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ON THE POINT-DISTINGUISHING CHROMATIC INDEX OF $K_{n,n}$

BY

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SUMMARY - Let $\chi_0(G)$ be the point-distinguishing chromatic index of a graph G.

We determine some properties of a point-distinguishing χ_0-coloring of $K_{n,n}$. In particular we determine the values of n for which $K_{n,n}$ has a point-distinguishing χ_0-coloring with a vertex whose incident lines are colored with the same color and in such cases we calculate $\chi_0(K_{n,n})$. In the other cases we determine a characterization of a point-distinguishing χ_0-coloring of $K_{n,n}$.

1. INTRODUCTION

The point-distinguishing (p. d.) chromatic index of a graph G = (V,E), denoted $\chi_0(G)$, is the minimum number of colors assignable to E so that no two distinct points are incident with the same color set of lines.

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In [1] tight bounds on this chromatic index are obtained for $K_{n,n}$. In fact it is proved that

$$\lceil \log_2 n \rceil + 1 \leq \chi_0(K_{n,n}) \leq \lceil \log_2 n \rceil + 2 \qquad (1)$$

This relation implies that, for $2^{k-2} \leq n \leq 2^{k-1}$, it is $\chi_0(K_{n,n})=k$ or $k+1$.

In [1] the problem of determining $\chi_0(K_{n,n})$ exactly for arbitrary n is posed.

We prove that a point-distinguishing χ-coloring of $K_{n,n}$ corresponds to a matrix of order n with elements belonging to $L = \{1, 2, \dots, \chi\}$, that determines a partition of the 2^{χ} distinct subsets of L on three particular collections of sets.

We give an algorithm to determine a p.d. coloring of $K_{2n,2n}$, knowing such a coloring of $K_{n,n}$.

We say that in a p.d. coloring of G a vertex is <u>monochromatic</u> if all the lines incident with it have the same color.

A p.d. coloring with a monochromatic vertex is called a p.d. <u>monocoloring.</u>

We prove that in a p.d. χ_0-coloring of $K_{n,n}$ only one vertex can be monochromatic, with only one exception.

Moreover we determine when $K_{n,n}$ has a p.d. χ_0-monocoloring and in this case we calculate $\chi_0(K_{n,n})$. In the other cases we give a characterization of a p.d. χ_0-coloring of $K_{n,n}$.

Denote by J_n the all one matrix of order n.

2. Denote by $V = \{v_1, v_2, \dots, v_n\}$ and $W = \{v_1', v_2', \dots, v_n'\}$ the classes of vertices of $K_{n,n}$.

Consider a p.d. coloring f of the edges of $K_{n,n}$ with χ colors, where $L = \{1, 2, \dots, \chi\}$ is the set of colors.

If $a_{ij} \in L$ is the color of the edge $v_i v_{j'}$, the matrix of order n $A = \left[a_{ij} \right]$ represents such a coloring.

The i-th row of A determines the color set of the edges incident with v_i, while the j-th column corresponds to the color set of the edges incident with $v_{j'}$.

Since f is a p.d. coloring, it follows that two distinct lines of A never determine the same color set.

If l is a line of A, we delote by $\{l\}$ the set of colors contained in A and by $\{\bar{l}\}$ the complementary set of $\{l\}$ with respect to L, that is $\{\bar{l}\} = L - \{l\}$.

We say that <u>a set H belongs to A</u> if a line l of A such that $H = \{l\}$ exists.

Finally we say that l <u>corresponds</u> to $\{l\}$.

PROPOSITION 2.1 - A p.d. χ-coloring of $K_{n,n}$ exists iff there exists a matrix of order n with elements belonging to $\{1, 2, \ldots, \chi\}$ such that distinct lines correspond to distinct sets.

The proof follows from the preceding considerations.$\square$

PROPOSITION 2.2 - A p.d. χ-coloring of $K_{n,n}$ implies a partition of distinct subsets of $L = \{1, 2, \ldots, \chi\}$ on three collections $\mathcal{C}$, $\mathcal{R}$ and $\mathcal{F}$ of order n, n, $2^{\chi}-2n$ respectively, such that if $H \in \mathcal{C}$, then $\bar{H}$ and all its subsets belong to $\mathcal{C}$ or to $\mathcal{F}$.

