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THE DUALITY BETWEEN INCIDENCE ALGEBRAS AND COALGEBRAS. A FEW REMARKS

BY

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Introduction

A set S with a suitable "decomposition law" gives rise to a coalgebra C(S), the so-called Incidence Coalgebra of S. Its dual algebra is the better known Incidence Algebra of S, A(S). In the last few years, a few particular cases of these structures have been studied in great detail (see [1],[2],[5]). These concepts have shown themselves to be powerful tools in enumerative combinatorial problems with which they are concerned. We are interested in them from a general point of view. In fact, in our opinion they are likely to be the source of a correct algebraic counterpart of combinatorial structures.

In the quoted cases, S is the set of the intervals of a locally finite ordered set (see [2]) or the set of the morphisms of a Moebius Category (see [1] and [5]). In both cases, the use of the so-called "standard topology" plays a central role. Nevertheless, this use is not essential. As we shall see, it may be substituted by the duality between A(S) and C(S). Making use of this duality, as well as of new results about Incidence Coalgebras, in the present work, we also generalize a few properties about Incidence Algebras due to Leroux [5].

§1. Decomposition law of a set

Let S be a given set. Let N[S] denote the free abelian monoid generated by S. A pair of applications:

$$d: S \longrightarrow N[S\times S]$$

$$e: S \longrightarrow N$$

is said to be a decomposition law of S if the coefficients $\left[{s \atop q,r}\right]$, defined by: $\sum_{q,r} \left[{s \atop q,r}\right](q,r) := d(s)$, satisfy the following equations:

1.1 $$\sum_r \left[{s \atop q,r}\right]\left[{r \atop t,v}\right] = \sum_r \left[{r \atop q,t}\right]\left[{s \atop r,v}\right]$$

1.2 $$\sum_q \left[{s \atop q,r}\right] e(q) = \sum_q \left[{s \atop r,q}\right] e(q) = \delta^s_r.$$

If this is the case, these non-negative coefficients $\left[{s \atop q,r}\right]$ are called <u>section coefficients of S</u>. We shall also say that the element $s \in S$ can be cut into the ordered pair (q,r) in exactly $\left[{s \atop q,r}\right]$ ways; moreover, if $\left[{s \atop q,r}\right] > 0$, the pair (q,r) is said to be a <u>decomposition</u> of s.

The following example of decomposition law has been studied in [1] and [5]. It can be also found, together with many others, in [4]. Let C be a small category and let S=Mor(C). Let us suppose that the set $\{(q,r) \mid r \circ q = s\}$ is finite for every $s \in S$. The decomposition law of S is now defined in the following way:

$$d: S \longrightarrow N[S\times S]$$
$$s \rightsquigarrow \sum_{r\circ q = s} (q,r)$$

$$e: S \longrightarrow N$$
$$s \rightsquigarrow \begin{cases} 1 & \text{if } s \text{ is an identity of } C \\ 0 & \text{otherwise} \end{cases}$$

In this case, we have: $\left[{s \atop q,r}\right] = \begin{cases} 1 & \text{if } r\circ q = s \\ 0 & \text{otherwise.} \end{cases}$

We denote $\left[{s \atop q,t,v}\right]$ the common value of both sides of 1.1. This is the number of ways we can cut s into the ordered triple (q,t,v). Reiterating 1.1 allows us to define more general coefficients $\left[{s \atop s_1 \ldots s_{n+1}}\right]$:

1.3 $$\left[{s \atop s_1 \ldots s_n\, s_{n+1}}\right] = \sum_t \left[{s \atop s_1 \ldots s_{n-1}\, t}\right]\left[{t \atop s_n\, s_{n+1}}\right]$$

as well as a map:

1.4 $$d^n: S \longrightarrow N[S\times \ldots \times S]$$
$$s \rightsquigarrow \sum_{s_1 \ldots s_{n+1}} \left[{s \atop s_1 \ldots s_{n+1}}\right](s_1, \ldots, s_{n+1})$$

Obviously, $d^1=d$.

If $[{s \atop s_1,\dots,s_n}]>0$ the n-tuple $(s_1,\dots,s_n)$ is called a <u>decomposition of degree n</u> of s. In particular, the 1-tuple (r) is said to be a decomposition of degree 1 of $s\in S$ if, and only if, r=s. The decomposition $(s_1,\dots,s_n)$ is called a <u>strict decomposition</u> if $d(s_i)\neq(s_i,s_i)$ for $1\le i\le n$. If each $s\in S$ admits a finite number of strict decompositions, we say that the decomposition law is <u>hereditarily finite</u>. In this case, the supremum of the set of degrees of strict decompositions with reference to a given element $s\in S$ is called the <u>length</u> of s and denoted l(s).

