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Set theory and infinitary binomial coefficients

by J.W. Degen - September, 1986

Introduction

First some definitions.
Let X and Y be sets. We write $X \leqslant Y$ to mean that there is an injection from X into Y , and $X \sim Y$ will mean that there is a bijection between X and Y . Occasionally we shall need the notation $X \leqslant^* Y$, by which we mean that either X is empty or Y can be surjected onto X .
The expression $\binom{X}{Y}$ denotes the set of all subsets Z of X such that $Z \sim Y$, and is called a **binomial coefficient**, with **basis X** and supponent Y . A binomial coefficient is called **finitary** (**infinitary**) if its basis is a finite (infinite) set. It is well known that infinitary binomial coefficients play **the** central role in infinitary combinatorics, notably in Ramsey theory.

Now, the present paper is **not** about Ramsey theory, that is: nothing like homogeneous sets w.r.t. partitions of $\binom{X}{Y}$ will be considered. The aim of our paper is a more modest one: viz. to establish **size comparisons** between binomial coefficients, and between them and other expressions. We shall restrict the range of our investigations even further: We shall consider only binomial coefficients with **finite supponents**, which are written as $\binom{X}{n}$, where n is a (settheoretic) natural number.

These delineations stated, the following question arises:
What nontrivial things can one say about the size of $\binom{X}{n}$, when X is infinite? For if X is infinite, and m, n are any positive integers, then $X \sim \binom{X}{m} \sim \binom{X}{n}$. (*)

Okay. The statement (*) certainly follows from the axiom of choice (AC), and we shall see below, that in turn AC is implied by the special case: For all infinite X : $X \sim \binom{X}{2}$. As an exercise, let us prove that AC implies (*). Let us denote by $\binom{X}{<\omega}$ the set of finite subsets of X . It suffices to show (using AC) that for all infinite X : $\binom{X}{<\omega} \leqslant X$.

Let $m \in \omega$, and let $f \in X^m$ (i.e. f is a function from m into X). Set $I(f) := \text{range}(f)$. Then I is a surjection of $\bigcup_{m\in\omega} X^m$ onto the set $\binom{X}{<\omega}$, i.e. we have $\binom{X}{<\omega} \leqslant^* \bigcup_{m\in\omega} X^m$.
Using AC we have $\binom{X}{<\omega} \leqslant \bigcup_{m\in\omega} X^m$. Furthermore, since X is infinite, we have, again by AC, $X \sim \aleph_\alpha$ for some $\alpha \in$ On. Since $\aleph_\alpha \sim \aleph_\alpha^n$ for all positive integers n, it follows

$$\bigcup_{m\in\omega} X^m \leqslant \aleph_\alpha \cdot \aleph_0 \sim \aleph_\alpha \sim X .$$

Probably there is a shorter and more elegant proof of (*) from AC. But our proof illuminates some typical arguments used in size comparisons between infinite sets (infinite cardinals):

1) The passage from $\leqslant^*$ to $\leqslant$.
2) The passage from X is infinite to $X \sim \aleph_\alpha$ for some α,
   or if we had not made **this** passage:
3) $X \cdot X \sim X$, if X is infinite,
4) $X \cdot \aleph_0 \sim X$, if X is infinite.

However, if we drop AC, many curious problems arise. So let ZF be the Zermelo-Fraenkel set theory without AC. Then which of the following assertions - which are all trivially ZF+AC-provable - will remain ZF provable?

a) $\forall X\ (X \text{ infinite} \rightarrow 100 \cdot \binom{X}{2} \leqslant \binom{X}{3})$
b) $\forall X\ (X \text{ infinite} \rightarrow \binom{X}{3} \leqslant 100 \cdot \binom{X}{2})$
c) $\forall X\ (X \text{ infinite} \rightarrow \binom{X}{2} \leqslant X^2)$, where X^2 is the cartesian square.
d) $\forall X\ (X \text{ infinite} \rightarrow X^2 \leqslant \binom{X}{2})$.
e) $\forall X\ (\omega \leqslant \binom{X}{2} \rightarrow \omega \leqslant X)$.

the answers to these questions will turn out as follows

a): YES ; b): NO ; c): NO ; d): NO ; e): NO .

Of course, the numbers 2,3 and 100 in a) and b) are only examples; and the harmony of mathematics demands that they can be subsumed under a general structure.

This demand can be indeed fulfilled, viz. by our **Main Theorem** below.

Let me close this introduction by mentioning that our choiceless theory of infinitary binomial coefficients with finite supponents has a recursion theoretic counterpart. After defining the recursion-theoretic analogs of the set-theoretic notions, we can, roughly, show:

A) If ZF proves something, then "it" is recursively true.