Proof. Let A be the matrix that represents a χ-coloring of $K_{n,n}$. Denote by $\mathcal{C}$, $\mathcal{R}$ and $\mathcal{F}$ the collections of distinct subsets of colors respectively contained in the columns, rows of A and not belonging to A.

Then, if $H \in \mathcal{C}$, $\bar{H}$ and all its subsets belong to $\mathcal{C}$ or to $\mathcal{F}$.

On the contrary, we have that, denoted by c a column of A, there exists a subset S of $\{\bar{c}\}$ that coincides with $\{r\}$, where r is a row of A. Then the element common to c and r belongs to $\{c\} \cap \{\bar{c}\}$; a contradiction.$\square$

REMARK 2.3 - The property given in Prop. 2.2 concerning $\mathcal{E}$ holds also for $\mathcal{R}$.

Proof. Let $R \in \mathcal{R}$ be. Then $\bar{R}$ and all its subsets belong to $\mathcal{R}$ or to $\mathcal{F}$. In fact if a subset $\{r\}$ of $\bar{R}$ belongs to $\mathcal{E}$, then R is a subset of $\overline{\{r\}}$ and it is contained in $\mathcal{R}$, a contradiction. $\square$

The condition given in Proposition 2.2 is not sufficient for the existence of a p.d. χ - coloring of $K_{n,n}$.

For example, let k=5 be and $L = \{1,2,3,4,5\}$.

We can consider the collections of subsets of L $\mathcal{E} = \{2345, 234, 235, 245, 345, 23, 45, 24, 35\}$, $\mathcal{R} = \{12345, 1345, 1245, 1235, 1234, 25, 134, 34, 125\}$ and $\mathcal{F} = \{\phi, 1, 2, 3, 4, 5, 12, 13, 14, 15, 145, 123, 135, 142\}$.

For every $H \in \mathcal{E}[\mathcal{R}]$, $\bar{H} \in \mathcal{E}[\mathcal{R}]$ or to $\mathcal{F}$ together with all its subsets. However there does not exist a matrix whose lines correspond to the sets of $\mathcal{E}$ and $\mathcal{R}$.

THEOREM 2.4 - If A represents a p.d. χ-coloring of $K_{n,n}$, then the matrix of order 2n

$$\begin{bmatrix} A & A \\ A & (\chi+1)J_n \end{bmatrix} \qquad (2)$$

represents a p.d. $(\chi+1)$-coloring of $K_{2n,2n}$.

Proof. We denote type a) the rows r_i and columns c_i of (2) for $1 \leq i \leq n$ and type b) for $n+1 \leq i \leq 2n$.

So two distinct lines of type a) determine the same distinct color sets of the corresponding lines of A.

A similar situation is for the lines of type b) since they determine the same color set of lines of A with the addition of the new color $\chi+1$. Finally, a line of type a) and a line of type b) determine two distinct color sets because the (χ +1)-color does not belong to both the lines. $\square$

For example the matrix

$$A_1 = \begin{bmatrix} 2 & 2 & 1 & 2 & 1 \\ 3 & 4 & 4 & 3 & 4 \\ 1 & 1 & 3 & 3 & 1 \\ 2 & 2 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 4 \end{bmatrix}$$

represents a 4-coloring of $K_{5,5}$. The families considered in Prop. 2.2 are $\mathcal{C} = \{123, 124, 134, 234, 14\}$, $\mathcal{R} = \{12, 34, 13, 24, 1234\}$ and $\mathcal{F} = \{\emptyset, 1, 2, 3, 4, 23\}$.

We remark that the matrix formed by the first q rows and q columns $3 \leq q \leq 5$, determines a p.d. 4-coloring of $K_{q,q}$.

