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OBTAINING GENERATING FUNCTIONS FROM ORDERED-PARTITION RECURRENCE FORMULAS

BY

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RÉSUMÉ. — En résolvant deux problèmes d'énumération en théorie chromatique des graphes, on a découvert que les formules de récurrence obtenues devaient se sommer sur des ensembles de partitions ordonnées. Utilisant une somme infinie, ces formules fournissent des fonctions génératrices, puis des expressions closes. Cette technique trouve son illustration dans le comptage du nombre de façons de placer des diagonales disjointes dans un polygone et aussi dans le problème de comptage des coloriages disjoints d'un cycle. Les suites de nombres obtenues ne sont pas sans rappeler d'autres suites déjà étudiées par Carlitz et Motzkin.

ABSTRACT. — In solving two enumeration problems in chromatic graph theory it was discovered that the natural recurrence formulas which developed included summing over ordered-partitions. Using an infinite sum these formulas can be turned into generating functions that lead to closed form expressions. This technique is illustrated on the problem of counting how many ways a set of some non-intersecting diagonals can be placed in an n-gon and on the problem of counting non-crossing colorings of a cycle. These sequences are reminiscent of some work of Carlitz and Motzkin.

1. Non-Intersecting Diagonals.

— In the investigation of certain structures in one proof of the Four Color Theorem [3] the following question arises : In how many ways can arbitrarily many non-intersecting diagonals be placed in a labeled n-gon? We wish to allow also sets of diagonals which do not triangulate a labeled n-gon in contradistinction to the famous problem of Euler solved by the Catalan numbers.

Let A_n be the number of such diagonalizations. A_4 and A_5 are shown in diagram 1. Note that we include the diagonalization using no diagonals. We also start with $A_3 = 1$.

We will find it useful to define another sequence. Let B_{n-2} be the number of ways of diagonalizing the labeled n-gon with non-intersecting diagonals in which no diagonal goes through vertex 1. In a large n-gon we may imagine a diagonal drawn from vertex 2 to vertex n forming an $(n-1)$-gon. This $(n-1)$-gon can be properly diagonalized in A_{n-1} ways. When extended back to the n-gon we can include the diagonal $\overline{2n}$ or not. So

$$B_{n-2} = 2A_{n-1} \quad \text{for } n \geq 4.$$

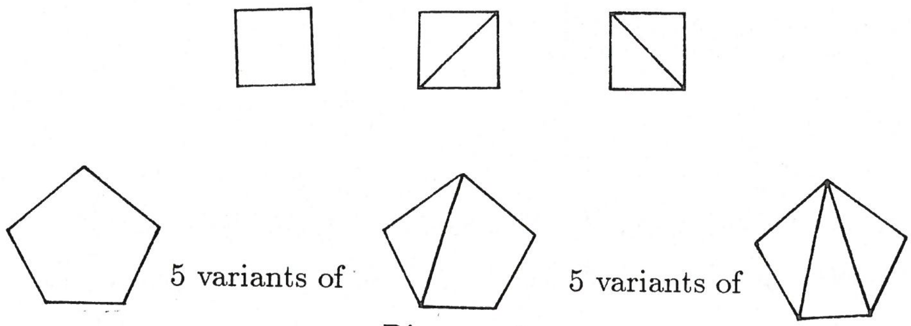


Diagram 1

So

$$B_{n-2} = 2A_{n-1} \quad \text{for } n \geq 4.$$

Obviously $B_1 = 1$ should be treated separately, but $B_2 = 2$, $B_3 = 6$, $B_4 = 22$.

2. The Recurrence Formula. — Let us consider the set of vertices connected to 1 by the diagonals in some diagonalization. Let these vertices be $2 < a_1 < \cdots < a_m < n$. Exactly $(m+1)$ regions are formed by these diagonals. The first region in the a_1-gon with vertices $1, 2, \ldots, a_1$. The second is the $(a_1 - a_2 + 2)$-gon with vertices $1, a_1, a_1 + 1, \ldots, a_2$, and so forth, until the last region, which is an $(n - a_m + 2)$-gon. The number of ways these regions can be filled in with non-intersection diagonals (none of which goes through vertex 1) is $B_{a_1-2}B_{a_2-a_1}B_{a_3-a_2}\cdots B_{n-a_m}$. The diagonalizations of A_n we are trying to enumerate fit into disjoint classes determined by which diagonals go through vertex 1.

