The Farey Graph

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This talk describes joint work with David Singerman and Keith Wicks. We have studied a set of graphs, closely related to the Farey graph, each having the modular group $\Gamma = PSL_2(\mathbb{Z})$ as a group of automorphisms. The eventual aim is to understand these graphs and to find number-theoretic, group-theoretic and combinatorial interpretations of their properties.

Let $\hat{\mathbb{Q}}$ be the rational projective line $\mathbb{Q} \cup \{\infty\}$. The <u>Farey graph</u> $\mathcal{F}$ has vertex-set $\hat{\mathbb{Q}}$, with vertices $v = r/s$ and $w = x/y$ adjacent if and only if $ry - sx = \pm 1$ (for $v, w \neq \infty$, this is equivalent to the condition that v and w should be adjacent terms in some Farey series — hence the name). $\mathcal{F}$ has several interesting properties:

Γ acts, by Möbius transformations, as a group of automorphisms of $\mathcal{F}$, transitively on vertices and edges;

geodesics from ∞ to v in $\mathcal{F}$ correspond to continued fraction expansions of v;

$\mathcal{F}$ can be imbedded in the upper half-plane to give a Γ-invariant triangular map which has every triangular map as a quotient.

More generally, if a group G acts on a set Ω, then each orbit of G on $\Omega \times \Omega$ is the edge-set of a directed graph $\mathcal{G}$, with $V(\mathcal{G}) = \Omega$ and $G \leqslant \mathrm{Aut}\,\mathcal{G}$ (these are Sims' suborbital graphs). If $G = \Gamma$ and $\Omega = \hat{\mathbb{Q}}$, we obtain graphs $\mathcal{G} = \mathcal{G}_{u,n}$ on $\hat{\mathbb{Q}}$ ($n \geqslant 1$, $1 \leqslant u \leqslant n$, $(u,n) = 1$) which generalise $\mathcal{F}$. Each $\mathcal{G}_{u,n}$ is a disjoint union of $\psi(n) = n \prod_{p|n} (1 + \frac{1}{p})$ copies of a graph $\mathcal{F}_{u,n}$ which can be imbedded in $\mathcal{F}$. For example, $\mathcal{G}_{1,1} = \mathcal{F}_{1,1} = \mathcal{F}$, while $\mathcal{G}_{1,2} \cong 3.\mathcal{F}_{1,2}$ where $\mathcal{F}_{1,2}$ (vertices $v = r/s$, r odd and s even, with r/s and x/y adjacent $\Leftrightarrow$ $ry - sx = \pm 2$) plays the same role for all maps as $\mathcal{F}$ does for triangular maps.

I shall describe some of the properties of these graphs, such as automorphisms, connectivity and circuits.