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COMPUTATION OF STANDARD BASES OF S_n-MODULES CORRESPONDING TO n-SUBSETS OF $\mathbb{N}^2$

BY

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All the classical module construction for the representation theory of S_n can be modelled in the polynomial ring $\mathbb{Q}[X_{i,j}]$ over $\mathbb{Q}$ in the n^2 commuting indeterminates $X_{i,j}$. We recall a definition for generalized *Specht modules*:

$$\pi \cdot X_{i,j} := X_{\pi(i),j}\,, \quad \pi \in S_n$$

defines a $\mathbb{Q}S_n$-algebra structure on $\mathbb{Q}[X_{i,j}]$. For n-subsets A of $\mathbb{N} \times \mathbb{N}$ one can define a cyclic $\mathbb{Q}S_n$-submodule S_A of $\mathbb{Q}[X_{i,j}]$ via *A-tableaux* $S, T : A \to \{1, .., n\}$, where S is bijective, and *bideterminants*. We give a paradigm for $A = \begin{array}{cccc} \times & \times & & \times \\ \times & & \times & \\ & \times & \times & \end{array}$.

$$\left(\begin{array}{cccc|cccc} 1 & 3 & & 7 & 1 & 1 & & 1 \\ 2 & & 5 & & 2 & & 2 & \\ & 4 & 6 & & & 3 & 3 & \end{array} \right) :=$$

$$det \begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix} det \begin{pmatrix} X_{3,1} & X_{3,3} \\ X_{4,1} & X_{4,3} \end{pmatrix} det \begin{pmatrix} X_{5,2} & X_{5,3} \\ X_{6,2} & X_{6,3} \end{pmatrix} X_{7,1}.$$

These bideterminants share the property:

$$\pi \cdot (S|T) = (\pi \cdot S|T), \quad \pi \in S_n.$$

Let $P := P_A : A \to \{1, .., n\}$ denote the projection onto the first coordinate.

$$S_A := \mathbb{Q}S_n \cdot (S|P) = \langle (\pi \cdot S|P) : \pi \in S_n \rangle_{\mathbb{Q}}$$

This is a generalization of the classical cases: If A is a diagram,then S_A is isomorphic to the classical Specht module involving Vandermonde determinants. If A is a skew diagram, then S_A is a *skew module*.
We remark that permuting rows and columns of A gives isomorphic modules. The smallest non-classical examples are

		×					
	×			*and*	×	×	
×	×	×			×		×
×						×	×

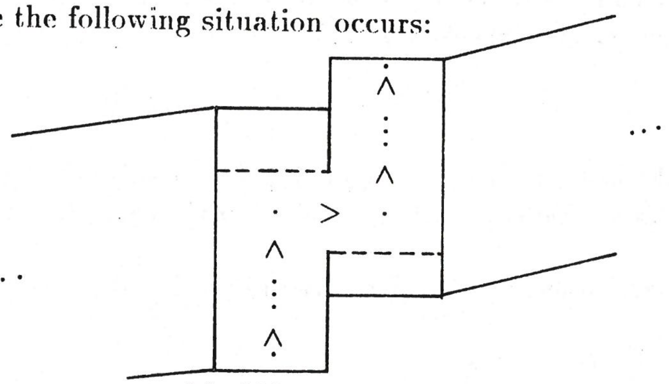
For a skew digram A a bijective A-tableau is called *standard*, if the entries are increasing from left to right in each row and from top to bottom in each column, e.g.:

					1	2	3	12
				4	5	9		
	6	7	8	10	13			
11	14							

Theorem. *If A is a skew diagram, then $\{(S|P_A) : S$ standard$\}$ is a $\mathbb{Q}$-basis of $\mathcal{S}_A$. This is even true over the integers $\mathbb{Z}$.*

Remark. *This basis is the column lexicographical smallest subset of $\{(S|P_A) : S : A \to \{1,..,n\}$ bij.$\}$, which is a $\mathbb{Q}$-basis of $\mathcal{S}_A$;*

otherwise somewhere the following situation occurs:



and *Laplace expansion* expresses this bideterminant as a sum of column lexicographical smaller bideterminants of shape A by *shuffle* permutations.
Therefore our aim is to construct the column-lexicographical smallest basis of $\mathcal{S}_A$ in general, the *A-standard* basis. The tool for this will be simultaneous Laplace expansions over more than 2 columns (because of "holes").

