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ON ω-RAMSEYAN SEMIGROUPS

BY

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Summary. We prove that a finitely generated semigroup is ω-ramseyan (see the following definition 1) iff it is finite.

The free semigroup (resp. free monoid) on an alphabet A is denoted by A^+ (resp. A^*). The elements of A^* are called words. A sequence (infinite word) on A is a map from $\mathbb{P}$ (the set of positive integers) into A. The set of sequences on A is denoted by A^ω. The length of a word w is denoted by $|w|$.

Let us introduce the following definition:

Definition 1. Given an alphabet A, a set E and a map f: $A^+ \longrightarrow E$, we say that the map f is ω-ramseyan iff each sequence s of A^ω admits a factorisation

$$s = t\, u_1 \ldots u_i \ldots$$

where $t \in A^*$, $u_i \in A^+$ and for $i, j \in \mathbb{P}$

$$f(u_i) = f(u_j) = f(u_i \ldots u_j)$$

We say that a semigroup S is ω-ramseyan iff every morphism f: $A^+ \longrightarrow S$, such that f(A) is finite, is ω-ramseyan.

For the notions of ramseyan, strongly repetitive, strongly ramseyan and ω-repetitive semigroup, we refer to [1], where the following theorem is proved.

Theorem 1. Let S be a finitely generated semigroup. The following conditions are equivalent:

1) S is finite;

2) S *is ramseyan*;

3) S *is strongly repetitive*;

4) S *is strongly ramseyan*;

5) S *is* ω-*repetitive*.

In this note, we present an improvement of the previous theorem. In fact, the following proposition holds.

Proposition 1. *A finitely generated semigroup is* ω-*ramseyan iff it is finite*.

Proof. Suppose that S is a finitely generated ω-ramseyan semigroup. Let G be a(finite) set of generators of S, $\bar{G}$ be a copy of G and f the morphism (from $\bar{G}^+$ into S) defined by

$$f(\bar{g}) = g$$

for each $\bar{g}$ of $\bar{G}$.

We say that a word w of $\bar{G}^+$ is *irreducible* iff for each v of $\bar{G}^+$ such that

$$f(w) = f(v)$$

we have

$$|w| \leq |v| .$$

Now, suppose, by way of contradiction, that S is infinite.

There is an infinite set of irreducible words in $\bar{G}^+$. By a well known combinatorial argument (see, for example, lemma 1.1 in [1]) there is a sequence s in $\bar{G}^\omega$ such that each factor of s is an irreducible word. Since S is ω-ramseyan there is a factorisation

$$s = t\, u_1\, u_2 \ldots$$

such that, in particular,

$$f(u_1) = f(u_1\, u_2)$$

and the length of u_1 is strictly less than $u_1\, u_2$. So, $u_1\, u_2$ cannot be an irreducible word. A contradiction.

Conversely, suppose that S is finite.

Consider an alphabet A, a morphism $f: A^+ \longrightarrow S$ and a sequence s of A^ω.

By a direct argument for morphisms from A^+ into a finite semigroup (or, by a result of Schützenberger, see [2]) we can prove that there exists a factorisation

$$s = t\, u_1 \ldots u_i \ldots u_j \ldots$$

where $t \in A^*$, $u_i \in A^+$ and for each i, $j \in \mathbb{P}$

$$f(u_i) = f(u_j).$$

Now , an elementary argument using the finiteness of S shows that there exists a positive integer p such that, for each integer $l \geqslant 1$,

$$e^p = e^p . e^p = e^{lp}$$

where e is the common image under f of the words u_i.

Consider the factorisation

$$s = t\, v_1 \ldots v_h \ldots v_k \ldots$$

where, for each integer h,

$$v_h = u_{(h-1)p+1} \ldots u_{hp}.$$

We have

$$\begin{aligned} f(v_h) &= f(u_{(h-1)p+1} \ldots u_{hp}) = \\ &= f(u_{(h-1)p+1}) \ldots f(u_{hp}) = \\ &= \underbrace{e \ldots\ldots e}_{p\text{-times}} = \\ &= e^p \end{aligned}$$

and

$$\begin{aligned} f(v_h \ldots v_k) &= f(v_h) \ldots f(v_k) = \\ &= \underbrace{e^p \ldots\ldots\ldots\ldots e^p}_{(k-h+1)\text{-times}} = \end{aligned}$$

$$= e^p.$$

So, S is ω-ramseyan. ■

Remark. An alternative proof of this proposition uses for the "if" part the (infinite version of) Ramsey theorem and for the "only if" part the lemma 2.1. of [1].

REFERENCES

1. J. Justin and G. Pirillo, On a natural extension of Jacob's ranks, Journal of Combinatorial Theory, Series A, Vol. 43, N. 2, 205-218 (1986).

2. M.P. Schützenberger, Quelques problèmes combinatoires de la théorie des automates, Cours professé à l'Institut de Programmation (Fac. Sciences de Paris) en 1966/67, rédigé par J.-F. Perrot.

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