1.5 PROPOSITION. For every hereditarily finite decomposition law of S if $[{s \atop q,r}]>0$ then $l(q)+l(r)\le l(s)$.

<u>Proof</u>. Let l(q)=j and l(r)=h; let $(q_1,\dots,q_j)$ and $(r_1,\dots,r_h)$ be strict decompositions of q and r respectively. We have:

$$[{s \atop q_1\dots q_j\, r_1\dots r_h}] = \sum_{u,v}[{s \atop u,v}][{u \atop q_1\dots q_j}][{v \atop r_1\dots r_h}]\ge[{s \atop q,r}][{q \atop q_1\dots q_j}][{r \atop r_1\dots r_h}]>0$$

Thus, $(q_1,\dots,q_j,r_1,\dots,r_h)$ is a strict decomposition of s and $l(q)+l(r)\le$ $\le l(s)$. □

If the decomposition law of S is hereditarily finite we put:

$$S_{(n)} = \{s\in S|l(s)=n\} \quad\text{and}\quad S_n = \bigcup_{k\le n} S_{(k)}.$$

Our study of decomposition law will need the following properties, due to Joyal [4], relative to <u>neutral elements</u> of S. Recalling that a neutral element of S is an element $s\in S$ such that e(s)=1, we have:

1.6 If $s\in S_0$, then s is a neutral element of S.

1.7 For each s in S there exists a unique pair of neutral elements $\partial_0(s)$ and $\partial_1(s)$ such that $[{s \atop \partial_0(s),s}]$ and $[{s \atop s,\partial_1(s)}]$ are both positve numbers. In particular, $[{s \atop \partial_0(s),s}]=[{s \atop s,\partial_1(s)}]=1$.

1.8 The following statements are equivalent:

i) s is neutral;

ii) e(s)>0;

iii) $\partial_0(s)=s$ (resp. $\partial_1(s)=s$).

As a consequence, we have: $e(s)=\begin{cases}1 & \text{if } s \text{ is a neutral element of } S\\ 0 & \text{otherwise}\end{cases}$

1.9 If $[{}^{s}_{q,r}]>0$ and $e(r)=s$ (resp. $e(q)=1$), then $s=q$ and $r=\partial_1(s)$ (resp. $s=r$ and $q=\partial_o(s)$).

1.10 If $[{}^{s}_{q,r}]>0$, then we have:

i) $\partial_o(s)=\partial_o(q)$

ii) $\partial_1(s)=\partial_1(r)$

iii) $\partial_1(q)=\partial_o(r)$.

1.11 If the decomposition law of S is hereditarily finite then for each neutral element s of S we have $d(s)=(s,s)$; i.e. $l(s)=0$.

If $r,s\in S_o$, we will denote $S_{(n)}(r,s)$ the set of all elements $u\in S_{(n)}$ such that $\partial_o(u)=r$ and $\partial_1(u)=s$. Furthermore we will denote $U^n_S(r,s)$ the set of the strict decompositions of degree n of the elements of $S_{(n)}(r,s)$

The set S will be said to be <u>finitely generated</u> if, for each pair $r,s\in S_o$,

i) $S_{(1)}(r,s)$ is finite;

ii) $[r,s]=\{q\in S_o \mid$ there exist u,v,w S with $[{}^{u}_{v,w}]>0$ and $\partial_o(u)=r$, $\partial_o(w)=q$, $\partial_1(u)=s\}$ is finite.

1.12 PROPOSITION. Let S be a set equipped with a hereditarily finite decomposition law and finitely generated. If, for each $n\geq 1$ and for each pair $r,s\in S_o$, the cardinality of $S_{(n)}(r,s)$ is less than or equal to the cardinality of $U^n_S(r,s)$ then $S_{(n)}(r,s)$ is a finite set.

<u>Proof</u> If n=1 the proposition is trivial. Let us assume that the conclusion holds for n. If $(u_1,\dots,u_{n+1})\in U^{n+1}_S(r,s)$ then $\sum_{v\in S}[{}^{v}_{u_1\dots u_n}][{}^{u}_{v,u_{n+1}}]=[{}^{u}_{u_1\dots u_{n+1}}]>0$ for a certain $u\in S_{(n+1)}(r,s)$. Thus there exists $q\in[r,s]$ and $w\in S_{(n)}(r,q)$ such that $u_{n+1}\in S_{(1)}(q,s)$ and $[{}^{w}_{u_1\dots u_n}]>0$. For the inductive hypothesis we have, for every $q\in[r,s]$, that $U^n_S(r,q)$ is finite. Therefore $U^{n+1}_S(r,s)$ is finite and this implies that $S_{(n+1)}(r,s)$ is finite. □

1.13 COROLLARY. With the hipotheses of the proposition 1.12, if S_0 is a finite set then $S_{(n)}$ is a finite set. □

With each set S, equipped with a hereditarily finite decomposition law and with each positve integer n, we may associate a directed graph $G(S_0,S_{(n)})$, the length n graph of S, assuming S_0 as its vertex-set and $S_{(n)}$ as its edge-set. Each arrow s of $S_{(n)}$ is directed from $\partial_0(s)$ to $\partial_1(s)$. Obviously, if $r,s\in S_0$ we can regard $S_{(n)}(r,s)$ as the subgraph of $G(S_0,S_{(n)})$ of the arrows $u\in S_{(n)}$ such that $\partial_0(u)=r$ and $\partial_1(u)=s$.