Laplace Duality Theorem. *Let $\phi : A \to B$ be a bijection of n-subsets of $\mathbb{N} \times \mathbb{N}$ with inverse ψ, let S, T be A-tableaux. Then $S' := S \circ \psi$ and $T' := T \circ \psi$ are tableaux of shape B, and the following holds:*

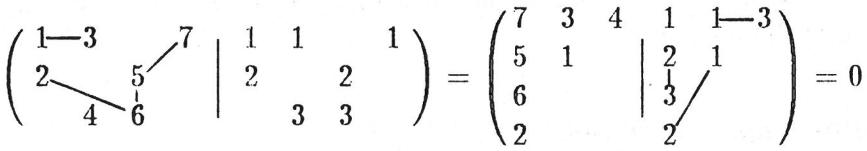
$$(S;\phi|T) := \sum_{\sigma \in \mathcal{V}(B)^{\psi} \bmod \mathcal{V}(B)^{\psi} \cap \mathcal{V}(A)} sign(\sigma)(S \circ \sigma|T)$$

$$= \sum_{\tau \in \mathcal{V}(A)^{\phi} \bmod \mathcal{V}(A)^{\phi} \cap \mathcal{V}(B)} sign(\tau)(S'|T' \circ \tau) =: (S'|T';\psi)$$

(Here $\mathcal{V}(B)^{\psi}$ mod $\mathcal{V}(B)^{\psi} \cap \mathcal{V}(A)$ denotes an arbitrary transversal of the left cosets of $\mathcal{V}(B)^{\psi} \cap \mathcal{V}(A)$ in $\mathcal{V}(B)^{\psi} := \psi \circ \mathcal{V}(B) \circ \psi^{-1}$.)

Example. *In the sequel we will identify the elements of an n–subset A with* $\{1, \ldots, n\}$ *reflecting the column lexicographical order. The corresponding basic tableau will be denoted by* $S_0 = S_0(A)$.

$$A = \begin{pmatrix} 1 & 3 & & 7 \\ 2 & & 5 & \\ & 4 & 6 & \end{pmatrix}, B = \begin{pmatrix} 1 & 5 & 7 \\ 2 & 6 & \\ 3 & & \\ 4 & & \end{pmatrix}, \psi = \begin{pmatrix} 1 & 2 & 3 & 4 & |5 & 6 & | & 7 \\ 7 & 5 & 6 & 2 & |3 & 1 & | & 4 \end{pmatrix}$$



Therefore the column lexicographic greatest bideterminant in the sum can be expressed as a linear combination of smaller ones, these are exactly those bideterminants whose left tableaux S satisfy

$$\neg((S(1) > S(3)) \wedge ((S(2) > S(6)) \wedge ((S(5) > S(7))$$

$$= ((S(1) < S(3)) \vee ((S(2) < S(6)) \vee ((S(5) < S(7)),$$

i.e. all ***standard*** A-tableaux have to satisfy these conditions.

Method. *Make systematic use of the Laplace duality theorem, right hand side equals 0, to construct conditions for* ***standard*** *A-tableaux.*

A combinatorial description of all diagrams B such that for all bijections $\phi : A \to B$ the right hand side equals 0 is given by the following.