In fact we may write

$$A_n = \sum B_{a_1-2}B_{a_2-a_1}\cdots B_{n-a_m}$$

summed over all sequences $2 < a_1 < a_2 < \cdots < a_m < n$. The subscripts form every possible set of partitions of $n-2$. Using this and the fact that $A_n = (1/2)B_{n-1}$ we may write

$$\frac{1}{2}B_{n-1} = \sum_{a_1+\cdots+a_m=n-2} B_{a_1}B_{a_2}\cdots B_{a_m}.$$

The summation extends over every ordering of each partition. For example

$$\frac{1}{2}B_4 = B_3 + B_1B_2 + B_2B_1 + B_1B_1B_1.$$

The second term on the right counts the further diagonalisations of (a) (referring to diagram 2) while the third term counts the extensions of (b).

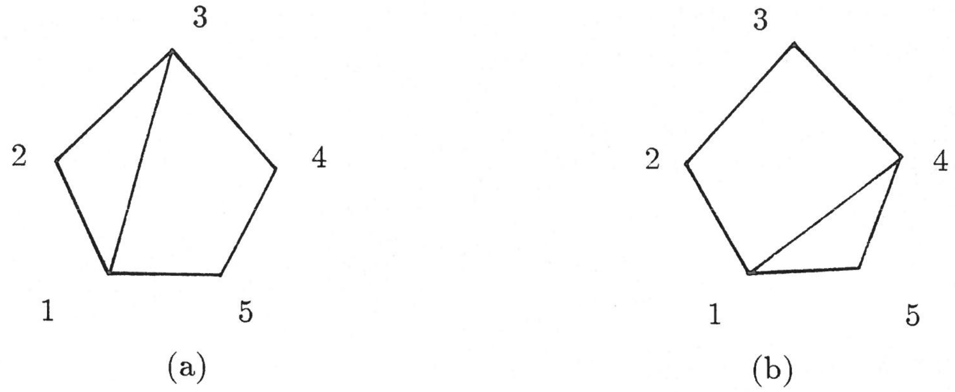


Diagram 2

formula applies is $n = 3$. We must still initialize the B's with $B_1 = 1$, $B_2 = 2$, $B_3 = 6$.

3. The Generating Function. — Define $f(x) = B_1x + B_2x^2 + B_3x^3 + \cdots$ Clearly, the coefficient of x^p in $\sum_{i=1}^{\infty} f(x)^i$ is the sum of all $B_{a_1}B_{a_2}\cdots B_{a_m}$ where the subscripts add up to p, i.e. $(1/2)B_{p+1}$. Hence

$$\begin{aligned}\sum_{i=1}^{i=\infty} f(x)^i &= B_1x + \frac{1}{2}B_3x^2 + \frac{1}{2}B_4x^3 + \cdots \\ &= \frac{1}{2}B_2x + \frac{1}{2}B_3x^2 + \frac{1}{2}B_4x^3 + \cdots \\ &= (f-x)/(2x).\end{aligned}$$

Therefore,

$$(1-f)(f-x) = 2xf,$$

which gives

$$f = -\frac{1}{2} + \frac{x}{2} \pm \frac{1}{2}\sqrt{x^2 - 6x + 1}.$$

We shall see in the next section that we must use the "$-$" sign in front of the square root. The functional equation also provides us with the additional recurrence formula

$$B_n = B_{n-1} + (B_1B_{n-1} + B_2B_{n-2} + \cdots + B_{n-1}B_1),$$

which is reminiscent of the formula for Catalan Numbers.