Moreover in

$$A_2 = \begin{bmatrix} A_1 & A_1 \\ A_1 & 5J_5 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 \\ 3 & 4 & 4 & 3 & 4 & 3 & 4 & 4 & 3 & 4 \\ 1 & 1 & 3 & 3 & 1 & 1 & 1 & 3 & 3 & 1 \\ 2 & 2 & 4 & 4 & 4 & 2 & 2 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 4 & 1 & 2 & 3 & 4 & 4 \\ 2 & 2 & 1 & 2 & 1 & 5 & 5 & 5 & 5 & 5 \\ 3 & 4 & 4 & 3 & 4 & 5 & 5 & 5 & 5 & 5 \\ 1 & 1 & 3 & 3 & 1 & 5 & 5 & 5 & 5 & 5 \\ 2 & 2 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 5 \\ 1 & 2 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 5 \end{bmatrix}$$

the matrix formed with the first 5+q rows and columns, $1 \leq q \leq 5$, represents a p.d. 5-coloring of $K_{5+q,5+q}$.

3. PROPOSITION 3.1 - If a matrix A of order n, $2^{k-2} < n \leq 2^{k-1}$, represents a k-coloring of $K_{n,n}$, then at most only one line of A is monochromatic.

Proof. Suppose A contains two monochromatic lines. Then they are parallel, for example like two rows.
Then all the columns contain two fixed colours. Apart from these two colors they determine sets with k-2 colors.
Then at most they can be 2^{k-2}.
Since $n > 2^{k-2}$, we have a contradiction. □

The value $n = 2^{k-2}$ is excluded. For example for k=4 we have $\chi_0(K_{4,4}) = 4$ and the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 3 & 3 \\ 2 & 4 & 3 & 4 \end{bmatrix}$$

gives a p.d. 4-coloring of $K_{4,4}$ with two monochromatic lines.

COROLLARY 3.2 - Let $k \geq 3$ be a positive integer. For $2^{k-1} - [\frac{k}{2}] < n \leq 2^{k-1}$, we have $\chi_0(K_{n,n}) = k+1$.

Proof. If it were $\chi_0(K_{n,n}) = k$, in the matrix that represents a k-coloring of $K_{n,n}$ the lines would be $2n > 2^k - k$.
By Proposition 3.1, in a k-coloring of $K_{n,n}$ at least k-1 monochromatic sets and the empty set ϕ are excluded from the lines of A.
Then there is the relation $2n \leq 2^k - k$; a contradiction. □

THEOREM 3.3 - If the matrix A of order n, $2^{k-2} \leq n \leq 2^{k-1}$, represents a k-coloring of $K_{n,n}$, then, for $[\frac{2^k}{3}] < n \leq 2^{k-1}$, A can not contain a monochromatic line.

Proof. Suppose A contains a monochromatic line, for example a row. Then the complement of the color set of every column can not obviously be contained in a row and neither in a column, because all the columns contain the fixed color.
Hence the color sets are at least 3n. So we have $2^k \geq 3n$. $\square$

THEOREM 3.4 - Let $k \geq 3$ be a positive integer and $\bar{n} = \left[\frac{2^k}{3} \right]$. We have that $\chi_o(K_{\bar{n},\bar{n}}) = k$ and $K_{\bar{n},\bar{n}}$ has a p.d. k-monocoloring.