Theorem. *Let A and B be n-subsets of* $\mathbb{N} \times \mathbb{N}$, *B a diagram with column lengths* $\mu \vdash n$. *For* $T : A \to \{1, .., n\}$ *with* $con(T) := (|T^{-1}(1)|, ..., |T^{-1}(n)|) \models n$ *the following two statements are equivalent:*
i) $(S; \phi | T) = 0$ *holds for all bijections* $\phi : A \to B$ *and for all bijections* $S : A \to \{1, .., n\}$.
ii) $\mu \ntrianglelefteq (con(T) \searrow)'$ ($\searrow$ *means reordering to get a proper partition)*

Example.

$$P = \begin{pmatrix} 1 & 1 & & 1 \\ 2 & & 2 & \\ & 3 & 3 & \end{pmatrix} \qquad con(P)' = (3, 2, 2)' = (3, 3, 1)$$

therefore $\{\mu \vdash n : \mu \ntrianglelefteq (3, 3, 1)\} = \{(4, 1, 1, 1), (4, 2, 1), ..., (7)\}$.

Remark. *However there exist (B,ϕ) such that $\mu \ntrianglelefteq (con(P)\searrow)'$ and $(S;\phi|P)=0$, e.g.*

$$\left(\begin{array}{cccc|cccc} 1 & 3 & & 7 & 1 & 1 & & 1 \\ 2 & & 5 & & 2 & & 2 & \\ & 4 & 6 & & & 3 & 3 & \end{array}\right) = \left(\begin{array}{ccc|ccc} 7 & 3 & 6 & 1 & 1 & 3 \\ 5 & 1 & & 2 & 1 & \\ 4 & 2 & & 3 & 2 & \end{array}\right) =$$

$$= \left(\begin{array}{cc|cc} 7 & 6 & 1 & 3 \\ 5 & & 2 & \\ 4 & & 3 & \end{array}\right) \cdot \left(\begin{array}{c|c} 3 & 1 \\ 1 & 1 \\ 2 & 2 \end{array}\right) = 0$$

because of $(3) \ntrianglelefteq (2,1)$, so this trivial example indicates how also "local" versions of the theorem may be derived.

Algorithm. *Input: n-subset A, $S_0 = S_0(A)$.*

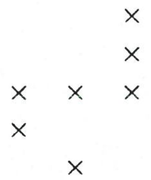
i) $\mathcal{R} := \{(i,i+1) : i, i+1$ *both in one column of* $S_0\}$

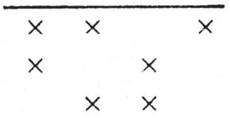
ii) Construct the set $\mathcal{J}$ of all pairs (i,j), where the column in which i appears is on the left of the column of j.

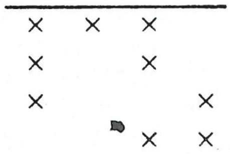
iii) For $v = 1$ to n div 2 do: To each v-subset $J \in \mathcal{J}$ construct (B,ψ) such that μ = column lengths of B is maximal in the dominance order and J is exactly the set of "jumps" of ψ: for each column of B there is a jump from the lowest element involved in one column to the highest element in the next involved column to the left. If $\mu \ntrianglelefteq (con(P)\searrow)'$ for at least one μ then add J to the list $\mathcal{R}$ of relations. More general one has to consider "local" versions, too. (interprete $J = ((i_1,j_1),(i_2,j_2),...)$ as $J = ((i_1 < j_1) \vee (i_2 < j_2) \vee ...)$.

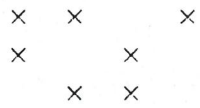
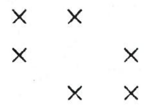
iv) Construct all $S : A \to \{1,..,n\}$, which satisfy the relations in $\mathcal{R}$. To do this, split the relations to get a Boolean normal form.

