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A COMBINATORIAL MODEL FOR HAHN POLYNOMIALS

BY

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Résumé. – Nous décrivons des modèles combinatoires pour les polynômes de Hahn, Krawtchouk, Jacobi, Meixner, Laguerre et Charlier, et démontrons de façon combinatoire les formules limite reliant ses familles de polynômes orthogonaux.

Abstract. – We describe combinatorial models for Hahn, Krawtchouk, Jacobi, Meixner, Laguerre and Charlier polynomials ,and prove combinatorially the limit formulas involving these families of orthogonal polynomials.

Introduction

Hahn polynomials were first studied in great details by S. Karlin and J.L. McGregor [12] and used in their analysis of birth and death processes. They also play an important role, along with other families of orthogonal polynomials, in representation theory ([1], [13], [20], [25]).

As pointed out by D. Foata [6], there is a combinatorial model for Hahn polynomials which contains the combinatorics of the other families appearing below them in R. Askey's chart ([2] , [16]) of hypergeometric orthogonal polynomials. All of these models are consistent with the arrows (limit formulas) between them. We would like to describe these facts. Chapter I will describe the models (first the configurations and then their weights) and chapter II the limit formulas. Roughly speaking the models are described by the following diagram where $A \rightarrowtail B$ means an injective map from A to A+B and $\circlearrowright$ a permutation :

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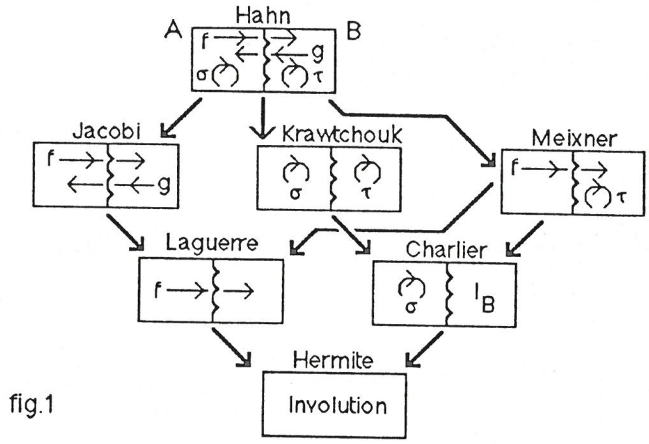

fig.1

Chapter I. The models

Given a 2-species **T** (also called bi-species or species on two sorts of points (see [11], [14], [15] or [26]); i.e. for every pair (A, B) of disjoint finite sets, **T**[A, B] is some set of combinatorial configurations) we define the associated 1-species T by :

$$T\,[S] = \{\, (A, B, t) \mid A \cup B = S,\ A \cap B = \phi \ \text{ and } \ t \in \mathbf{T}[A, B] \,\}.$$

We now introduce several 2-species:

(Charlier configurations) $\mathbf{C}[A, B] = \mathbf{S}[A] \times \{1_B\}$
where **S**[A] is the set of permutations of A;
(Laguerre configurations) $\mathbf{L}[A, B] = \{ f : A \to A+B \mid f \text{ is injective} \}$;
(Meixner configurations) $\mathbf{M}[A, B] = \mathbf{L}[A, B] \times \mathbf{S}[B]$;
(Krawtchouk configurations) $\mathbf{K}[A, B] = \mathbf{S}[A] \times \mathbf{S}[B]$;
(Jacobi configurations) $\mathbf{P}[A, B] = \mathbf{L}[A, B] \times \mathbf{L}[B, A]$;
(Hahn configurations) $\mathbf{Q}[A, B] = \mathbf{L}[A, B] \times \mathbf{S}[A] \times \mathbf{L}[B, A] \times \mathbf{S}[B]$
$= \mathbf{M}[A, B] \times \mathbf{M}[B, A] = \mathbf{P}[A, B] \times \mathbf{K}[A, B]$

Note that if (f, σ, g, τ) is a Hahn configuration on (A, B) then $(\sigma, 1_B)$ (resp. f, (f, τ), (σ, τ), (f, g)) is a Charlier (resp. Laguerre, Meixner, Krawtchouk, Jacobi) configuration on (A, B).