4. The Non-Recursive Expression. — First write

$$\left(1+(x^2-6x)\right)^{1/2} = 1 + \frac{x^2-6x}{2} + \sum_{k\geq 2}(-1)^{k+1}\frac{1}{k}\binom{2k-2}{k-1}\frac{1}{2^{2k-1}}(x^2-6x)^k.$$

Noting that the coefficient of x^{n-k} in $(x-6)^k$ is $\binom{k}{n-k}(-6)^{2k-n}$ we see that for $n > 3$ the coefficient of x^n in $f(x)$ is then

$$\sum_{n\geq k\geq 2} (-1)^{n+k}\frac{1}{2^n}\frac{1}{k}\binom{2k-2}{k-1}\binom{k}{n-k}3^{2k-n}$$
$$= \sum_{n\geq k\geq n/2} (-1)^{n+k}\frac{3^{2k}}{6^n}\frac{(2k-2)!}{(k-1)!\,(2k-n)!\,(n-k)!}.$$

From this we can calculate

$$B_5 = 90, \quad B_6 = 394, \quad B_7 = 1086, \quad B_8 = 8558,$$
$$A_6 = 45, \quad A_7 = 197, \quad A_8 = 903, \quad A_9 = 4279.$$

The sequence B_n is listed in [5] as counting "lattice paths with diagonal steps of, of type 1."

5. The Relevance to Graph Coloring. — Let us consider a graph whose outer perimeter is an n-gon. If we consider any edge of the graph whose vereices do not lie on the outer cycle we can form two new graphs, one by deleting this edge and another by contracting it. A celebrated method of Birkhoff [1] shows how this deletion and contraction can be used to relate the number of ways of λ-coloring one graph to the ways of λ-coloring two simple graphs. In the approach to the Four Color Theorem in [3] we decompose all graphs in this way while preserving the outer cycle. Thus, every graph can be represented as a linear combination of the forms enumerated above.

6. Non-Crossing Colorings of a Cycle. — In our work on the Four Color Problem [3], the following problem also arises : in how many ways can the n vertices of a labeled cycle be partitioned into arbitrarily many sets such that :

(i) no adjacent pair of vertices is in the same set, and

(ii) the sets do not cross; by which we mean that if a and b are in one set and c and d are in another, then the diagonals ab and cd do not intersect.

We will call a partition which satisfies (i) and (ii) a *non-crossing coloring* because of the analogy to restrictions on adjacent vertices. This time let A_n be the number of non-crossing colorings for the n-cycle $v_1v_2\ldots v_n$ (diagrams 3, 4, and 5).

Notice that the adjective "labeled" when applied to the cycle means that we shall consider the last two examples in diagram 4 (where $A_4 = 3$) to be distinct.

$A_3 = 1$

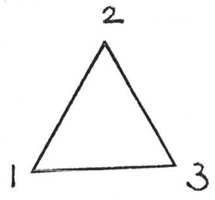


diagram 3

$A_4 = 3$

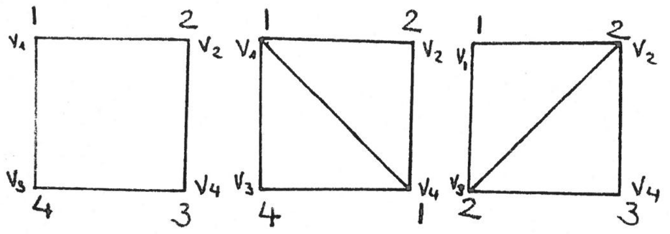
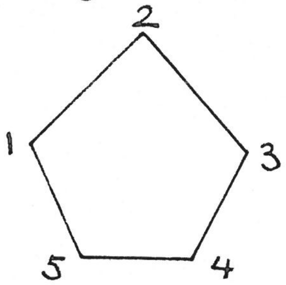


diagram 4



$A_5 = 6$

and five variants of

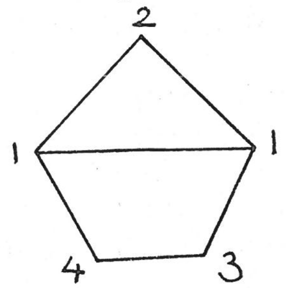


diagram 5

The sequence continues : $A_6 = 15$, $A_7 = 36$, $A_8 = 91$, $A_9 = 232$, $A_{10} = 603, \ldots$ We note that the sequence A_n is not contained in [5]. We can convert these diagrams into sequences of numbers as follows :

(1) Label every vertex which is in a part by itself with the number 0.