The sets S and T with hereditarily finite decomposition laws are said to have isomorphic presentations if, for each $n\in N$, the length n graphs $G(S_0,S_{(n)})$ and $G(T_0,T_{(n)})$ are isomorphic graphs. Thus, S and T have isomorphic presentations if, and only if, there exists a bijection $\varphi_0:S_0\longrightarrow T_0$ such that the sets $S_{(n)}(\partial_0(s),\partial_1(s))$ and $T_{(n)}(\varphi_0(\partial_0(s)),\varphi_0(\partial_1(s)))$ have the same cardinality for each n and for each $s\in S_{(n)}$.

§2. Incidence coalgebras

Each decomposition law (d,e) of a set S allows us to define a coalgebra over a characteristic zero field K. Let us associate a variable x_s to each $s\in S$ and denote K[S] the K-vector space spanned by x_s's. Owing to 1.1 and 1.2, the linear maps

2.1
$$\Delta_S:K[S]\longrightarrow K[S]\otimes K[S]$$
$$x_s \longmapsto \sum_{q,r}\left[{s \atop q,r}\right]x_q\otimes x_r$$

2.2
$$\varepsilon_S:K[S]\longrightarrow K$$
$$x_s \longmapsto \begin{cases}1 & \text{if s is a neutral element of S}\\ 0 & \text{otherwise}\end{cases}$$

satisfy the properties required for diagonalization and counit map in a coalgebra. The corresponding coalgebra $C(S)=(K[S],\Delta_S\varepsilon_S)$ is said to be the Incidence Coalgebra of S.

We will assume throughout that the decomposition law of S is hereditarily finite. Under this hypothesis it is easy to check that

$\Delta_S(K[S_n]) \subseteq K[S_n] \otimes K[S_n]$ for each $n \in N$. Thus, if we restrict both Δ_S and ε_S to $K[S_n]$, we obtain a subcoalgebra $C(S_n)$ of $C(S)$.

We come now to prove a proposition about grouplike elements of $C(S)$, i.e. elements c satisfying the conditions $\Delta_S(c)=c\otimes c$ and $\varepsilon_S(c)=1$. Statement 1.11 tells us that if s is a neutral element of S then x_s is a grouplike element of $C(S)$.

2.3 PROPOSITION. If c is a grouplike element of $C(S)$ then $c=x_s$ for some $s\in S_o$.

<u>Proof</u>. Let $c=\sum_s k^s x_s$ be a grouplike element of $C(S)$. We have:

$$\Delta_S(c)=\sum_s k^s \Delta_S(x_s)=\sum_s k^s \sum_{q,r}\left[{s \atop q,r}\right] x_q\otimes x_r=\sum_{q,r}\left(\sum_s k^s \left[{s \atop q,r}\right]\right) x_q\otimes x_r=\sum_{q,r} k^q k^r x_q \otimes x_r .$$

Let x_t be a generator of $C(S)$ occurring in c such that $l(t)\geqslant l(s)$ for every x_s wich occurs in c. Then $k^t k^t=\sum_s k^s \left[{s \atop t,t}\right]\neq 0$. Thus there exists $s\in S$ such that $k^s\neq 0$ and $\left[{s \atop t,t}\right]\neq 0$; from prop.1.5 it follows $l(s)\geqslant 2l(t)$. Hence $l(s)=l(t)=0$. Thus, if x_s occurs in c then $s\in S_o$. Since grouplike elements of a coalgebra over a field are linearly independent (see[7]), we have $c=k^t x_t, t\in S_o$. Owing to $\varepsilon_S(c)=1$ we deduce $k^t=1$. This completes the proof. □

Let $C(S)$, $C(T)$ be incidence coalgebras. We recall that a lin ear map $\varphi:C(S)\longrightarrow C(T)$ is a <u>coalgebras map</u> if $\Delta_T\circ\varphi=(\varphi\otimes\varphi)\circ\Delta_S$ and $\varepsilon_T\circ\varphi=\varepsilon_S$

2.4 PROPOSITION. Let $\varphi:C(S)\longrightarrow C(T)$ be a coalgebra map. Then, for every $s\in S_o$ there exists $t\in T_o$ such that $\varphi(x_s)=x_t$.