B) If something is recursively false, then ZF refuses to prove "it".

- the "something" refers, of course, only to a certain very special kind of statements about our binomial coefficients.

Let us begin with the detailed development.

All of the consistency results we need can be found at least implicitly in **Jech:** The axiom of choice, 1973.

As mentioned in the Introduction, our background theory is ZF without any further choice principles. First we need some further definitions. Once and for all the notion "infinite" is used as follows: A set X is called **infinite**, if $n \leqslant X$ for all $n \in \omega$. Otherwise X is called **finite** . In ZF we can prove that the following are equivalent for a set X :

X is infinite,

not $n \sim X$ for all $n \in \omega$,

$n \leqslant^* X$ for all $n \in \omega$.

A set X is called $\mathcal{D}$**-infinite** (**Dedekind-infinite**), if $\omega \leqslant X$.
In ZF the $\mathcal{D}$-infinity of X is equivalent to the existence of a nonsurjective injection from X into X .

X is called $\mathcal{D}$**-finite**, if X is not $\mathcal{D}$-infinite. Clearly, any $\mathcal{D}$-infinite set is infinite. On the other hand we have

1. Fact

It is consistent with ZF that there is an infinite set X such that $\mathcal{P}(X)$, the power set of X , is $\mathcal{D}$-finite. (Of course, then X itself is $\mathcal{D}$-finite).

Let X be an infinite set such that $X \neq X_1 + X_2$ (disjoint union) for all infinite sets X_1 and X_2 . Then X is called **amorphous**.

2. Fact

The existence of amorphous sets is consistent with ZF .

Let X be an infinite set and n a positive integer. If $\binom{X}{n} \leqslant X$ we say that X is **n-fold infinite**.

We are now ready for our first theorem, whose proof already exemplifies the kind of tricks to be used later on.

3. Theorem

3.1 $ZF \vdash X$ is 2-fold infinite $\rightarrow X$ is $\mathcal{D}$-infinite.

3.2 $ZF \nvdash X$ is $\mathcal{D}$-infinite $\rightarrow X$ is 2-fold infinite.

Proof

Ad 3.1

Let X be 2-fold infinite and let $I : \binom{X}{2} \rightarrow X$ be injective. Take three different elements a,b and c from X . Define the function J as follows

$$
\begin{array}{llcl}
 & & J & \\
 & a & \longmapsto & \{a,b\} \\
 & b & \longmapsto & \{b,c\} \\
 & c & \longmapsto & \{a,c\} \\
\text{new} & \cdots & \cdots\cdots\cdots & \cdots \\
 & x & \longmapsto & \{x,c\} \\
 & y & \longmapsto & \{y,c\} \\
 & z & \longmapsto & \{z,c\}
\end{array}
$$

And so on for all of $X \setminus \{a,b,c\}$

Applying first J and then I we can inject X into a proper subset of itself.

(The reader may convince himself, that this argument uses indeed only the means available in ZF besides the supposed injection I.)

Ad 3.2

We have to show that consistently with ZF there can be a $\mathcal{D}$-infinite set X which is not 2-fold infinite.

Using Fact 2, it suffices to show in ZF: (*) If A is an amorphous set such that $A \cap \omega = \emptyset$, then $X := A \cup \omega$ is not 2-fold infinite. For if we have an amorphous set, we can make it trivially disjoint with ω , and clearly X is $\mathcal{D}$-infinite.

So let us show (*).
Suppose X is 2-fold infinite via the injection $I : \binom{X}{2} \rightarrow X$.
We distinguish the following cases:

Case 1. There are infinitely many $n \in \omega$ such that $I^{-1}(n) \in \binom{A}{2}$. Then we have an ω-sequence $(P_i)_{i\in\omega}$ of termwise distinct elements of $\binom{A}{2}$. From $(P_i)_{i\in\omega}$ we can easily define an ω-sequence $(q_i)_{i\in\omega}$ of termwise **disjoint** elements of $\binom{A}{2}$. Then $\bigcup_{i\in \text{even}} q_i$ and $\bigcup_{i\in \text{od}} q_i$ split a subset of A into two infinite sets. This is a contradiction to the amorphousness of A .

Case 2. There are infinitely many $b \in A$ such that $I^{-1}(b) \in \binom{A}{2}$. Within Case 2 we distinguish two further cases.

Case 2.1. There are infinitely many $c \in A$ such that $I(\{0,a\}) = c$ for some $a \in A$.