Let C, L, M, K, P and Q be the associated species. We will make these into weighted species with weights in the ring $\mathbb{Q}[\alpha, \beta, a^{-1}, p, c, x]$ (where α, β, a, p, c are the various parameters of the orthogonal families) by giving every configuration a weight (or valuation) in that ring. The n^{th}-polynomial (or some renormalization of it) will be the total weights of the corresponding configurations on any finite set S with $|S| = n$ (S will usually be $[n] = \{1, 2, \ldots, n\}$). For a weighted set X, $|X|$ denotes the total weight of its elements.

§1. CHARLIER polynomials , $C_n^{(a)}(x)$, are defined by either :

(1.1) $$\sum_{n\geq n} C_n^{(a)}(x)\frac{t^n}{n!} = e^t\left(1-\frac{t}{a}\right)^x$$

(1.2) $$C_n^{(a)}(x) = {}_2F_0\left[\frac{-n,-x}{\quad};-\frac{1}{a}\right] = \sum_{0\leq k\leq n}\binom{n}{k}\left(\frac{1}{a}\right)^k(-x)_k$$

where $(a)_m$, $m\geq 0$, denotes the rising factorial, i.e. $(a)_{m+1}=(a)_m\cdot(a+m)$, $(a)_0=1$, and ${}_rF_s$ will denote the generalized hypergeometric series

$$ {}_rF_s\left[\begin{matrix}a_1, a_2,\dots, a_r\\ b_1, b_2,\dots, b_s\end{matrix};x\right] = \sum_{m\geq 0}\frac{(a_1)_m(a_2)_m\cdots(a_r)_m}{(b_1)_m(b_2)_m\cdots(b_s)_m}\frac{x^m}{m!}.$$

Given $(\sigma, 1_B)\in \mathbf{C}[A,B]$, we set

(1.3) $w_1(\sigma, 1_B) = (-x)^{\text{cyc}(\sigma)}(a^{-1})^{|A|}$ where σ has cyc (σ) cycles .

<u>Proposition 1</u> We have $|C[n]| = C_n^{(a)}(x)$.

Proof. The formula follows easily from either (1.1) or (1.2).

§2. LAGUERRE polynomials , $L_n^{(\alpha)}(x)$, are defined by either:

(2.1) $$\sum_{n\geq 0} L_n^{(\alpha)}(x)t^n = (1-t)^{-1-\alpha}\exp(-xt(1-t)^{-1})$$

(2.2) $$n!\,L_n^{(\alpha)}(x) = (\alpha+1)_n\,{}_1F_1\left[\begin{matrix}-n\\ \alpha+1\end{matrix};x\right] = \sum_{i+j=n}\binom{n}{i}(\alpha+1+j)_i(-x)_j$$

Given $f\in \mathbf{L}[A,B]$, we set

(2.3) $w_2(f) = (1+\alpha)^{\text{cyc}(f)}(-x)^{|B|}$ where cyc (f) is the number of cycles (in A) of the injective map f and $|B|$ is also the number of "chains" of f.

<u>Lemma 1</u> We have $|\,\mathbf{L}[A,B]\,| = (1+\alpha+|B|)_{|A|}(-x)^{|A|}$.

Proof. This is a now classical combinatorial lemma (see [9] lemma (2.1); [6] lemma (3.1); [8] lemma 3) which we will use again and again. See also [17] for a short proof using 2-species.

<u>Proposition 2</u> We have $|L[n]| = n!\,L_n^{(\alpha)}(x)$.

