Matroids with Coefficients

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There are matroids which are not representable over any ring. However, A. Dress introduced the concept of a fuzzy ring by weakening the ring axioms such that every matroid becomes representable over some fuzzy ring.

Definition 1:

A fuzzy ring $K = (K;+;\cdot;\varepsilon;K_o)$ consists of a set K together with two compositions " $+ : K \times K \to K$ " and " $\cdot : K \times K \to K$ ", a specified element $\varepsilon \in K$ and a specified subset $K_o \subseteq K$ such that

(FR0) $(K,+)$ and $(K,\cdot)$ are abelian semigroups with neutral elements 0 and 1, respectively;

(FR1) $0\cdot\kappa = 0$ for every $\kappa \in K$;

(FR2) $\alpha\cdot(\kappa_1+\kappa_2) = \alpha\cdot\kappa_1 + \alpha\cdot\kappa_2$ for all $\kappa_1, \kappa_2 \in K$ and $\alpha \in K^\times := \{\alpha \in K \mid 1 \in \alpha\cdot K\}$;

(FR3) $\varepsilon^2 = 1$;

(FR4) $K_o + K_o \subseteq K_o$, $K\cdot K_o \subseteq K_o$, $0 \in K_o$, $1 \notin K_o$;

(FR5) for $\alpha \in K^\times$ one has $1 + \alpha \in K_o$ iff $\alpha = \varepsilon$;

(FR6) $\kappa_1, \kappa_2, \lambda_1, \lambda_2 \in K$ and $\kappa_1+\lambda_1$, $\kappa_2+\lambda_2 \in K_o$ implies $\kappa_1\cdot\kappa_2 + \varepsilon\cdot\lambda_1\cdot\lambda_2 \in K_o$;

(FR7) $\kappa, \lambda, \kappa_1, \kappa_2 \in K$ and $\kappa + \lambda\cdot(\kappa_1+\kappa_2) \in K_o$ implies $\kappa + \lambda\cdot\kappa_1 + \lambda\cdot\kappa_2 \in K_o$.

Examples:

i) If R is a commutative ring with $1 \in R$, then $(R;+;\cdot;-1;\{0\})$ is a fuzzy ring. Vice versa, if $(K;+;\cdot;\varepsilon;K_o)$ is a fuzzy ring and if $K_o = \{0\}$, then K is a commutative ring with $\varepsilon = -1$.

ii) If R is a commutative ring with $1 \in R$ and if $U \leq R^\times$, then we can define a "quotient structure" $R/U := (P(R)^U;+;\cdot;-U;P(R)^U_o)$ with
$P(R)^U := \{T \subseteq R \mid U\cdot T = T\}$ and $P(R)^U_o := \{T \in P(R)^U \mid 0 \in T\}$
where $T_1 \dot{+} T_2 := \{\kappa_1 \dot{+} \kappa_2 \mid \kappa_i \in T_i\}$ for $T_1, T_2 \in P(R)^U$.

iii) As a particular case of ii) we put $R = \mathbb{R}$, $U = \mathbb{R}^\times$. Then $\mathbb{R}/\mathbb{R}^\times = \{\{0\}, \mathbb{R}^\times, \mathbb{R}\}$. We obtain the following tables where $0 = \{0\}$, $1 = \varepsilon = \mathbb{R}^\times$, $u = \mathbb{R}$, $K_o = \{0,u\}$:

+	0	1	u
0	0	1	u
1	1	u	u
u	u	u	u

·	0	1	u
0	0	0	0
1	0	1	u
u	0	u	u

In the sequel let E denote a finite set and $(K;+;\cdot;\varepsilon;K_o)$ some fuzzy ring. For $\mathfrak{R} \subseteq K^E$ we put $\mathfrak{R}_o := \{r \in \mathfrak{R} \mid \emptyset \neq E \setminus r^{-1}(0) = r^{-1}(K^\times)\}$ and $(\mathfrak{R}_o)_{min} := \{r \in \mathfrak{R}_o \mid s^{-1}(K^\times) \subseteq r^{-1}(K^\times)$ for $s \in \mathfrak{R}_o$ implies $s^{-1}(K^\times) = r^{-1}(K^\times)\}$.

Definition 2:

A matroid M defined on E is representable over K, if there exists some $R \subseteq K^E$ satisfying

i) for $f \in E$ and $r_1, r_2 \in R$ we have $r_1 \underset{f}{\wedge} r_2 \in R$ where

$$(r_1 \underset{f}{\wedge} r_2)(e) := \begin{cases} r_2(f)\cdot r_1(e) + \varepsilon r_1(f)\cdot r_2(e) & \text{for } e \in E \setminus \{f\} \\ 0 & \text{for } e = f ; \end{cases}$$

ii) for every $r \in R$ and $e \in E$ with $r(e) \notin K_o$ there exists some $s \in R_o$ such that $e \in E \setminus s^{-1}(0) = s^{-1}(K^x) \subseteq E \setminus r^{-1}(0)$;

iii) a subset $C \subseteq E$ is a circuit in M iff there exists some $r \in (R_o)_{min}$ satisfying $C = E \setminus r^{-1}(0) = r^{-1}(K^x)$.

Example:

Every matroid M defined on E is representable over $K := \mathbb{R}/\mathbb{R}^x$. If $\langle \ldots \rangle$ denotes the closure operator of M, then $R := \{r \in K^E \mid r(e) = 1$ implies $e \in \langle E \setminus (r^{-1}(0) \cup e)\rangle\}$ satisfies the conditions in Definition 2.

Theorem (Grassmann-Plücker-Relations for Fuzzy Rings):

A matroid M of rank m, defined on E, is representable over the fuzzy ring K iff there exists a map $\chi : E^m \to K^x \cup \{0\}$ such that

i) $\chi(e_1,\ldots,e_m) \neq 0$ iff $\{e_1,\ldots,e_m\}$ is some base of M ;

ii) $\chi(e_{\pi(1)},\ldots,e_{\pi(m)}) = \varepsilon\cdot\chi(e_1,\ldots,e_m)$ for all $e_1,\ldots,e_m \in E$ and every odd permutation $\pi \in \Sigma_m$;

iii) for all $e_o, e_1,\ldots,e_m, f_2,\ldots,f_m \in E$ we have

$$\sum_{i=0}^{m} \varepsilon^i \cdot \chi(e_o,\ldots,\hat{e}_i,\ldots,e_m)\cdot\chi(e_i,f_2,\ldots,f_m) \in K_o .$$

A map χ which satisfies the conditions of the theorem will be called a Grassmann-Plücker-map.

Example:

If M is representable over some ring R, then there exists a representation $f : E \to R^n$, i.e. we have

$$\det(f(e_1),\ldots,f(e_m)) \in \begin{cases} \{0\} & \text{if } \{e_1,\ldots,e_m\} \text{ is not a base} \\ R^x & \text{else .} \end{cases}$$

Then a Grassmann-Plücker-map $\chi : E^m \to R^x \cup \{0\}$ is given by $\chi(e_1,\ldots,e_m) := \det(f(e_1),\ldots,f(e_m))$ for $e_1,\ldots,e_m \in E$.

References

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