(2) Starting at v_1 and proceeding in order, label all the vertices in each new partition part with the next lowest unused integer. That is, the first unlabeled vertex (maybe v_1) will be labeled 1, and so will all of the other vertices in its partition set. The next unlabeled veertex (maybe v_1) must be labeled 2, and so will of the others in its set, and so on, until all vertices are numbered. We then convert the non-crossing colorings into sequences by writing down the labels for v_1 through v_n, in order.

These sequences have the following properties. They are sequences of n terms of nombers 0 through m (for some $m < n$) in which :

(i) every non-zero number appears more than once;

(ii) the first occurrence of i is before the first occurrence of j if $i < j$;

(iii) if i and j are non-zero numbers, $i < j$, then no two j's have an i between them;

(iv) the sequences cannot begin and end with a 1.

Every n-sequence obeying these four conditions corresponds to a non-crossing coloring. For example, the following sequences are acceptable :

0010202010340431;   1012023450054030.

The sequence 01012021 corresponds to diagram 6.

In order to calculate the A's we will make use of another sequence of integers $B_1, B_2, B_3, \ldots$ where B_n is the number of sequences satisfying

$\{v_1\}$ $\{v_2, v_4, v_8\}$ $\{v_3\}$

$\{v_5, v_7\}$ $\{v_6\}$

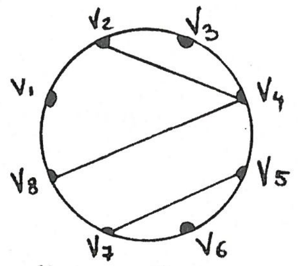


diagram 6

conditions (i), (ii), and (iii) above, but not necessarily (iv). That means the B's count sequences which can begin and end with 1's.

$B_1 = 1$ $\{0\}$
$B_2 = 1$ $\{00\}$
$B_3 = 2$ $\{000, 101\}$
$B_4 = 4$ $\{0000, 0101, 1001, 1010\}$
$B_5 = 9$ $\{00000, 00101, 01001, 01010, 10001, 10010, 10100, 10101, 12021\}$
$B_6 = 21$.

7. The Recurrence Formula for B. — If the sequence of type-B starts with a 0 it can continue in B_{n-1} ways. On the other hand if 1 is the first term there must be some other 1's as well. Let there be p more of them. If the B-sequence does not end with a 1 there are p gaps between the 1's. Let them be of lengths $m_1, m_2, m_3, \dots, m_p$. Then

$$\text{gaps} + 1'\text{s} = n\,; \qquad \sum m_i = n - p.$$

The gap of size m_i can be filled in B_{m_1} ways. This is the advantage of using B's instead of A's. The number of ways a B-line can start with a 1 but not end with a 1 is :

$$\sum_{p\geq 2} \sum_{\substack{m_1+m_2+\cdots+m_p=n-p \\ m_i\geq 1}} B_{m_1} B_{m_2} \dots B_{m_p}.$$

Contrast this with the formula in Section 2.

If the B-sequence starts and ends with a 1, then the sizes of the gaps add to $n-p-1$. If $p = 1$, then there are only the two 1's on the ends and the inside can be filled in B_{n-2} ways. Altogether there are

$$B_{n-2} + \sum_{p>2} \sum_{\substack{m_1+m_2+\cdots+m_p=n-p-1 \\ m_i\geq 1}} B_{m_1} B_{m_2} \dots B_{m_p}$$

sequences with 1's on the ends. The recurrence equation is :

$$B_n = B_{n-1} + \sum_{\Sigma m_i = n-p} \prod B_{m_i} + B_{n-2} + \sum_{\Sigma m_i = n-p-1} \prod B_{m_i}.$$