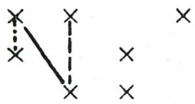
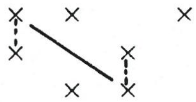
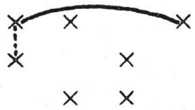
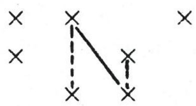
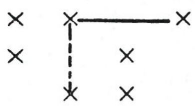
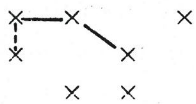
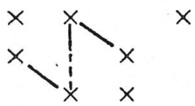
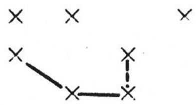
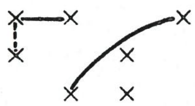
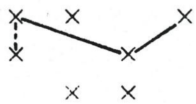
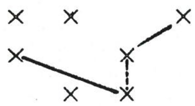
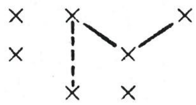
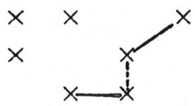
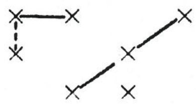
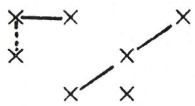
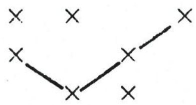
Examples:	dimension		number of standard tableaux "R.H.S.= 0"
$\begin{array}{ccc} \times & \times & \\ \times & & \times \\ & \times & \times \end{array}$	42	=	42
$\begin{array}{ccc} & & \times \\ \times & \times & \times \\ \times & & \\ & \times & \end{array}$	47	=	47
$\begin{array}{ccc} \times & & \times \\ \times & \times & \times \\ \times & & \\ & \times & \end{array}$	56	=	56

 127 = 127

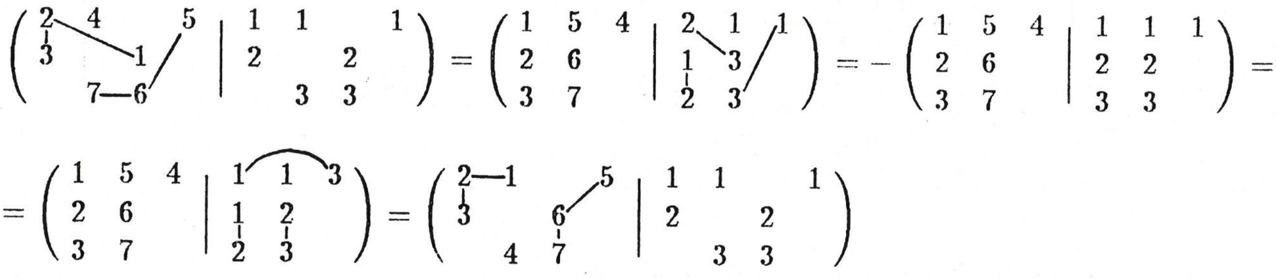
 126 < 137

 1278 < 1299

As an illustration we give the list of relations for the case $A =$  which of course covers the case $A =$  ,too:

	$(1 < 4)$		$(1 < 6)$
	$(1 < 7)$		$(3 < 6)$
	$(3 < 7)$		$(1 < 3) \vee (3 < 5)$
	$(2 < 4) \vee (3 < 5)$		$(2 < 4) \vee (4 < 6)$
	$(1 < 3) \vee (4 < 7)$		$(1 < 5) \vee (5 < 7)$
	$(2 < 6) \vee (5 < 7)$		$(3 < 5) \vee (5 < 7)$
	$(4 < 6) \vee (5 < 7)$		$(1 < 3) \vee (4 < 6) \vee (6 < 7)$
	$(1 < 3) \vee (4 < 5) \vee (5 < 7)$		$(2 < 4) \vee (4 < 5) \vee (5 < 7)$

The table above showes that these relations are not enough. Therefore one has to develop combinatorial descriptions of more complicated constructions. For example one can use the fact that $\begin{matrix} 1 & 1 & 1 \\ 2 & 2 & \\ 3 & 3 & \end{matrix}$ is the only column increasing tableau of shape $(3,3,1)'$ and content $(3,2,2)$.



As $S(3) < S(4)$ holds for all S we can add relation $((1,5),(4,6),(6,7))$ to $\mathcal{R}$. Similarly we receive $((2,3),(3,5),(4,6),(6,7))$.

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