§3. JACOBI polynomials , $P_n^{(\alpha,\beta)}(x)$ are defined by either :

(3.1) $$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n,\ n+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{1-x}{2}\right]$$

(3.2) $$n!\,P_n^{(\alpha,\beta)}(x) = \sum_{i+j=n}\binom{n}{i}(\alpha+1+j)_i\,(\beta+1+i)_j\left(\frac{x+1}{2}\right)^i\left(\frac{x-1}{2}\right)^j$$

Given $(f, g) \in \mathbf{P}[A, B]$, we set

(3.3) $$w_3(f,g) = (\alpha+1)^{\mathrm{cyc}\, f} (\beta+1)^{\mathrm{cyc}\, g} \left(\frac{x+1}{2}\right)^{|A|} \left(\frac{x-1}{2}\right)^{|B|}.$$

Proposition 3 We have $|P[n]| = n! \cdot P_n^{(\alpha,\beta)}(x)$.
Proof. See [8] or [19].

§4. MEIXNER polynomials, $m_n(x; \beta, c)$, are defined by either :

(4.1) $$\sum_{n \geq 0} m_n(x;\beta,c) \frac{t^n}{n!} = \left(1 - \frac{t}{c}\right)^x (1-t)^{-x-\beta}$$

(4.2) $$m_n(x;\beta,c) = (\beta)_n \ {}_2F_1\left[\begin{matrix} -n, -x \\ \beta \end{matrix}; 1 - c^{-1}\right]$$
$$= \sum_{i+j=n} \binom{n}{i} (\beta+j)_i (-x)_j (c^{-1}-1)^j$$

Given $(f, \tau) \in \mathbf{M}[A, B]$, we set

(4.3) $$w_4(f,\tau) = \beta^{\mathrm{cyc}(f)} (-x)^{\mathrm{cyc}(\tau)} (c^{-1}-1)^{|B|}$$

Proposition 4 We have $|M[n]| = m_n(x; \beta, c)$.
Proof. See [7].

Remark. Note that we can obtain another combinatorial model, $\overline{M}$, for Meixner polynomials by taking : $\overline{\mathbf{M}}[A, B] = \mathbf{S}[A] \times \mathbf{S}[B]$ and $w_4(\sigma,\tau) = (-x)^{\mathrm{cyc}(\sigma)} (x+\beta)^{\mathrm{cyc}(\tau)} (c^{-1})^{|A|}$; this follows from (4.1) which gives :

$$m_n(x;\beta,c) = \sum_{i+j=n} \binom{n}{i} (-x)_i (x+\beta)_j (c^{-1})^i .$$

§5. KRAWTCHOUK polynomials, $K_n(x; p, N)$, $0 \leq n \leq N$, are defined by :

(5.1) $$K_n(x;p,N) = {}_2F_1\left[\begin{matrix} -n, -x \\ -N \end{matrix}; \frac{1}{p}\right] \quad \text{where} \quad 0<p<1$$

(5.2) $$(-1)^n (-N)_n K_n(x;p,N) = \sum_{i+j=n} \binom{n}{i} (-x)_i (N-n+1)_j \left(\frac{1}{p}\right)^i$$

But in [23] one finds $(1+qt)^x (1-pt)^{N-x}$, where $p+q=1$, as a generating function for Krawtchouk polynomials. More precisely we have:

(5.3) $$\sum_{n \geq 0} (-N)_n p^n K_n(x;p,N)\, t^n/n! = (1+qt)^x (1-pt)^{N-x}$$

This shows that for $(\sigma, \tau) \in \mathbf{K}[A, B]$ by setting :

(5.4) $$w_5(\sigma,\tau) = (-x)^{\mathrm{cyc}(\sigma)} (x-N)^{\mathrm{cyc}(\tau)} (-q/p)^{|A|}$$

we obtain a model , K , for Krawtchouk polynomials. More precisely:

Proposition 5 We have $| K\ [n] | = (-N)_n K_n(x; p, N)$.