Proof. Let $L = \{1, 2, \ldots, k\}$ be. Determine three collections $\mathcal{C}$, $\mathcal{R}$ and $\mathcal{F}$ of subsets of L of orders $\bar{n}, \bar{n}$ and $\bar{n}+i$, where $3\bar{n}+i = 2^k$ and $i=1$ or $i=2$ respectively for even or odd k, such that, for every $H \in \mathcal{C}$, $\bar{H}$ and all its subsets belong to $\mathcal{F}$.
Consider as elements of $\mathcal{F}$ all the non-empty subsets of $L - \{1\} = \{2, 3, \ldots, k\}$ of increasing order $1, 2, \ldots, \frac{k-1}{2}$ for odd k and $1, 2, \ldots, \frac{k-2}{2}$ for even k with the addition of some arbitrary sets of order $\frac{k+1}{2}$ or $\frac{k}{2}$ respectively, so that the total number is $\bar{n}$.
This is possible because the subsets of $L - \{1\}$, of order $\neq 0, k-1$, are $2^{k-1} - 2$ and it is $\bar{n} \leq 2^{k-1} - 2$.
The elements of $\mathcal{C}$ are the complements of these sets of $\mathcal{F}$ with respect to L.
We add to $\mathcal{F}$ the empty set for $i=1$, the empty set and its complement for $i=2$.
The elements of $\mathcal{R}$ are the remaining sets; we note that $\mathcal{R}$ contains the monochromatic set $\{1\}$.
So, by construction, if $H \in \mathcal{C}$, $\bar{H}$ and all its subsets belong to $\mathcal{F}$.
Hence, if $C \in \mathcal{C}$ and $R \in \mathcal{R}$, it is $C \cap R \neq \emptyset$.
Let $\mathcal{C} = \{C_1, C_2, \ldots, C_{\bar{n}}\}$ be and $\mathcal{R} = \{R_1, R_2, \ldots, R_{\bar{n}}\}$.
Now we construct a matrix $A = [a_{ij}]$ of order $\bar{n}$ whose lines correspond to the sets of $\mathcal{C}$ and $\mathcal{R}$.

We can select a_{ij} as an element of $R_i \cap C_j$ so that the i-th row (j-th column) exactly corresponds to R_i (C_j) , $1 \leq i,j \leq n$.
In fact, denoted $R_i = \{\gamma_1, \gamma_2, \cdots, \gamma_s\}$, $s \leq k$, by construction $\mathcal{C}$ contains s sets $H_1, H_2, \ldots, H_s$, among the first k, such that γ_t belongs to H_t, $1 \leq t \leq s$.
When $C_j = H_t$, we write $a_{ij} = \gamma_t$. In this way every row r_i contains all the colors of R_i.
Moreover, by construction $\mathcal{R}$ contains at least $[\frac{k}{2}]$ sets and their complements.
So, denoted $C_j = \{\delta_1, \delta_2, \cdots, \delta_v\}$, $v \leq k$, $\mathcal{R}$ contains v sets $D_1, D_2, \ldots, D_v$ such that $\delta_u \in D_u$, $1 \leq u \leq v$.
When $C_j = H_t$, an element of C_j, for example δ_1, coincides with γ_t and we can suppose $D_1 = R_i$.
In this case the element $a_{ij} = \gamma_t \in R_i \cap C_j$ is already determined.
For $\delta_u \neq \gamma_t$ or when $C_j \neq H_t$, in correspondence to the rows $R_i = D_u$ we write $a_{ij} = \delta_u$.
So in every column c_j, $1 \leq j \leq n$, there are all the colors contained in C_j.
Every other undetermined element $a_{ij} \in R_i \cap C_j$ can be arbitrarily taked.
As the lines of this matrix correspond to the sets of $\mathcal{C}$ and $\mathcal{R}$ and these sets are distinct by construction, by Prop. 2.1 the theorem follows. $\square$

COROLLARY 3.5 - Let $k \geq 3$ be a positive integer and $\bar{n} = [\frac{2^k}{3}]$. For every t, $2^{k-2} \leq t \leq \bar{n}$, $\chi_{\mathcal{G}}(K_{t,t}) = k$ and $K_{t,t}$ has a p.d. k-monocoloring.

Proof. It suffices take the first t sets of $\mathcal{C}$ and $\mathcal{R}$ considered in Theorem 3.4 and construct the matrix A whose lines correspond to these sets. The construction of A is the same as that followed in Theorem 3.4. $\square$

4. Throughout this section we denote $L = \{1, 2, \dots, k\}$ and suppose $k > 3$.

PROPOSITION 4.1 - Let $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ be, $\mathcal{R}$ and $\mathcal{F}$ three families of subsets of L, whith $\mathcal{R}$ and $\mathcal{F}$ of orders n, $2^k - 2n$ and $\left[\frac{2^k}{3}\right] < n \leq 2^{k-1}$, such that if $H \in \mathcal{C}$, then $\bar{H}$ and all its subsets belong to $\mathcal{C}$ or to $\mathcal{F}$.

Then $\left|\bigcup_{i=1}^{n} C_i\right| = k$.

