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THE TUTTE-GROUP OF A MATROID

BY

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With the intent to construct an algebraic theory of matroids A. Dress introduced the concept of the Tutte-group. In particular this group enables an algebraic approach to those matroids which are not representable over any field.

Let M denote a matroid defined on the finite set E of rank n with rank function ρ, and let $\mathcal{H}$ denote the set of its hyperplanes.

<u>Definition of the Tutte-group:</u>

Let $\mathbb{F}_M$ denote the free abelian group generated by the symbols ε and $X_{H,a}$ for $H \in \mathcal{H}$, $a \in E \setminus H$, and let $\mathbb{K}_M$ denote the subgroup generated by ε^2 and all elements of the form

$$\varepsilon \cdot X_{H_1,a_2} \cdot X^{-1}_{H_1,a_3} \cdot X_{H_2,a_3} \cdot X^{-1}_{H_2,a_1} \cdot X_{H_3,a_1} \cdot X^{-1}_{H_3,a_2}$$

with $H_1,H_2,H_3 \in \mathcal{H}$, $L := H_1 \cap H_2 \cap H_3 = H_i \cap H_j$ for $i \neq j$, $\rho(L) = n-2$ and $a_i \in H_i \setminus L$ for $i \in \{1,2,3\}$.

Then the <u>Tutte-group</u> $\mathbb{T}_M$ of M is defined by

$$\mathbb{T}_M := \mathbb{F}_M \,/\, \mathbb{K}_M \,.$$

Let $T_{H,a}$ denote the image of $X_{H,a}$ and ε_M the image of ε in $\mathbb{T}_M$, respectively.

<u>Proposition 1:</u>

i) Assume M is representable over a field K with hyperplane functions $f_H : E \to K$. Then a well-defined homomorphism $\psi : \mathbb{T}_M \to K^*$ is given by $\psi(\varepsilon_M) := -1$; $\psi(T_{H,a}) := f_H(a)$ for $H \in \mathcal{H}$, $a \in E \setminus H$.

ii) Assume M is binary.

If K is a field and there exists a homomorphism $\psi: \mathbb{T}_M \to K^*$ with $\psi(\varepsilon_M) = -1$, then M is representable over K, and a system of hyperplane functions $(f_H)_{H \in \mathcal{H}}$ is given by

$$f_H(a) := \begin{cases} 0 & \text{for } a \in H \\ \psi(T_{H,a}) & \text{for } a \notin H. \end{cases}$$

Definition of the truncated Tutte-group:

Let $\mathbb{H}_M$ denote the subgroup of $\mathbb{T}_M$ generated by all elements of the form

$$\varepsilon_M \cdot T_{H_1,a} \cdot T_{H_1,b}^{-1} \cdot T_{H_2,b} \cdot T_{H_2,a}^{-1}$$

with $H_1, H_2 \in \mathcal{H}$, $\rho(H_1 \cap H_2) = n-2$, $a,b \in E \setminus (H_1 \cup H_2)$, $\rho((H_1 \cap H_2) \cup \{a,b\}) = n$. Then

$$\overline{\mathbb{T}}_M := \mathbb{T}_M \,/\, \mathbb{H}_M$$

is called the truncated Tutte-group of M. For $T \in \mathbb{T}_M$ let $\overline{T}$ denote its image in $\overline{\mathbb{T}}_M$.

Remarks: i) If M is binary, then $\mathbb{T}_M = \overline{\mathbb{T}}_M$.

ii) Let M denote the projective plane over the field $\mathbb{F}_3$. Then also $\mathbb{T}_M = \overline{\mathbb{T}}_M$

Proposition 2:

Assume $\psi : \mathbb{T}_M \to \mathbb{F}_3^*$ is a homomorphism satisfying $\psi(\varepsilon_M) = -1$. Then the following two statements are equivalent:

i) M is representable over $\mathbb{F}_3$ and a system of hyperplane functions is given by

$$f_H(a) := \begin{cases} 0 & \text{for } a \in H \\ \psi(T_{H,a}) & \text{for } a \notin H. \end{cases}$$

ii) ψ induces a homomorphism $\overline{\psi} : \overline{\mathbb{T}}_M \to \mathbb{F}_3^*$, i.e. $\mathbb{H}_M \subseteq \ker \psi$.

By applying Tutte's homotopy theory one proves

Proposition 3:

Let z denote the number of connected components of M and put $r = \#(E) - z + \#(\mathcal{H})$. Then

$$\overline{\mathbb{T}}_M \cong \langle \overline{\varepsilon_M} \rangle \times \mathbb{Z}^r .$$

Furthermore the following four statements are equivalent.

i) M is ternary,

ii) $\overline{\varepsilon_M} \notin \overline{\mathbb{T}}_M^2 := \{\overline{T}^2 \mid T \in \mathbb{T}_M\}$,

iii) $\overline{\varepsilon_M} \neq 1$,

iv) The Fano-Matroid, its dual, $U_{2,5}$ and $U_{3,5}$ are no minors of M.

Here $U_{k,m}$ denotes the uniform matroid of rank k with m elements; this means every subset containing k elements is a base.

Corollary:

If M is binary, then

$$\mathbb{T}_M \cong \langle \varepsilon_M \rangle \times \mathbb{Z}^r \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}^r & \text{if the Fano-Matroid and its dual are no minors of } M \\ \mathbb{Z}^r & \text{else .} \end{cases}$$

In general the structure of the non-truncated Tutte-group may be more complicated. In particular there exist infinitely many minimal matroids M with $\varepsilon_M \in \mathbb{T}_M^2 \setminus \{1\}$ and also infinitely many minimal matroids M satisfying $\varepsilon_M = 1$.

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