Given $(\sigma, \tau) \in \mathbf{K}[A, B]$, if we rather set :

(5.5) $\overline{w}_5\ (\sigma, \tau) = (N - | A | - | B | + 1)^{\mathrm{cyc}\,(\sigma)}\ (- x)^{\mathrm{cyc}\,(\tau)}\ (1/p)^{| B |}$

Then using (5.2), we get a second model, $\overline{K}$, for Krawtchouk polynomials. More precisely:

Proposition 6 We have $| \overline{K}\ [n] | = (- 1)^n (- N)_n\ K_n\ (x; p, N)$.

Remark. By writing

$$(1 + qt)^x (1 - pt)^{N - x} = \left(\frac{1}{1 - t(1 + qt)^{-1}}\right) \left(\frac{1}{1 - pt}\right)^{-N}$$

F.Bergeron (in [3]) defined the first combinatorial model for Krawtchouk polynomials as "assemblies of blue and red octopuses" .

Remark There is another model (a 2-species), for $N!\, K_n(x; p, N)$, where this time n and N are not mere parameters but rather "numbers of points" in the configuration. Let $\overline{\overline{\mathbf{K}}}$ be the 2-species defined by :

$$\overline{\overline{\mathbf{K}}}[A, B] = \{(S, \sigma) \mid S \subseteq A,\ \sigma \in S[A + B] \text{ and } \sigma(S) = S\}$$

$$\overline{\overline{w}}_5(S, \sigma) = (-x)^{\mathrm{cyc}\,\sigma_1}\left(\frac{1}{p}\right)^{| S |} \quad \text{where } \sigma_1 = \sigma|_S,\ \sigma_2 = \sigma|_{A + B - S}$$

Proposition 7 We have $|\overline{\overline{\mathbf{K}}}[A, B]| = N!\, K_n(x; p, N)$ if $| A | = n$, $| B | = N\text{-}n$.

Proof. This follows immediately from the formula (see fig.2):

$$N!\ K_n(x; p, N) = \sum_{k=0}^{n} \binom{n}{k} (N - k)!\, (- x)_k \left(\frac{1}{p}\right)^k .$$

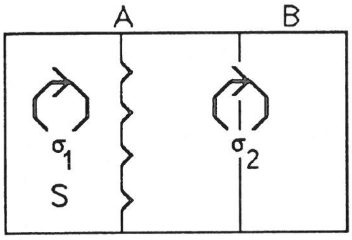

fig.2

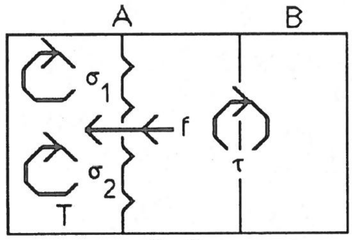

fig.3

§6. HAHN polynomials, $Q_n(x;\alpha,\beta,N)$, $0 \le n \le N$, are defined by :

$$(6.1)\quad Q_n(x;\alpha,\beta,N) = {}_3F_2\left[\begin{matrix} -n,\ n+\alpha+\beta+1,\ -x \\ \alpha+1,\ -N \end{matrix}\ ;1\right].$$

We also have the following expression ([10 page 390 and [6] page 1550)

$$(6.2)\qquad (\alpha+1)_n(-N)\,Q_n(x;\alpha,\beta,N)$$

$$= \sum_{i+j=n} \binom{n}{i} (\alpha+1+j)_i(\beta+1+i)_j(x-N)_i(-x)_j(-1)^j$$

Given $(f,\sigma,g,\tau) \in \mathbf{Q}[A,B]$, we set

$$(6.3)\quad w_6(f,\sigma,g,\tau)=(\alpha+1)^{\mathrm{cyc}\,(f)}(\beta+1)^{\mathrm{cyc}\,(g)}(x-N)^{\mathrm{cyc}\,(\sigma)}(-x)^{\mathrm{cyc}\,(\tau)}(-1)^{|B|}$$