Case 2.2. There are infinitely many $n \in \omega$ such that $I(\{0,a\}) = n$ for some $a \in A$.

Now in both subcases we can split A into two infinite subsets. Again a contradiction. $\square$

As mentioned in the Introduction, $\binom{X}{2}$ and X^2 are in general $\leqslant$-incomparable on the basis of ZF . We shall prove this below (Theorems 5 and 6). Nevertheless, there are cases where $\binom{X}{2}$ and X^2 behave the same way, if X is infinite. Recall, that one can prove in ZF:

$$AC \longleftrightarrow \forall \text{infinite } X : X^2 \leqslant X .$$

(To be precise, one needs $X^2 \sim X$ for the direction $\longleftarrow$. However, since $X \leqslant X^2$ is trivial, $X^2 \sim X$ follows by the ZF-provable Schröder-Bernstein-theorem from $X^2 \leqslant X$.)

4. Theorem

$ZF \vdash (\forall \text{infinite } X : \binom{X}{2} \leqslant X) \rightarrow AC$. i.e. AC is ZF-implied by "every infinite set is 2-fold infinite".

Proof:

First we show

(1) $ZF \vdash X$ is infinite $\rightarrow 2 \cdot X \leqslant \binom{X}{2}$.

To prove (1), take a five element subset $A = \{a_o, \ldots, a_4\}$ from X . Then obviously $2 \cdot A \leqslant \binom{A}{2}$. We have to inject into $\binom{X}{2} \setminus \binom{A}{2}$ the remaining set $2 \cdot X \setminus 2 \cdot A$. Let $x \in X \setminus A$. Then we map $\langle 0, x \rangle$ to $\langle x, a_o \rangle$ and $\langle 1, x \rangle$ to $\langle x, a_1 \rangle$. This proves (1).

(2) We use now the hypothesis that every infinite set is 2-fold infinite. Using (1) we have: For every infinite set X :

$$\binom{X}{1} + \binom{X}{2} \leqslant \binom{X}{2} .$$

To complete our proof, it is sufficient to show that $X^2 \leqslant X$ for an arbitrary infinite set X .

The cartesian square X^2 of X is the set $\{\{\{x\}, \{x,y\}\} : x,y \in X\}$.

Thus X^2 is a certain subset of

$$Z := \binom{\binom{X}{1} + \binom{X}{2}}{1} + \binom{\binom{X}{1} + \binom{X}{2}}{2} .$$

Using (2) we get the following chain:

$$Z \leqslant \binom{\binom{X}{1} + \binom{X}{2}}{2} \leqslant \binom{X}{1} + \binom{X}{2} \leqslant \binom{X}{2} \leqslant X .$$

Thus, since $X^2 \subseteq Z$, $X^2 \leqslant X$.

5. Theorem

It is consistent with ZF that there is an infinite set X such that **not** $X^2 \leqslant \binom{X}{2}$. Moreover, X can be chosen as a set of real numbers.

Proof:

First we show in ZF:

(1) If X is an infinite totally ordered set (say by $\sqsubset$), then $2\cdot\binom{X}{2} \leqslant X^2$.

This is so, because an element of $2\cdot\binom{X}{2}$ is either of the form $<0,\{a,b\}>$ or $<1,\{a,b\}>$ and because we can use the total order $\sqsubset$ on X as follows:

If $a \sqsubset b$, we map $<0,\{a,b\}>$ to $<a,b>$ and $<1,\{a,b\}>$ to $<b,a>$.

If $b \sqsubset a$, we procede just dually, of course.

Next we show in ZF:

(2) Let X be an infinite $\mathfrak{D}$-finite set which carries a total order, say $\sqsubset$. Then $\binom{X}{2}$ is also an infinite $\mathfrak{D}$-finite set.

(Proof of (2): $\binom{X}{2}$ is indeed $\mathfrak{D}$-finite. For suppose we could inject ω into $\binom{X}{2}$, then we could first construct an ω-sequence of termwise disjoint elements $\in\binom{X}{2}$. Then using the total order $\sqsubset$ we could chose the $\sqsubset$-smallest in each term of this ω-sequence, thereby manufacturing an ω-sequence in X and proving the contradiction that X is $\mathfrak{D}$-infinite.)

(3) Fact

"There is an infinite $\mathfrak{D}$-finite subset of the reals" is consistent with ZF. Call this set X; X is totally ordered by $\sqsubset$ = the restriction of the usual ordering of the reals to X.