<u>Proof</u> Let $s\in S_o$; then $\Delta_T(\varphi(x_s))=(\varphi\otimes\varphi)\circ\Delta_S(x_s)=\varphi(x_s)\otimes\varphi(x_s)$ and $\varepsilon_T(\varphi(x_s))=$ $=\varepsilon_S(x_s)=1$. Thus $\varphi(x_s)$ is a grouplike element of $C(T)$. Hence, by prop 2.3 $\varphi(x_s)=x_t$ where $t\in T_o$ □

2.5 PROPOSITION. Let $\varphi:C(S)\longrightarrow C(T)$ be a coalgebra map. If x_t, $t\in T$, occurs in $\varphi(x_s)$ then there exist two decompositions (q,r) and (u,v) of s such that:

i) x_t occurs in $\varphi(x_r)$ and $x_{\partial_o(t)}=\varphi(x_{\partial_o(r)})$;

ii) x_t occurs in $\varphi(x_u)$ and $x_{\partial_1(t)}=\varphi(x_{\partial_1(u)})$.

<u>Proof</u> By prop.2.4, the conclusion holds when $l(s)=0$. Arguing by induction on $l(s)=n$, let us assume that it holds for each element of S with a length less than n. If $\varphi : C(S) \longrightarrow C(T)$ is a coalgebra map and if $\varphi(x_s) = \sum_p k_s^p x_p$ then, for each pair $w,z \in T$, we have: $\sum_{a,b\in S} k_a^w k_b^z \left[{s \atop a,b}\right] = \sum_{p\in T} k_s^p \left[{p \atop w,z}\right]$. So, for $w=\partial_0(t)$ and $z=t$ we have $\sum_{a,b} k_a^{\partial_0(t)} k_b^t \left[{s \atop a,b}\right] = k_s^t \neq 0$ (why x_t occurs in (x_s)). Hence there exists a decomposition (i,j) of s such that $k_i^{\partial_0(t)} \neq 0$ and $k_j^t \neq 0$. If $i=\partial_0(s)$ then $j=s$, thus, by prop.2.4, $x_{\partial_0(t)} = \varphi(x_{\partial_0(s)})$. If $l(j)<l(s)$ then, by the induction hipothesis, there exists a decomposition (h,r) of j such that x_t occurs in $\varphi(x_r)$ and $x_{\partial_0(t)} = \varphi(x_{\partial_0(r)})$. Now we obtain the proof of the existence of the decomposition (q,r) of s by: $0 < \left[{s \atop i,j}\right]\left[{j \atop h,r}\right] \leq \left[{s \atop i,h,r}\right] = \sum_w \left[{w \atop i,h}\right]\left[{s \atop w,r}\right]$. Similary we can prove ii). $\square$

If we write $\overset{n}{\otimes} K[S]$ for the n-fold tensor product of K[S] with itself, making use of 1.4 we may define two new linear maps:

$$\Delta_S^n : K[S] \longrightarrow \overset{n+1}{\otimes} K[S]$$
$$x_s \longmapsto \sum_{s_1 \ldots s_{n+1}} \left[{s \atop s_1 \ldots s_{n+1}}\right] x_{s_1} \otimes \ldots \otimes x_{s_{n+1}}$$

and

$$\overline{\Delta}_S^n : K[S] \longrightarrow \overset{n+1}{\otimes} K[S]$$
$$x_s \longmapsto \sum \left[{s \atop s_1 \ldots s_{n+1}}\right] x_{s_1} \otimes \ldots \otimes x_{s_{n+1}}$$

where the last sum ranges over all the strict decompositions $(s_1 \ldots s_{n+1})$ of degree n+1 of s. Thus, if $P_S : K[S] \longrightarrow K[S]$ is the linear map defined by

$$P_S(x_s) = \begin{cases} 0 & \text{is } s \in S_0 \\ 1 & \text{otherwise} \end{cases}$$

we have $\overline{\Delta}_S^n = (\overset{n+1}{\otimes} P_S) \circ \Delta_S^n$ where $\overset{n+1}{\otimes} P_S$ is the n-fold tensor product of P_S with itself. Obviously, $\Delta_S^1 = \Delta_S$. It is easy to check that:

2.6 $$\Delta_S^{n+1} = (\Delta_S^n \otimes I) \circ \Delta_S = (\Delta_S \otimes (\overset{n}{\otimes} I)) \circ \Delta_S^n$$

where $I : K[S] \longrightarrow K[S]$ is the identity map. So, if $\varphi : C(S) \longrightarrow C(T)$ is a coalgebra map, proceeding by induction on n, from 2.6 we deduce:

2.7 $$\Delta_T^n \circ \varphi = (\overset{n+1}{\otimes} \varphi) \circ \Delta_S^n$$

2.8 PROPOSITION. If $\varphi: C(S) \longrightarrow C(T)$ is a coalgebra map then:

$$\bar{\Delta}_T^n \circ \varphi = (\overset{n+1}{\otimes} P_S) \circ (\overset{n+1}{\otimes} \varphi) \circ \bar{\Delta}_S^n.$$