For example

$$B_6 = B_5 + (B_1B_3 + B_2B_2 + B_3B_1 + B_1B_1B_1) + B_4 + (B_1B_2 + B_2B_1).$$

8. The Generating Function for B. — Let us define the generating function $y = \sum_{j\geq 1} B_j x^j$. The coefficient of x^n in the expression $x^p y^p$ is $\sum_{\Sigma m_i = n-p} \prod B_{m_i}$ and similarly the coefficients of x^n in $x^{p+1}y^p$ is $\sum_{\Sigma m_i = n-p-1} \prod B_{m_i}$. Therefore the recurrence formula converts to the following functional equation :

$$y - x = xy + \sum_{p\geq 2} x^p y^p + x^2 y + \sum_{p\geq 2} x^{p+1} y^p.$$

The left-hand adjustment $(-x)$ is due to the lack of terms of degree 1 on the right-hand side.

We can now calculate :

$$y - x = xy + \frac{x^2y^2}{1-xy} + x^2y + \frac{x^3y^2}{1-xy} = (1+x)\left(xy + \frac{x^2y^2}{1-xy}\right),$$

so that

(1) $$\frac{1+x}{y-x} = \frac{1}{xy} - 1$$

or

$$y - x = xy + xy^2.$$

Thus

$$y = \frac{1 - x \pm \sqrt{(1-x)^2 - 4x^2}}{2x},$$

which implies

$$\begin{aligned} 2xy + x - 1 &= -\sqrt{1-(2+3x)x} \\ &= -1 + \sum_{j\geq 1} \frac{1}{2^{2j-1}} \frac{1}{j} \binom{2j-2}{j-1} (2+3x)^j x^j. \end{aligned}$$

The coefficients of x^n on the right-hand side is :

$$\sum_{n\geq j\geq n/2} \frac{1}{2^{2j-1}} \frac{1}{j} \binom{2j-2}{j-1} \binom{j}{n-j} 3^{n-j} 2^{2j-n}$$

or

(2) $$B_{n-1} = \frac{3^n}{2^n} \sum_{n \geq j \geq n/2} \frac{1}{j3^j} \binom{2j-2}{j-1} \binom{j}{n-j}.$$

This should be contrasted with the generalized ballot number of Carlitz [2].

9. The Motzkin Connection. — This sequence of B's 1, 1, 2, 4, 9, 21, 51, ... has already been analyzed by Theodore Motzkin in [2] wherein the claim is made that this sequence counts "the number c_n of divisions of n points in a circle into sets of l points without crossing... for $l = 1$ or 2 we have $y - 1 = xy + x^2y^2$ (*sic*) and the sequence c_n begins 1, 2, 4, 9, 21, 51, ..."

If we interpret $l = 1$ to mean an isolated vertex and $l = 2$ to mean two vertices connected by a chord, then c_n counts the partitions of the n vertices into some collection of non-intersecting diagonals (not even meeting at a common vertex), where not every vertex need be part of a chord. For $n = 4$, we have $c_4 = 9 = B_5$. (diagram 7)