Proof. As $n > 2^{k-2}$ and the sets on k-2 elements are 2^{k-2}, we have $\left|\bigcup_{i=1}^{n} C_i\right| \geq k-1$. We prove that it can not be $\left|\bigcup_{i=1}^{n} C_i\right| = k-1$.

In fact, suppose that an element, for example 1, does not belong to $\bigcup_{i=1}^{n} C_i$.

Then, if $H \in \mathcal{C}$, $\bar{H}$ contains 1; hence $\bar{H}$ belongs to $\mathcal{F}$.

So the order of $\mathcal{F}$ is greater than n and we obtain the impossible relation $2^k = 2n + |\mathcal{F}| > 3n > 2^k$. $\square$

PROPOSITION 4.2 - Let $\mathcal{C}$, $\mathcal{R}$ and $\mathcal{F}$ be three families of distinct subsets of L, of orders n, n, 2^k-2n respectively, where $\left[\frac{2^k}{3}\right] < n \leq 2^{k-1}$, such that if $H \in \mathcal{C}$, then $\bar{H}$ and all its subsets belong to $\mathcal{C}$ or to $\mathcal{F}$.

Then every element of L belongs to at least two distinct sets of $\mathcal{C}$.

Proof. By Prop. 4.1 every element of L belongs to at least one set of $\mathcal{C}$.

Suppose that the element 1 belongs to only one set, for example C_1.

Then, if $H \in \mathcal{C}$ and $H \neq C_1$, $\bar{H}$ contains 1; so $\bar{H}$ belongs to $\mathcal{F}$.

There are two cases:

1) $\bar{C}_1 \notin \{C_2, \dots, C_n\}$.

Then $\mathcal{F}$ contains at least the n-1 sets $\bar{C}_i$, $2 \leq i \leq n$, and the empty set $\emptyset$.

So $|\mathcal{F}| \geq n$ and we have the impossible relation

$$2^k = 2n + |\mathcal{F}| \geq 3n > 2^k . \tag{4}$$

2) $\bar{C}_1 \in \{C_2, \dots, C_n\}$.

Let $C_2 = \bar{C}_1$ be. If n is odd, there is a set $R \in \mathcal{R}$ such that $\bar{R} \in \mathcal{F}$.
So $\mathcal{F}$ contains the n-2 sets $\bar{C}_i$, $3 \leq i \leq n$, $\emptyset$ and $\bar{R}$; then $|\mathcal{F}| \geq n$ and (4) holds again.

Let n be even. Remark that $\bar{\emptyset} \notin \mathcal{C}$. So, if $\bar{\emptyset} \in \mathcal{F}$, $\mathcal{F}$ contains the n-2 sets $\bar{C}_i$, $3 \leq i \leq n$, $\emptyset$ and $\bar{\emptyset}$ and we have the preceding situation.
If $\bar{\emptyset} \notin \mathcal{F}$, then $\bar{\emptyset} \in \mathcal{R}$. So there exists a set $R \in \mathcal{R}$, such that $\bar{R} \in \mathcal{F}$ and again $|\mathcal{F}| \geq n$. []

PROPOSITION 4.3 - Let $\mathcal{C}$, $\mathcal{R}$ and $\mathcal{F}$ be three families of distinct subsets of L, of orders n, n, 2^k-2n respectively and $\left[\frac{2^k}{3}\right] < n \leq 2^{k-1}$, such that, if $H \in \mathcal{C}$, then $\bar{H}$ and all its subsets belong to $\mathcal{C}$ or to $\mathcal{F}$.

Then there exists a family of k sets of $\mathcal{C}$ that has a transversal.