Proof $\bar{\Delta}_T^n \circ \varphi = (\overset{n+1}{\otimes} P_T) \circ \Delta_T^n \circ \varphi = (\overset{n+1}{\otimes} P_T) \circ (\overset{n+1}{\otimes} \varphi) \circ \Delta_S^n = (\overset{n+1}{\otimes} (P_T \circ \varphi)) \circ (\bar{\Delta}_S^n + \Delta_S^n - \bar{\Delta}_S^n)$. But, by 2.4 and by the definition of P_T, $(\overset{n+1}{\otimes} (P_T \circ \varphi)) \circ (\Delta_S^n - \bar{\Delta}_S^n)$ is the zero map; so $\bar{\Delta}_T^n \circ \varphi = (\overset{n+1}{\otimes} P_T) \circ (\overset{n+1}{\otimes} \varphi) \circ \bar{\Delta}_S^n$. □

We observe that $K[S_n] \subseteq Ker(\bar{\Delta}_S^n)$. If $Ker(\bar{\Delta}_S^n) = K[S_n]$ we say that C(S) is an <u>n-regular coalgebra</u>; moreover C(S) will be said to be <u>regular</u> if it is n-regular for each natural number $n \geq 1$.

Making use of the foregoing proposition we con prove the following basic result:

2.9 PROPOSITION. Let $\varphi: C(S) \longrightarrow C(T)$ be a coalgebra map. If C(T) is an n-regualar coalgebra then, for each $m \leq n$, $\varphi(C(S_m)) \subseteq C(T_n)$

Proof The special case where n=0 has been considered in prop.2.4. Let now $n>0$. If $x_s \in K[S_m]$, $m \leq n$, then $\bar{\Delta}_S^n(x_s) = 0$; hence, by prop.2.8, $\bar{\Delta}_T^n(\varphi(x_s)) = 0$ Thus, owing to the n-regularity of C(T), $\varphi(x_s) \in K[T_n]$. □

2.10 COROLLARY. Let $\varphi: C(S) \longrightarrow C(T)$ be a coalgebra map and let C(T) be regular. If x_t occurs in $\varphi(x_s)$ and $l(t) = l(s)$ then $x_{\partial_0(t)} = \varphi(x_{\partial_0(s)})$ and $x_{\partial_1(t)} = \varphi(x_{\partial_1(s)})$. □

2.11 PROPOSITION. If C(S) and C(T) are two isomorphic regular incidence coalgebras then S and T have isomorphic presentations.

Proof If $\varphi: C(S) \longrightarrow C(T)$ is a coalgebra isomorphism then, by prop.2.4, we can define a bijection $\varphi_o: S_o \longrightarrow T_o$ putting $\varphi_o(s) = t$ whenever $\varphi(x_s) = x_t$. Moreover, by prop.2.9, φ restricts to a coalgebra isomorphism between $C(S_n)$ and $C(T_n)$, for each $n \in N$. Thus, if $\varphi(x_s) = \sum_t k_s^t x_t$ and if $l(s) = n$, putting $\varphi_n(x_s) = \sum_{l(t)=n} k_s^t x_t$ we obtain a linear one-to-one correspondence between the vector spaces $K[S_{(n)}]\simeq K[S_n]/K[S_{n-1}]$ and $K[T_{(n)}]\simeq K[T_n]/K[T_{n-1}]$ Now, by corollary 2.10, for each $s\in S_{(n)}$, φ_n restricts to a linear one-to--one corrispondence between $K[S_{(n)}(\partial_0(s),\partial_1(s))]$ and $K[T_{(n)}(\varphi_0(\partial_0(s)),\varphi_0(\partial_1(s))]$. Thus the sets $S_{(n)}(\partial_0(s),\partial_1(s))$ and $T_{(n)}(\varphi_0(\partial_0(s)),\varphi_0(\partial_1(s)))$ have the same cardinality. □

We observe that the hypothesis of regularity is necessary in prop.2.11 as the following example shows. Let $S=\{u_1,u_2,u_3,p,q,r,s\}$ be. If $S_0=\{u_1,u_2,u_3\}$, $S_{(1)}=\{p,q,r\}$, $S_{(2)}=\{s\}$ with $u_1=\partial_0(p)=\partial_0(r)=\partial_0(s)$, $u_2=\partial_1(p)=\partial_0(q)$, $u_3=\partial_1(q)=\partial_1(r)=\partial_1(s)$ and $[{}^{s}_{p,q}]=2$ then C(S) is a regular incidence coalgebra. While the set $T=\{u_1',u_2',u_3',p',q',r',s'\}$ with $T_0=\{u_1',u_2',u_3'\}$, $T_{(1)}=\{p',q'\}$, $T_{(2)}=\{r',s'\}$, $u_1'=\partial_0(p')=\partial_0(r')=\partial_0(s')$, $u_2'=\partial_1(p')=\partial_0(q')$, $u_3'=\partial_1(q')=\partial_1(r')=\partial_1(s')$ and $[{}^{s'}_{p',q'}]=[{}^{r'}_{p',q'}]=1$ gives rise to a non-regular incidence coalgebra C(T). For $x_{s'}-x_{r'}\in Ker(\bar{\Delta}^1_T)$ and $x_{s'}-x_{r'}\notin K[T_1]$. Obviously S and T have not isomorphic presentations while the linear map $\varphi:C(S)\longrightarrow C(T)$ defined by $\varphi(x_{u_i})=x_{u_i'}$, i=1,2,3, $\varphi(x_p)=x_{p'}$, $\varphi(x_q)=x_{q'}$, $\varphi(x_s)=x_{s'}+x_{r'}$ and $\varphi(x_r)=x_{s'}-x_{r'}$ is a coalgebra isomorphism.