Proposition 8 We have $|\,Q[n]\,| = (\alpha+1)_n(-N)_n\,Q_n(x;\alpha,\beta,N)$

Remark. There is another combinatorial model, for $(\alpha+1)_n\,N!\,Q_n(x;\alpha,\beta,N)$ this time, based on the formula :

$$(\alpha+1)_n\,N!\,Q_n(x;\alpha,\beta,N) = \sum_{i+j=n} \binom{n}{i}(N-i)!\,(\alpha+1+i)_j(\alpha+\beta+n+1)_i(-x)_i.$$

Let $\overline{\mathbf{Q}}$ be the two species defined by (see fig.3):

$\overline{\mathbf{Q}}[A,B] = \{\,(T,f,\sigma_1,\sigma_2,\tau) \mid T \subseteq A,\ f \in \mathbf{L}[A-T,T],\ \sigma_1,\sigma_2 \in \mathbf{S}[T],\ \tau \in \mathbf{S}[A+B-T]\,\}$

$$\overline{w}_6(T,f,\sigma_1,\sigma_2,\tau) = (\alpha+1)^{\mathrm{cyc}\,f}(-x)^{\mathrm{cyc}\,\sigma_1}(\alpha+\beta+|A|+1)^{\mathrm{cyc}\,\sigma_2}.$$

Proposition 9 We have $|\,\overline{\mathbf{Q}}[A,B]\,| = (\alpha+1)_n\,N!\,Q_n(x;\alpha,\beta,N)$ where $|A| = n$ and $|B| = N-n$.

Chapter II. The limits.

We would like to prove combinatorially the following limit formulas:

Theorem I We have

(Q.-P.) $(\alpha+1)_n \lim\limits_{N\to\infty} Q_n(Nx;\alpha,\beta,N) = n!\,P_n^{(\alpha,\beta)}(1-2x)$

(Q.-M.) $(\beta)_n \lim\limits_{N\to\infty} Q_n(x;\beta-1,\gamma N,N) = m_n(x;\beta,c)$ where $\gamma = c^{-1}-1$

(Q.-K.) $\lim\limits_{t\to\infty} Q_n(x;pt,qt,N) = K_n(x;p,N)$ where $p+q=1$.

(P.-L.) $\lim\limits_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right) = L_n^{(\alpha)}(x)$

(M.-C.) $\lim\limits_{\beta\to\infty} \frac{1}{(\beta)_n} m_n\left(x;\beta,\frac{a}{\beta}\right) = C_n^{(a)}(x)$

(M.-L.) $\lim\limits_{c\to 1} m_n\left(\frac{cx}{1-c};\beta,c\right) = n!\,L_n^{(\beta-1)}(x)$

(K.-C.) $\lim\limits_{N\to\infty} K_n\left(x;\frac{a}{N},N\right) = C_n^{(a)}(x)$.

Remark. These are all proved using the following technique.We consider a model for the left hand side (before taking the limit); when we do take the limit, most configurations are killed (i.e. their weight tends to zero), the only surviving configurations (with their limiting weights) form precisely our model for the right hand side. This method, due to D. Foata, was first used in [7] to prove (M.-L.) combinatorially. In other words these limits correspond to the seven arrows (above Hermite) in Figure 1. These arrows are obvious forgetful epimorphisms for which in the inverse image above any given configuration everything is killed except precisely one (degenerate) configuration with the right limiting weight.

