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PFAFF-SAALSCHÜTZ REVISITED

BY

JIANG ZENG

Generalizing SURÁNYI's proof [Sur] of LE JEN SHOO's formula, SZÉKELY [Sz] obtained the following binomial coefficient identity with five independent parameters

$$(1)\quad \binom{a+c+d+e}{c+d}\binom{b+c+d+e}{b+d} = \sum_n \binom{a+b+c+d+e-n}{a+b+d+e}\binom{a+d}{n+d}\binom{b+e}{n+e}$$

and said that it was *a common generalization of several cubic identities.* As the right-hand side of (1) contained only a product of three binomial coefficients, it was very likely that identity (1) was nothing but the Pfaff-Saalschütz formula.

Let $(a)_n$ be the ascending factorial defined by

$$(a)_0 = 1, \qquad (a)_n = a(a+1)\cdots(a+n-1), \quad (n \geq 1)$$

and let

$${}_rF_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s\end{matrix}; x\right) = \sum_{n\geq 0} \frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n}\frac{x^n}{n!}$$

be the generalized hypergeometric series. Then, the Pfaff-Saalschütz identity reads (see [Bai, p. 9]) :

$$(2)\qquad {}_3F_2\left(\begin{matrix} a,b,-n \\ c,1+a+b-c-n\end{matrix}; 1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}.$$

Note that ASKEY [Ask] found that identity (2) was already derived by PFAFF [Pfa], ninety three years before SAALSCHÜTZ [Sa], to whom it is traditionally referred. See also a recent paper by ROY [Ro]. Short proofs of (2) have been given by several authors, such as DOUGALL [Do], NANJUNDIAH [Na], GESSEL-STANTON [Ge-St], and others... See [Ca-Fo] and [Fo] for two combinatorial derivations.

Now rewrite (1) by expressing the binomial coefficients as ratios of ascending factorials. This leads to :

$$(3)\quad {}_4F_3\left(\begin{matrix}-a,-b,-c,1\\ -a-b-c-d-e,d+1,e+1\end{matrix};1\right) = \frac{(c+d+e+1)_a(b+d+e+1)_a}{(b+c+d+e+1)_a(e+1)_a} \times \frac{a!\,b!\,c!\,d!\,(b+d+e)!\,(c+d+e)!}{(a+d)!\,(b+d)!\,(b+e)!\,(c+d)!\,(c+e)!},$$

a formula that apparently reduces to the Pfaff-Saalschütz identity for $d=0$. However, the elementary transformation

$$(4)\quad {}_4F_3\left(\begin{matrix}-n,a_1,a_2,a_3\\ b_1,b_2,b_3\end{matrix};x\right) = \frac{(a_1)_n(a_2)_n(a_3)_n}{(b_1)_n(b_2)_n(b_3)_n}(-x)^n \times {}_4F_3\left(\begin{matrix}-n,1-n-b_1,1-n-b_2,1-n-b_3\\ 1-n-a_1,1-n-a_2,1-n-a_3\end{matrix};\frac{1}{x}\right)$$

applied to the ${}_4F_3$ occurring in (3) yields

$$\begin{aligned}{}_4F_3&\left(\begin{matrix}-a,-b,-c,1\\ -a-b-c-d-e,d+1,e+1\end{matrix};1\right)\\ &= \frac{(-b)_a(-c)_a(1)_a}{(-a-b-c-d-e)_a(d+1)_a(e+1)_a}(-1)^a \times {}_4F_3\left(\begin{matrix}-a,1+b+c+d+e,-a-d,-a-e\\ 1-a+b,1-a+c,-a\end{matrix};1\right)\\ &= \frac{(-b)_a(-c)_a(1)_a}{(-a-b-c-d-e)_a(d+1)_a(e+1)_a}(-1)^a \times {}_3F_2\left(\begin{matrix}1+b+c+d+e,-a-d,-a-e\\ 1-a+b,1-a+c\end{matrix};1\right).\end{aligned}$$

But, applying the Pfaff-Saalschütz identity to the last ${}_3F_2$ gives back identity (3). Thus, (1) and (2) are truly equivalent.

Now introduce the q-ascending factorial

$$(a;q)_0=1,\quad (a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1}),\quad (n\geq 1)$$

the Gaussian polynomial

$$\begin{bmatrix}n\\ m\end{bmatrix} = \begin{cases}(q;q)_n(q;q)_m^{-1}(q;q)_{n-m}^{-1}, & \text{if } 0\leq m\leq n;\\ 0, & \text{otherwise;}\end{cases}$$

and the basic hypergeometric series (see [Bai, p. 65])

$$ {}_r\Phi_s\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,x\right] = \sum_{n\geq 0} \frac{(a_1;q)_n\cdots(a_r;q)_n}{(b_1;q)_n\cdots(b_s;q)_n}\frac{x^n}{(q;q)_n}. $$

Formula (1) can be q-ified in an obvious manner as :

$$ (5)\quad \begin{bmatrix} a+c+d+e \\ c+d \end{bmatrix}\begin{bmatrix} b+c+d+e \\ b+d \end{bmatrix} = \sum_n q^{(d+n)(e+n)} \begin{bmatrix} a+b+c+d+e-n \\ a+b+d+e \end{bmatrix}\begin{bmatrix} a+d \\ n+d \end{bmatrix}\begin{bmatrix} b+e \\ n+e \end{bmatrix}. $$

Using an analogous argument, especially the q-version of the elementary transformation (4), identity (5) can be shown to be equivalent to the q-Pfaff-Saalschütz identity (see [Bai, p. 68]) :

$$ (6)\qquad {}_3\Phi_2\left[\begin{matrix} a,b,q^{-n} \\ c,abq^{1-n}/c \end{matrix};q,q\right] = \frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/ab;q)_n}. $$

Thus even the q-version of identity (1) brings nothing really new.

Several authors (e.g. [Wr, Go]) have rediscovered the q-Pfaff-Saalschütz identity in a form that involves products of Gaussian polynomials, as in (5).

Recently, ZEILBERGER [Zei] gave a very ingenious combinatorial proof of the q-Pfaff-Saalschütz formula (other proofs are due to ANDREWS-BRESSOUD [An-Br] and GOULDEN [Gou]), by "chineseing" the proof derived by CARTIER-FOATA [Ca-Fo] for the ordinary Pfaff-Saalschütz formula. A slight modification of ZEILBERGER's argument can be used to give a combinatorial proof of identity (5). So this proof will not be reproduced in this note.

Finally, it must be mentioned that the inverse bijection found by SZÉKELY [Sz] to prove identity (1) is in fact equivalent to the bijection constructed by ZEILBERGER [Zei]. However the latter has been able to implement a new ingredient to prove the q-case.

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Jiang ZENG,
Département de mathématique,
Université Louis-Pasteur,
7, rue René-Descartes,
F-67084 Strasbourg, France.