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Aspects of asymptotic graph theory

by

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ABSTRACT

The aim of this note is to shed some light on the link between enumeration problems for labeled graphs and orders and those for unlabeled ones.

1. Introduction and the main result

Usually, it is much easier to enumerate a class $\mathcal{C}$ of labeled structures than the corresponding class $\mathcal{C}^u$ of unlabeled structures, i.e., the isomorphism types in $\mathcal{C}$. So quite often in order to derive (at least) an asymptotic formula for the number of elements in $\mathcal{C}^u$, the enumeration problem is solved in $\mathcal{C}$ and then it is proved that "almost all" structures in $\mathcal{C}$ are rigid, which means they have no nontrivial automorphisms. A typical example for proceeding along these lines is counting the class of all graphs. Obviously, the number $G(n)$ of all labeled graphs on n vertices is $2^{\binom{n}{2}} = 2^{\frac{n^2}{2}-\frac{n}{2}}$. Now using the fact that almost all graphs are rigid, this is to say that the quotient of $G(n)$ and the number of graphs in $G(n)$ which allow only the trivial automorphism tends to 1 as n goes to infinity, one gets an asymptotic formula for the number $G^u(n)$ of unlabeled graphs on n vertices, viz. $G^u(n) \sim \frac{G(n)}{n!}$. This formula was already known to Pólya (compare Ford and Uhlenbeck, 1957). For related results see also Oberschelp (1967), Wright (1971) and Fagin (1977).

We will see, loosely speaking, that whenever $\mathcal{C}$ is a class of labeled structures with one binary relation, then almost all structures in $\mathcal{C}$ are rigid, provided that C is "rich" enough. This result covers classes C, like the class of all graphs, the class of all directed graphs or the class of all tournaments, for which this behavior is well-known. But it covers also classes, for example the ℓ-colorable graphs for $\ell \geq 2$ or the partial orders, for which this was not yet known. Using the rigidity of these classes we obtain some new results in the asymptotic enumeration of the corresponding unlabeled classes and we derive also some other consequences.

Throughout this note all structures are structures provided with exactly one binary relation. Let C be an infinite class of finite labeled structures which is closed under (induced) substructures and isomorphisms. Every element in C is assumed to be defined on $n = \{0, \ldots, n-1\}$ for some n. Let C^u be the class of unlabeled structures correspoinding to C, i.e., C^u is the set of all isomorphism-types of structures in C. We denote by $C(n)$ the number of structures in C defined on n and by $C^u(n)$ the number of structures in C^u on n elements. Obviously, $\frac{C(n)}{n!} \leq C^u(n)$.

All logarithms throughout this paper are logarithms to the base 2. Using these notations and conventions we can formulate the main result.

Theorem

Let C be an infinite class of finite labeled structures (provided with exactly one binary relation) which is closed under substructures and isomorphisms. Assume that C satisfies the growth condition

$$\log C(n) < cn^2 + dn + o(n)$$

for all n where $c > 0$, and d is arbitrary. Then there is constant s such that for all n

$$C^u(n) \leq \frac{C(n)}{n!}(1 + \frac{s}{2^{cn}}) .$$

The proof of this result will appear in Prömel [1986].

Notice that the essential term in the growth condition of C is the $c > 0$. If the dn could be replaced by some (finally) concave up function $\omega(n)$, where $\omega(n) = o(n^2)$, i.e., $\log C(n) = cn^2 + \omega(n) + o(n)$. Moreover, considering structures with a fixed number of different binary relations would also not change the result of the Theorem. The lack of applications prevents us from including these slight generalizations in the Theorem. Observe that the Theorem is somewhat stronger than the fact that the automorphism group of almost every structure in C is trivial.

2. Applications of the Theorem

2.1 Graphs and digraphs

Of course, the Theorem applies immediately, for example, to the classes of all graphs and all digraphs and we get

$$G^u(n) \leq \frac{G(n)}{n!}(1 + \frac{s'}{2^{n/2}}) \text{ for all } n \text{ ,}$$

for the class of all graphs and

$$D^u(n) \leq \frac{D(n)}{n!}(1 + \frac{s''}{2^n}) \text{ for all } n \text{ ,}$$

for the class of all digraphs (where a digraph is an irreflexive binary relation). Tighter bounds for these special cases can be found, e.g., in Oberschelp (1967).