Proof of (Q.-P.). We will prove

$$\lim_{N\to\infty} \frac{1}{(-N)^n} (-N)_n (\alpha+1)_n Q_n(Nx;\alpha,\beta,N) = n!\, P_n^{(\alpha,\beta)}(1-2x)$$

which is equivalent to (Q.-P.) since $\lim_{N\to\infty} (-N)_n (-N)^{-n} = 1$. By Prop. 8, Hahn configurations form a model for $(\alpha+1)_n (-N)_n Q_n(Nx;\alpha,\beta,N)$; we put an additional multiplicative weight of $-1/N$ on each of the n points of the configuration, i.e. the weight of the Hahn configuration $(f,\sigma,g,\tau) \in Q[A,B]$ is now :

$$(\alpha+1)^{\mathrm{cyc}\, f} (\beta+1)^{\mathrm{cyc}\, g} (Nx-N)^{\mathrm{cyc}\,\sigma} (-Nx)^{\mathrm{cyc}\,\tau} (-1)^{|B|} \left(\frac{-1}{N}\right)^{|A|} \left(\frac{-1}{N}\right)^{|B|}$$

When $N\to\infty$, this configuration dies unless $\mathrm{cyc}(\sigma) = |A|$ and $\mathrm{cyc}(\tau) = |B|$ (i.e. $\sigma = 1_A$, $\tau = 1_B$) in which case $(f,\sigma,g,\tau) = (f, 1_A, g, 1_B)$ is really a Jacobi configuration. Moreover the value of the limit is $(\alpha+1)^{\mathrm{cyc}(f)}(\beta+1)^{\mathrm{cyc}(g)} (1-x)^{|A|} (-x)^{|B|}$ which is precisely $w_3(f,g)$ (with x replaced by $(1-2x)$).

Proof of Q.-M. Again we use $\lim_{N\to\infty} (-N)_n (-N)^{-n} = 1$. By Prop. 8, Hahn configurations form a model for $(\beta)_n (-N)_n Q_n(x;\beta-1,\gamma N,N)$ if we set $w_6(f,\sigma,g,\tau) = \beta^{\mathrm{cyc}(f)}(\gamma N+1)^{\mathrm{cyc}(g)} (x-N)^{\mathrm{cyc}(\sigma)} (-x)^{\mathrm{cyc}(\tau)} (-1)^{|B|}$. Again if we put an additional multiplicative weight of $-1/N$ on each of the n points and let $N\to\infty$, the only surviving Hahn configurations are those with $g = 1_B$ and $\sigma = 1_A$, i.e. Meixner configurations of weight $w_4(f,\tau) = \beta^{\mathrm{cyc}(f)} (-x)^{\mathrm{cyc}(\tau)} \gamma^{|B|}$. These add up to $m_n(x;\beta,c)$ by Prop. 4.

Proof of Q.-K. By Prop. 8, setting $\alpha = pt$ and $\beta = (1-p)t = qt$, we have a combinatorial model for : $(pt+1)_n (-N)_n Q_n(x; pt, qt, N)$. Put an additional multiplicative weight of $(pt)^{-1}$ on each of the n points. The weight of the Hahn configuration $(f,\sigma,g,\tau) \in Q[A,B]$ is now :

$$(pt+1)^{\mathrm{cyc}\, f} (qt+1)^{\mathrm{cyc}\, g} (x-N)^{\mathrm{cyc}\,\sigma} (-x)^{\mathrm{cyc}\,\tau} \left(\frac{1}{pt}\right)^{|A|} \left(\frac{-1}{pt}\right)^{|B|}.$$

When $t\to\infty$, the only surviving configurations are those with $\mathrm{cyc}(f) = |A|$ and $\mathrm{cyc}(g) = |B|$, i.e. of the form $(1_A, \sigma, 1_B, \tau)$ with limiting weight

$$(-x)^{\mathrm{cyc}\,\sigma} (-N+x)^{\mathrm{cyc}\,\tau} \left(\frac{-q}{p}\right)^{|A|}$$

These are Krawtchouk configurations . Since $\lim_{t\to\infty} (pt+1)_n (pt)^{-n} = 1$, we have $\lim_{t\to\infty} (-N)_n Q_n(x; pt, qt, N) = (-N)_n K_n(x; p, N)$ by prop.5.