§3. Incidence algebras

In enumeration problems relative to a set S with a decomposition law, the tool usually used is not the Incidence Coalgebra C(S) but its dual algebra $C(S)^*$. This algebra is obtained by C(S) defining the convolution product $f*g$ of the elements $f,g\in K[S]^*=Hom(K[S],K)$ in the following way:

3.1 $$(f*g)(x_s):=m_0(f\otimes g)\circ\Delta_S(x_s)=\sum_{q,r}[{}^{s}_{q,r}]f(x_q)g(x_r)$$

where $m:K\otimes K\longrightarrow K$ is the product over K. It is plain that this convolution product is associative and that the linear form ε_S is the two-sided identity. The vector space $K[S]^*$ together with the convolution product is called the Incidence Algebra of S. It will be denoted A(S). The invertible elements of incidence algebra of S are characterized by the following proposition.

3.2 PROPOSITION. An element $f\in A(S)$ is invertible if and only if, for each $s\in S_0$, $f(x_s)$ is an invertible element of K.

We recall that if A,B are two R-modules, a family $(f_i)_{i\in I}$ of elements of $Hom_R(A,B)$ is said to be a summable family if, for every $a\in A$, the set $\{i\in I \mid f_i(a)\neq 0\}$ is finite. Given a summable family $(f_i)_{i\in I}$ of elements of $Hom_R(A,B)$ we obtain a new element $\sum_{i\in I} f_i$ of $Hom_R(A,B)$ putting, for each $a\in A$, $(\sum_{i\in I} f_i)(a) := \sum_{i\in I} f_i(a)$. Thus, if $A=K[S]$ and $B=K$ the set $(x^s)_{s\in S}$ where

$$x^s : K[S] \longrightarrow K$$
$$x_t \rightsquigarrow \delta_t^s$$

is a summable family. Obviously if f is an arbitrary element of $K[S]^*$ then family $(f(x_s)x^s)_{s\in S}$ is a summable family and we have: $\sum_{s\in S} f(x_s)x^s = f$.

3.3 PROPOSITION. Let $\psi : K[T]^* \longrightarrow K[S]^*$ be a linear map. If $(\psi(x^t))_{t\in T}$ is a summable family of $K[S]^*$ and if, for every family $(k_t)_{t\in T}$ of elements of K, $\psi(\sum_{t\in T} k_t x^t) = \sum_{t\in T} k_t \psi(x^t)$ then there exists a linear map $\varphi : K[S] \longrightarrow K[T]$ such that $\psi(f) = f\circ\varphi$ (i.e. ψ is the dual map of φ).

Proof We put $(\psi(x^t))(x_s) = k_s^t$. Since $(\psi(x^t))_{t\in T}$ is a summable family, $k_s^t = 0$ for all but a finite number of $t\in T$. So, the linear map:

$$\varphi : K[S] \longrightarrow K[T]$$
$$x_s \rightsquigarrow \sum_{t\in T} k_s^t x_t$$

is well defined and, for each $f\in K[T]^*$, $\psi(f) = f\circ\varphi$. In fact, if $f = \sum_{t\in T} f(x_t)x^t$ is any element of $K[T]^*$ and $x_s\in K[S]$ we have:

$(f\circ\varphi)(x_s) = f(\sum_{t\in T} k_s^t x_t) = \sum_{t\in T} k_s^t f(x_t)$ and

$$\psi(f)(x_s) = \psi(\sum_{t\in T} f(x_t)x^t)(x_s) = (\sum_{t\in T} f(x_t)\psi(x^t))(x_s) = \sum_{t\in T} f(x_t)\psi(x^t)(x_s) =$$
$$= \sum_{t\in T} f(x_t)k_s^t .$$ □

We observe that if we equip $K[T]^*$ and $K[S]^*$ with the standard topology then the continous linear maps $\psi : K[T]^* \longrightarrow K[S]^*$ are exactly the same linear maps keeping the summable families. Thus the prop.3.3 shows that if ψ is continous, with respect to the standard topology, then ψ is the dual map of a linear map $\varphi : K[S] \longrightarrow [T]$.

3.4 PROPOSITION. If the algebra map $\psi:A(T)\longrightarrow A(S)$ is the dual map of a linear map $\varphi:C(S)\longrightarrow C(T)$ then φ is a coalgebra map.