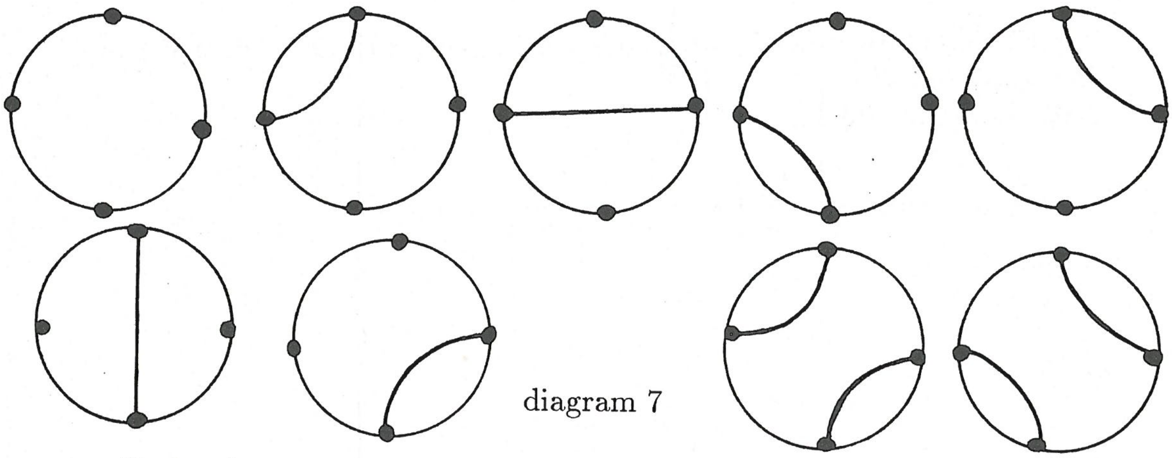
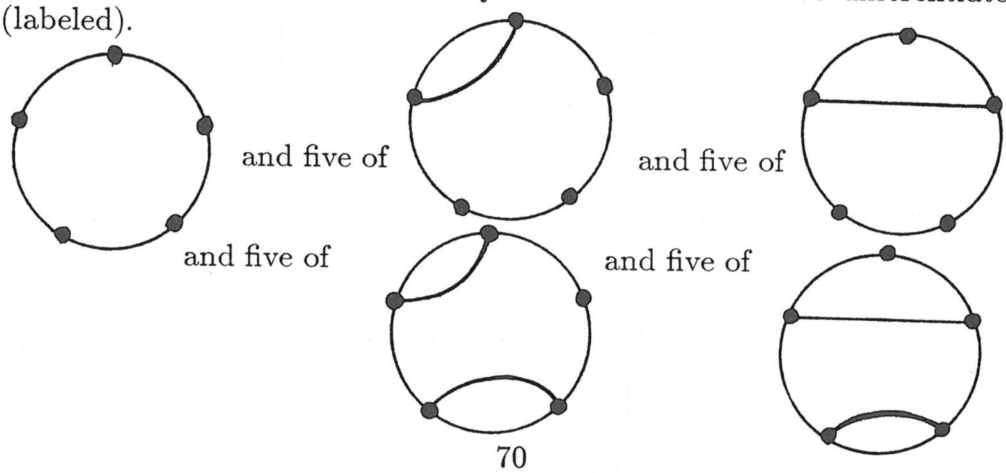


diagram 7

Notice that we must allow a pair of consecutive vertices to be in an l-set together. To demonstrate that we have the correct interpretation for c_n we observe that c_5 counts the patterns pictured in diagram 8 ($c_5 = 21 = B_6$) Notice that the vertices on the cycles are assumed to be differentiated (labeled).



Interestingly, Motzkin's question is different from ours, yet similar. He also works with non-crossing sets and labeled cycles, yet his sequence of c's corresponds to our non-cyclic linear form, type B, not type A. Our vertices can be used for more than one diagonal, his cannot. Of course, there must be a bijection between the objects that c_n counts, and the objects that B_{n+1} counts.

10. The Recurrence Formula Redux. — Let us turn our attention now to the A's. In determining the recurrence formula for the B's we enumerated the A's *en passant* :

$$A_n = B_{n-1} + \sum_{p \geq 2} \sum_{\Sigma m_i = n-p} \prod B_{m_i}.$$

From the recurrence formula we see that

$$A_n = B_n - B_{n-1} + B_{n-2} - B_{n-3} + \cdots \pm B_1 \tag{3}$$

We could derive this formula directly by inclusion-exclusion. From this we derive

$$A_n + A_{n-1} = B_n, \quad \text{for } n \geq 3$$

which we could again argue combinatorially. All B_n sequences are either A_n sequences or else (if they end in 1) when we remove the final term they are A_{n-1} sequences. Let f be the generating function for the A's :

$$f = \sum_{j \geq 2} A_j x^j = x^2 + x^3 + 3x^4 + 6x^5 + \cdots$$

If we proceed to note that

$$f + xf = y - x = x^2 + 2x^3 + 4x^4 + 9x^5 + \cdots$$

and therefore not that $f + xf - x$ satisfies equation (1), we eventually find that

$$f = \frac{1}{2x} - \frac{1}{2} - \frac{1}{2x(1+x)}\sqrt{1-(2+3x)x}$$

which is no better than using (2) and (3).

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