Proof of (P.-L.) By Prop. 3, Jacobi configurations with weight

$$w_3(A, B, f, g) = (\alpha+1)^{\mathrm{cyc}\, f} (\beta+1)^{\mathrm{cyc}\, g} \left(1-\frac{x}{\beta}\right)^{|A|} \left(-\frac{x}{\beta}\right)^{|B|}$$

form a model for $P_n^{(\alpha+\beta)}\left(1-\frac{2x}{\beta}\right)$.

We have $\lim_{\beta\to\infty} w_3(A, B, f, g) = 0$ unless $\mathrm{cyc}\, g = |B|$, i.e. $g = 1_B$, in which case $\lim_{\beta\to\infty} w_3 (A, B, f, 1_B) = (\alpha+1)^{\mathrm{cyc}\,(f)} (-x)^{|B|}$ which is the weight $w_2(f)$ of this Laguerre configuration.

Proof of (M.-C.) We will prove $\lim_{\beta\to\infty} \beta^{-n} M_n(x; \beta, a/\beta) = C_n^{(a)}(x)$ which, since $\lim_{\beta\to\infty} (\beta)_n \beta^{-n} = 1$, is equivalent to (M.-C.). If we put an additional multiplicative weight of $1/\beta$ on each points of the Meixner configurations on $[n]$; the weight of (A, B, f, τ) becomes $\beta^{-n} w_4(A, B, f, \tau) = \beta^{-n} \beta^{\mathrm{cyc}\,(f)}(-x)^{\mathrm{cyc}\,(\tau)} (\beta/a-1)^{|B|}$. When $\beta\to\infty$ this tends to zero unless $\mathrm{cyc}\,(f) + |B| = n$ (i.e. $f = 1_A$ and $\tau \in S[B]$) and $(A, B, 1_A, \tau)$ is reallly a Charlier configuration (B, τ) of weight

$$(-x)^{\mathrm{cyc}\,(\tau)}(1/a)^{|B|} = \lim_{\beta\to\infty} \beta^{-n} w_4(A, B, 1_A, \tau) \qquad \text{Q.E.D}$$

Using the same technique, (M.-L.) is proved in [7] and (K.-C.) is proved in [3] (with a different model (see remark below Prop. 6) for Krawtchouk polynomials). We can also prove combinatorially :

$$\lim_{\alpha\to\infty} \alpha^{-n} P_n^{(\alpha,\beta)}(x) = \frac{1}{n!}\left(\frac{x+1}{2}\right)^n$$

Conclusion

At the ${}_2F_1$ level in Askey's chart there is also the Meixner-Pollaczek polynomials, $p_n^{\lambda}(x;\varphi)$, for which several combinatorial models are described in [18] and used to prove limit formulas relating them to Laguerre and Hermite polynomials.

$$\lim_{\varphi\to 0} p_n^{\frac{1+\alpha}{2}}\left[\frac{(1+\alpha)(1-\cos\varphi)-x}{2\sin\varphi} ; \varphi\right] = L_n^{(\alpha)}(x)$$

$$n!\lim_{\lambda\to\infty} \lambda^{-n/2} p_n^{\lambda}\left[\frac{x\lambda^{1/2}-\lambda\cos\varphi}{\sin\varphi} ; \varphi\right] = H_n(x).$$

These limit formulas are proved using a different technique which also works (in [17]) to give a combinatorial proof of the so called "italian limit formula" :

$$2^n n! \lim_{\beta\to\infty} \beta^{-n} L_n^{(\beta^2/2)}\left(\frac{\beta^2}{2}-\beta x\right) = H_n(x).$$