<u>Proof</u> By $\psi(f*g)=\psi(f)*\psi(g)$ we deduce that $(f*g)\circ\varphi=(f\circ\varphi)*(g\circ\varphi)$. Hence:

$$m\circ(f\otimes g)\circ\Delta_T\circ\varphi=m\circ((f\circ\varphi)\otimes(g\circ\varphi))\circ\Delta_S=m\circ(f\otimes g)\circ(\varphi\otimes\varphi)\circ\Delta_S.$$

If $(\Delta_T\circ\varphi)(x_s)=\sum_{u,v} h_s^{uv} x_u\otimes x_v$ and $(\varphi\otimes\varphi)\circ\Delta_S(x_s)=\sum_{p,q} k_s^{pq} x_p\otimes x_q$, putting $f=x^r$, $g=x^t$ we have $m\circ(x^r\otimes x^t)\circ\Delta_T\circ\varphi(x_s)=h_s^{rt}$ and $m\circ(x^r\otimes x^t)\circ(\varphi\otimes\varphi)\circ\Delta_S(x_s)=k_s^{rt}$ Thus $\Delta_T\circ\varphi=(\varphi\otimes\varphi)\circ\Delta_S$. Moreover: $\varepsilon_T\circ\varphi=\psi(\varepsilon_T)=\varepsilon_S$ □

We come now to the study of the algebra A(S) considering the subsets:

$$J_n(S)=\{f\in A(S)\,|\,f(x_s)=0 \text{ for every } s\in S_{n-1}\}.$$

All these sets are ideals of A(S) and, in particular, $J_1(S)$ is its Jacobson radical. In fact it is possible to prove that $f\in J_1(S)$ if and only if, for every pair of elements $g,h\in A(S)$, $\varepsilon_S-g*f*h$ is an invertible element of A(S). Moreover, for each $n\geq 1$, we have

3.5 $$J_{n+1}(S)\subseteq J_n(S) \quad\text{and}\quad (J_1(S))^n\subseteq J_n(S).$$

Now we want to give a condition about the algebra A(S) which is a consequence of the notion of n-regularity.

3.6 PROPOSITION. If C(S) is an n-regular incidence coalgebra then, for every pair of sequences $s_1,\dots,s_m\in S$, with $l(s_i)>n$, and $k_1,\dots,k_m\in K$ then exists a finite family of scalars $h_{r_1\dots r_{n+1}}$, where $(r_1,\dots,r_{n+1})$ is a strict decomposition of degree n+1 of s_i, i=1,...,m, such that:

$$f=\sum h_{r_1\dots r_{n+1}} x^{r_1}*\dots*x^{r_{n+1}}$$

with $f(x_{s_i})=k_i$.

<u>Proof</u> If C(S) is an n-regular incidence coalgebra then the space spanned by $\bar\Delta_S^n(x_{s_i})$'s, with $l(s_i)>n$ and $i=1,\dots,m$, has not a dimension lower than m. Hence, if $\bar\Delta_S^n(x_{s_i})=\sum\left[{s_i \atop r_1\dots r_{n+1}}\right]x_{r_1}\otimes\dots\otimes x_{r_{n+1}}$, there exist $h_{r_1\dots r_{n+1}}$ such that $\sum\left[{s_i \atop r_1\dots r_{n+1}}\right]h_{r_1\dots r_{n+1}}=k_i$. Thus $f=\sum h_{r_1\dots r_{n+1}}x^{r_1}*\dots*x^{r_{n+1}}$ satisfies the conditions $f(x_{s_i})=k_i$. □

3.7 PROPOSITION. Let C(S) be a regular incidence coalgebra. If S is finitely generated then, for each $n \geq 1$ and for each pair $r,s \in S_o$,

$$x^r * J_n(S) * x^s = x^r * (J_1(S))^n_* x^s.$$

Proof Since $J_1^n(S) \subseteq J_n(S)$, we will only prove that $x^r * J_n(S) * x^s \subseteq x^r * (J_1(S))^n_* x^s$. The proof is by induction on n. The conclusion clearly holds when n=1. Thus we suppose that the conclusion holds for n. For prop.1.12, since C(S) is regular, the sets $S_{(m)}(r,s)$ are finite for each $m \geq 1$. Thus, if $f \in x^r * J_{n+1}(S) * x^s$, for prop.3.6, we can find a sequence of linear forms $g_{n+1}, g_{n+2}, \ldots, g_{n+i} \in (J_1(S))^{n+i}$, such that $g_{n+i}(x_q) = f(x_q) - \sum_{j=1}^{i-1} g_{n+j}(x_q)$ for every $q \in S_{(n+i)}(r,s)$ and $g_{n+i} = \sum h_{r_1 \ldots r_{n+i}} x^{r_1} * \cdots * x^{r_{n+i}}$ where the sum ranges over all the strict decompositions of degree n+1 of the elements of $S_{(n+i)}(r,s)$. Obviously the family $(g_{n+i})_{i \geq 1}$ is a summable family and we have $f = \sum_{i \geq 1} g_{n+i}$. Therefore:

$$f = \Big(\sum_{\substack{u \in [r,s] \\ v \in S_{(1)}(r,u)}} x^v \Big) * \Big(\sum h_{r_1 \ldots r_{n+i}} x^{r_2} * \ldots * x^{r_{n+i}} \Big)$$

where the second sum ranges over all the strict decompositions $(r_1, \ldots, r_{n+i})$ of elements of $\bigcup_{i \geq 1} S_{(n+i)}(r,s)$ with $r_1 = v$. Thus:

$$f = \sum_{\substack{u \in [r,s] \\ v \in S_{(1)}(r,u)}} x^v * g_{u,v}$$

whith $g_{u,v} \in x^u * J_n(S) * x^s = x^u * (J_1(S))^n_* x^s$ by the induction hypothesis. Since the set of pairs (u,v) such that $u \in [r,s]$ and $v \in S_{(1)}(r,u)$ is finite, we can conclude that $f \in x^r * (J_1(S))^{n+1}_* x^s$. □

3.8 COROLLARY. With the hypotheses of proposition 3.7, if S_o is a finite set then $J_n(S) = (J_1(S))^n$.

Proof By corollary 1.13 $S_{(n)}$ is a finite set and we have:

$$J_n(S) = \sum_{r,s \in S_o} x^r * J_n(S) * x^s = \sum_{r,s \in S_o} x^r * (J_1(S))^n_* x^s = (J_1(S))^n. \quad ||$$

In order to prove a proposition, for incidence algebras, similar to the prop.2.11 we need two results which we give without proof. These results have been proved by Leroux (see [5]) in a particular case, but it is possible to repeat the same proof in our case.

3.9 PROPOSITION. If $\psi: A(S) \longrightarrow A(T)$ is an algebra map then $\psi(J_1(S)) \subseteq J_1(T)$. □

3.10 PROPOSITION. If $\psi: A(S) \longrightarrow A(T)$ is an algebra isomorphism then there exists an inner automorphism ι of $A(T)$ such that $\iota \circ \psi$ restricts to bijection from $(x^s)_{s \in S_o}$ to $(x^t)_{t \in T_o}$ □

3.11 PROPOSITION. Let S be a finitely generated set with regular incidence coalgebra and let $\psi: A(S) \longrightarrow A(T)$ be an algebra map. If S_o is finite or ψ restricts to a bijection from $(x^s)_{s \in S_o}$ to $(x^t)_{t \in T_o}$ then ψ is the dual map of a coalgebra map φ from $C(T)$ to $C(S)$.

Proof Let us suppose S_o is finite. By corollary 3.8 and prop.3.9 we have $\psi(J_n(S)) = \psi(J_1(S)^n) = (\psi(J_1(S)))^n \subseteq (J_1(T))^n = J_n(T)$. Therefore, if $t \in T_{(m)}$ then:

i) $s \in S_{(n)}$ and $\psi(x^s)(x_t) \neq 0$ implie $n \leq m$, so, since S_m is a finite set, $\psi(x^s)_{s \in S}$ is a summable family;

ii) for every family $(k_s)_{s \in S}$ of elements of K,

$$\psi(\sum_{s \in S} k_s x^s)(x_t) = \psi(\sum_{s \in S_m} k_s x^s)(x_t) + \psi(\sum_{s \in S - S_m} k_s x^s)(x_t) = (\sum_{s \in S_m} k_s \psi(x^s))(x_t);$$

in fact S_m is finite and, since $\sum_{s \in S - S_m} k_s x^s \in J_{m+1}(S)$, $\psi(\sum_{s \in S - S_m} k_s x^s) \in J_{m+1}(T)$

Thus, by prop.3.3 and prop.3.4 ψ is the dual map of a coalgebra map $\varphi: C(T) \longrightarrow C(S)$.

If ψ restricts to a bijection from $(x^s)_{s \in S_o}$ to $(x^t)_{t \in T_o}$ we put, for $s \in S_o$, $\psi_o(s) = t \in T_o$ whenever $\psi(x^s) = x^t$, then, by prop.3.7 and 3.9, we have:

$$\psi(x^r * J_n(S) * x^s) = \psi(x^r * (J_1(S))^n_* \, x^s) \subseteq \psi(x^r) * J_n(T) * \psi(x^s) = x^{\psi_o(r)} * J_n(T) * x^{\psi_o(s)}.$$

Now, arguing as above, we can conclude the proof. □

3.12 COROLLARY. Let C(S), C(T) be regular incidence coalgebras and let S, T be finitely generated sets. If A(S) and A(T) are isomorphic incidence algebras then S and T have isomorphic presentations.

Proof If $\psi : A(S) \longrightarrow A(T)$ is an algebra isomorphism then, by prop.3.10, there exists an inner automorphism ι of A(T) such that $\iota \circ \psi$ restricts to a bijection from $(x^s)_{s \in S_0}$ to $(x^t)_{t \in T_0}$. Thus C(S) and C(T) are isomorphic coalgebras and, by prop.2.11, we get the conclusion. □

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