Proof. Let r be the maximum number of sets of $\mathcal{C}$ with a transversal.
Let $[a_1, a_2, \dots, a_r]$ be such a transversal.
It can not be $r \leq k-2$. In fact, in this case, all the other sets of $\mathcal{C}$ would have elements belonging to $\{a_1, a_2, \dots, a_r\}$, so their number n-r satisfies the relation $n-r < 2^r \leq 2^{k-2}$. But this is impossible because $n > \left[\frac{2^k}{3}\right]$ and $r \leq k-2$.
Moreover it cannot be r=k-1.
In fact, let $H_1, H_2, \dots, H_{k-1}$ be the sets with the transversal $[a_1, a_2, \dots, a_{k-1}]$ and $D_1, D_2, \dots, D_{n-k+1}$ the remaining sets.
Then the elements of D_i, $1 \leq i \leq n-k+1$, belong to $L' = \{a_1, a_2, \dots, a_{k-1}\}$.
Let a_k be the color that does not belong to L'.

By Prop. 4.2, there exist at least two sets, for example H_1 and H_2, that contain a_k apart from a_1 and a_2 respectively.

Moreover there exists a set D_j, $j \in [1, n-k+1]$, that contains a_1 or a_2.

On the contrary, it holds $\left|\bigcup_{i=1}^{n-k+1} D_i\right| = k-3$; but this is impossible because the number of sets D_i, $n-k+1$, is greater than 2^{k-3}.

Suppose that D_j contains a_1.

Then, the sets $H_1, H_2, \ldots, H_{k-1}, D_j$ have the transversal $[a_k, a_2, \ldots, a_{k-1}, a_1]$ and $r=k$. $\square$

The preceding considerations about the elements of $\mathcal{C}$ hold also for the elements of $\mathcal{R}$.

In fact we have the following

Corollary 4.4 - Let $\mathcal{C}$, $\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$ and $\mathcal{F}$ three families of distinct subsets of L, of orders n, n, $2^k - 2n$ respectively, where $[\frac{2^k}{3}] < n \leq 2^{k-1}$, such that if $H \in \mathcal{C}$, then $\overline{H}$ and all its subsets belong to $\mathcal{C}$ or to $\mathcal{F}$.

Then the following statements hold:

1) $\left|\bigcup_{i=1}^{n} R_i\right| = k$,

2) every element of L belongs to at least two distinct sets of $\mathcal{R}$,

3) there exists a family of k sets of $\mathcal{R}$ that has a transversal.

The proof follows immediately from Remark 2.3 and Prop. 4.1, 4.2 and 4.3. $\square$

THEOREM 4.5 - A p.d. k-coloring of $K_{n,n}$, for $[\frac{2^k}{3}] < n \leq 2^{k-1}$ exists iff there exists a partition of the subsets of L on three families of distinct sets $\mathcal{C}$, $\mathcal{R}$ and $\mathcal{F}$, of orders n, n, $2^k - 2n$, such that, if $H \in \mathcal{C}$, then $\overline{H}$ and all its subsets belong to $\mathcal{C}$ or to $\mathcal{F}$.

Proof. The condition is necessary by Prop. 2.2.

We prove that it is also sufficient.

Let $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ be and $\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$.

We construct a matrix $A = [a_{ij}]$ of order n whose lines correspond to the sets of $\mathcal{C}$ and $\mathcal{R}$.

By Prop. 4.3 (Corollary 4.4) $\mathcal{C}$ $(\mathcal{R})$ contains a family of k sets with a transversal.

Then, denoted $R_i = \{\gamma_1, \gamma_2, \ldots, \gamma_s\}$, $s \leq k$, $\mathcal{C}$ contains s sets H_1, $H_2, \ldots,$ H_s such that $\gamma_t \in H_t$, $1 \leq t \leq s$.

Moreover, denoted $C_j = \{\delta_1 \delta_2 \ldots \delta_v\}$, $v \leq k$, $\mathcal{R}$ contains v sets $D_1, D_2, \ldots, D_v$ such that $\delta_u \in D_u$, $1 \leq u \leq v$.

The procedure for the construction of A is the same as that followed in Theorem 3.5; in this case, by Theorem 3.3 A does not contain a monochromatic line. $\square$

REFERENCES

[1] F. Harary and M. Plantholt, The point-distinguishing chromatic index, Graphs and Applications (Proc.), A Wiley-Interscience Publication (1985), 147-162.

[2] F. Harary, Graph Theory, Addison-Wesley, Reading (1969)

[3] D. Welsh, Matroid Theory, Academic Press, London (1976)

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