0 0     0 0 0 1 0     0 0 0 0 0     0 0 0 0 0
0 0 0 0 1     0 0 0 0 1     1 0 0 0 0     0 0 0 0 1.
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The recursion for the basis of $const_{n+1}(R)$ clearly shows that $const_{n+1}(R)$ has a basis consisting of n**2 + 1 permutation matrices.

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