2.2 Partial orders

Kleitman and Rothschild [1975] established an asymptotic formula for the number of (labeled) partial orders on a finite set. Let $P(n)$ denote the number of (labeled) partial orders on $n = \{0, \ldots, n-1\}$. Then they showed that

$$P(n) = (1 + 0(\frac{1}{n}))(\sum_{i=1}^{n}\sum_{j=1}^{n-i}\binom{n}{i}\binom{n-i}{j}(2^i - 1)^j(2^i - 1)^{n-i-j})$$

which becomes, using a recent result of Davison (1986):

$$P(n) = (1 + 0(\frac{1}{n}))\alpha\sqrt{\frac{2}{\pi}}2^{n^2/4+3n/2-1/2 \ log \ n},$$

$$\text{where } \alpha = \alpha(n) = \begin{cases} \sum_{k=-\infty}^{\infty} 2^{-(\frac{k+1}{2})^2} & \text{if } n \text{ is even} \\ \sum_{k=-\infty}^{\infty} 2^{-k^2} & \text{if } n \text{ is odd.} \end{cases}$$

Using this and the Theorem we obtain

Corollary 2.2 *Let $P^u(n)$ denote the number of unlabeled partial orders on an n-element set. Then there exists a constant s such that for all n*

$$P^u(n) \leq \frac{P(n)}{n!}(1 + \frac{s}{2^{n/4}}) .$$

□

As an immediate consequence of Corollary 2.2 we derive

Corollary 2.2a *Almost all partial orders are rigid, i.e., have no nontrivial automorphism.*

□

This answers a question of R. Möhring (1985). Another consequence of Corollary 2.2 is of course

Corollary 2.2b *The number $P^u(n)$ of unlabeled partial orders on an n-element set (or, equivalently, the number of T_0-topologies on an n-element set) is given by*

$$P^u(n) = (1 + O(\frac{1}{n}))\frac{\alpha}{\pi}2^{n^2/4 - n \log n + (3/2 + \log e)n - \log n}$$

where α is as above.

□

2.3 $K_{\ell+1}$-free graphs

The next two applications rely heavily on results obtained in Kolaitis, Prömel and Rothschild (1985).

Extending a former result of Erdös, Kleitman and Rothschild (1976), it is shown in Kolaitis, Prömel and Rothschild (1985) that almost all labeled $K_{\ell+1}$-free graphs are already ℓ-colorable. More precisely: A graph is $K_{\ell+1}$-free if it does not contain a complete graph $K_{\ell+1}$ with $\ell + 1$ vertices as a subgraph. Now let $L_\ell(n)$ denote the number of labeled ℓ-colorable graphs on n vertices, say on $n = \{0, \ldots, n-1\}$, and let $S_\ell(n)$ denote the number of labeled $K_{\ell+1}$-free graphs on $\{0, \ldots, n-1\}$. Then for every polynomial $p(n)$ there exists a constant c such that for all n we have $S_\ell(n) \leq L_\ell(n)(1 + \frac{c}{p(n)})$. Observe that $L_\ell(n) \leq S_\ell(n)$ is trivially true. Using the Theorem we are able to show that also almost all unlabeled $K_{\ell+1}$-free graphs are ℓ-colorable.

Corollary 2.3 *Let $\ell \geq 2$. Let $S_\ell^u(n)$ denote the number of unlabeled $K_{\ell+1}$–free graphs on n vertices and let $L_\ell^u(n)$ denote the number of unlabeled ℓ–colorable graphs on n vertices. Then for any polynomial $q(n)$ there is a constant d such that for all n*

$$L_\ell^u(n) \leq S_\ell^u(n) \leq L_\ell^u(n)(1 + \frac{d}{q(n)}) .$$

□

Using the asymptotic formula for labeled ℓ–colorable graphs (Prömel, 1986a) one obtains immediately an asymptotic formula as well for the number of unlabeled $K_{\ell+1}$–free graphs as for the number of unlabeled ℓ–colorable graphs, viz.