References

1. R. ASKEY, T.H. KOORNWINDER and W. SCHEMPP (editors), Special Functions : Group Theoretical Aspects and Applications (Reidel 1984).
2. R. ASKEY and J.A. WILSON, "Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials", Memoirs Amer. Math. Soc., March 1985, vol. 54, no. 319.
3. F. BERGERON, "Modèles combinatoires de familles de polynômes orthogonaux", Europ. J. Combin. (1987)
4. T.S. CHIHARA, "On Introduction to Orthogonal Polynomials" (Gordon and Breach, New York, 1978)
5. A. ERDELYI, "Higher Transcendental Functions" (McGraw Hill, New York1953), vol. 2
6. D. FOATA, "Combinatoire des identités sur les polynômes orthogonaux", Internat. Congress Math. (Warsaw, Poland, 1983).
7. D. FOATA and J. LABELLE, "Modèles combinatoires pour les polynômes de Meixner", Europ. J. Combin. 4 (1983) 305-311.
8. D. FOATA and P. LEROUX, "Polynômes de Jacobi, interprétation combinatoire et fonction génératrice", Proc. Amer. Math. Soc. 87 (1983) 47-53.
9. D. FOATA and V. STREHL, "Combinatorics of the Laguerre Polynomials", Proc. Waterloo Silver Jubilee, 1982, in : D.M. Jackson and S.A. Vanstone eds., Enumeration and Design (Academic Press, Toronto, 1984) 123-140.
10. G. GASPER, "Positivity and special function", in : R. Askey eds. Theory and applications of special functions (Acad. Press, 1975).
11. A. JOYAL, "Une théorie combinatoire des séries formelles", Adv. in Math. 42 (1981) 1-82.
12. S. KARLIN and J.L. McGREGOR, "The Hahn polynomials; formulas and application", Spripta Math., 26 (1961), 33-46.
13. T.H. KOORNWINDER, "Krawtchouk polynomials, a unification of the different group theoretic interpretations", SIAM. J. Math. Anal., vol. 13, No. 6, November 1982.
14. G. LABELLE, "On combinatorial differential equations", J. Math. Anal. Appl. 113 (1986), no. 2, 344-381.
15. J. LABELLE, "Applications diverses de la théorie des espèces de structures", Ann. Sci. Math. Québec 7 (1983) 59-94.
16. J. LABELLE, "Tableau d'Askey", in : Polynômes Orthogonaux et Applications, Proc. Bar-le-Duc 1984, C. Brezinski, A. Draux, A.P. Magnus, P. Maroni and A. Ronveaux, eds. (Springer-Verlag, Lecture Notes no. 1171) XXXVI-XXXVII.
17. J. LABELLE and Y.N. YEH, "The combinatorics of Laguerre, Charlier and Hermite polynomials revisited", (to appear).
18. J. LABELLE and Y.N. YEH, "Combinatorial proofs of some limit formulas involving orthogonal polynomials", (to appear).
19. P. LEROUX and V. STREHL, "Jacobi polynomials : combinatorics of the basic identities", Discrete Mathematics 57 (1985) 167-187.
20. W. MILLER, "Symmetry and separation of variables", Encyclopedia of Mathematics, Ed. G.-C. Rota (Addison-Wesley, 1977).
21. E.D. RAINVILLE, "Special Functions (Macmillan, New York, 1960).
22. J. RIORDAN, "An introduction to combinatorics" (Wiley, 1958).
23. V. STREHL, "Combinatorics of Jacobi-configurations I : complete oriented matchings", in : G. Labelle and P. Leroux eds., Proc. Coll. Combin. Enumérative, Montréal 1985 (Springer-Verlag, Lecture Notes no 1234).
24. G. SZEGO, "Orthogonal Polynomials", AMS Colloquium Publications, vol. 23, Providence, 1975 (4th ed.).
25. N. Ya VILENKIN, "Special Functions and the Theory of Group Representations", Amer. Math. Soc., Providence, R.I., 1968.
26. Y.N. YEH, "The calculus of virtual species and K-species", in : G. Labelle and P. Leroux eds., Proc. Coll. Combin. Enumérative, Montréal 1985 (Springer-Verlag, Lecture Notes 1234).

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