Corollary 2.3a *Let $\ell \geq 2$. Then*

$$L_\ell^u(n) \sim S_\ell^u(n) = (1 + 0(\frac{1}{n}))\frac{\beta}{\ell!}\left(\sqrt{\frac{\ell}{\pi}}\right)^\ell 2^{\frac{r^2}{2\ell} - \frac{\ell}{2}} \cdot 2^{\frac{\ell-1}{2\ell}n^2 - n \log n + n \log e\ell - \frac{\ell}{2} \log n}$$

where

$$\beta = \beta(n) = \sum_{k=0}^{\infty} 2^{rk + \frac{\ell k^2}{2}} \sum_{P(k,\ell,r)} 2^{-\frac{1}{2}\sum_{i\in\ell} m_i^2} ,$$

with $r = n \bmod \ell$ and $P(k,\ell,r)$ denotes the set of all ordered partitions $m_0, \ldots, m_{\ell-1}$ of $\ell k + 1$ with at least one part vanishing.

□

2.4 0–1 laws for some classes of graphs

Let $\mathcal{K}$ be an infinite class of finite labeled undirected graphs and let φ be a property of graphs expressible by a sentence of first–order logic. Moreover, let $\mathcal{K}(n)$ denote those graphs in $\mathcal{K}$ on n vertices, i.e., on $\{0, \ldots, n-1\}$ and let $\mu_n(\varphi)$ be the fraction of graphs in $\mathcal{K}(n)$ satisfying φ . Then the (labeled) asymptotic probability $\mu(\varphi)$ of φ on $\mathcal{K}$ is given by $\mu(\varphi) = lim_{n\to\infty} \mu_n(\varphi)$, provided that this limit exist.

Let $\mathcal{K}^u$ be a class of representatives from isomorphism classes in $\mathcal{K}$, i.e., $\mathcal{K}^u$ is the class of unlabeled graphs corresponding to $\mathcal{K}$. Then the (unlabeled) asymptotic probability of $\nu(\varphi)$ of φ on $\mathcal{K}^u$ is defined in the same way as $\mu(\varphi)$.

Of particular interest are classes $\mathcal{K}$, $\mathcal{K}^u$ resp., of graphs which have the property that for any first order property φ the asymptotic probabilities $\mu(\varphi), \nu(\varphi)$ resp., exist and are either 0 or 1. In this case $\mathcal{K}$, $\mathcal{K}^u$ resp., is said to have a $0-1$ law. Fagin (1976) showed, for example, that as well the class $\mathcal{G}$ of all labeled graphs as the class $\mathcal{G}^u$ of all unlabeled graphs have a $0-1$ law.

Let $\ell \geq 2$ and let $\mathcal{S}_\ell$ denote the class of all labeled $K_{\ell+1}$-free graphs. In Kolaitis, Prömel and Rothschild (1985) it is proved that the class $\mathcal{S}_\ell$ has a $0-1$ law. Using this result in connection with the Theorem we obtain immediately:

Corollary 2.4 *Let $\ell \geq 2$. Then the class $\mathcal{S}_\ell^u$ of unlabeled $K_{\ell+1}$-free graphs has a $0-1$ law.*

□

Observe that $\mathcal{S}_\ell^u$ is a class of graphs which is closed under induced subgraphs and which has the amalgamation property. A complete classification of all classes of graphs having these properties is given in Lachlan and Woodrow (1980). For a detailed discussion of these classes, see also Kolaitis, Prömel and Rothschild (1985). For those classes which are 'slowly growing' and which are closed under disjoint unions and components $0-1$ laws follow from the work of Compton (1984). As mentioned before, Fagin (1976) proved a $0-1$ law for the class $\mathcal{G}^u$ of all unlabeled graphs. Beyond these classes, there are essentially two possibilities for classes of unlabeled graphs closed under induced subgraphs and having the amalgamation property, namely the classes $\mathcal{S}_\ell^u$ of unlabeled $K_{\ell+1}$-free graphs, which are covered by Corollary 2.4, and the classes $\mathcal{E}_\ell^u$ of unlabeled equivalence graphs with at most ℓ components.

Since it is easy to see that for every $\ell \geq 1$ the class $\mathcal{E}_\ell^u$ has a $0-1$ law (cf. Prömel, 1986) we can conclude

Theorem 2.4a *Let $\mathcal{C}^u$ be any infinite class of finite undirected unlabeled graphs having the amalgamation property and closed under induced subgraphs. Then $\mathcal{C}^u$ has a $0-1$ law.*

□

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