Suppose now we had $X^2 \leqslant \binom{X}{2}$. By (1) it would follow $2\cdot\binom{X}{2} \leqslant \binom{X}{2}$. This means, that $\binom{X}{2}$ can be injected into a proper subset of itself. Thus $\binom{X}{2}$ would be Dedekind infinite, which is a contradiction to (2). $\square$

Remark

Supplement to Thm 5:

By the same methods as in the proof just given one may show:
It is consistent with ZF that there is an infinite subset Y of the reals such that for all $n \geqslant 2$: **not** $X^n \leqslant \binom{Y}{n}$.

On the other hand, we leave it as an exercise to prove in ZF:
If X is an infinite totally ordered set and $n \in \omega$, then $\binom{X}{n} \leqslant X^n$.

Thus to refute $\binom{X}{2} \leqslant X^2$ we must look for sets which cannot be totally ordered.

6. Theorem

It is consistent with ZF that there is an infinite set X such that not $\binom{X}{2} \leqslant X^2$.

Proof:

Transferring the second Fraenkel model (Jech, p. 48) to a symmetric model of ZF we have:

(1) It is consistent with ZF that there exists an ω-sequence $(P_i)_{i\in\omega}$ of pairwise disjoint unordered pairs $P_i = \{a_i, b_i\}$ such that $X := \bigcup_{i\in\omega} P_i$ is $\mathcal{D}$-finite (although trivially infinite). We consider this set **X** .

(2) $ZF \vdash X^2$ is also $\mathcal{D}$-finite (exercise)

(3) $ZF \vdash \binom{X}{2}$ is $\mathcal{D}$-infinite (immediate from the def. of X).

From (2) and (3) we have $ZF \vdash$ **not** $\binom{X}{2} \leqslant X^2$. $\square$

Incidentally, the proof of Thm. 6 yields another interesting fact which we state as our next theorem.

7. Theorem

$ZF \nvdash \forall X : \omega \leqslant \binom{X}{2} \rightarrow \omega \leqslant X^2$

$ZF \nvdash \forall X : \omega \leqslant \binom{X}{2} \rightarrow \omega \leqslant X$.

Exercise:

Show in ZF:

$$\omega \leqslant \binom{X}{2} \rightarrow \omega \leqslant^* X .$$

Remark

It is an open problem whether:

$ZF \vdash \forall X : \omega \leq^* \binom{X}{2} \rightarrow \omega \leq^* X$.

By now we have given exactly the announced answers to questions c), d), e). We now turn to questions a) and b).

We introduce first some new notions. Let X be a variable and $k_o, \ldots, k_n$ with $k_n \neq 0$ be natural numbers (more precisely: **terms** for natural numbers). Then we call the expression

$$k_o \cdot \binom{X}{0} + k_1 \cdot \binom{X}{1} + \ldots + k_n \cdot \binom{X}{n}$$

a **Pascal polynomial (over ω) of degree n.**

If X is interpreted as a set, then the Pascal polynomial gets the expected interpretation: + as disjoint union, $\cdot$ as cartesian product, and $\binom{X}{\ell}$ as the set of all ℓ-element subsets of X . If X is a natural number, then the value of the polynomial is taken to be without much ado as the natural number representing the cardinality of the resulting finite set.

8. MAINTHEOREM

Let E and F be two Pascal polynomials such that $E(n) < F(n)$ (strictly) for sufficiently large $n \in \omega$. Then we have

8.1 $ZF \vdash \forall \text{infinite } X : E(X) \leq F(X)$.

8.2 $ZF \nvdash \forall \text{infinite } X : F(X) \leq E(X)$.

Proof of the Main Theorem:

We need the

Lemma I

In ZF one can prove:

Let k, $n \in \omega$ and let X be an infinite set. Then $k \cdot \binom{X}{n} \leq \binom{X}{n+1}$.

Proof of Lemma I:

If either k or n is zero, the statement is trivial. Thus we may assume that k and n are both positive.
We shall make use of the following two easy properties of **finitary binomial coefficients:**

(A) $\quad k\cdot\binom{k\cdot(n+1)+n}{n} = \binom{k\cdot(n+1)+n}{n+1}$

(B) $\quad r \leqslant n-1 \rightarrow k\cdot\binom{k\cdot(n+1)+n}{r} \leqslant \binom{k\cdot(n+1+n}{r+1}$

Let now the infinite set X be given. We take a set $A = \{a_1,\ldots,a_\ell\}$ of $\ell = k\cdot(n+1)+n$ elements from X . By (A) we have $k\cdot\binom{A}{n} \leqslant \binom{A}{n+1}$.

The remaining task is the following:
We have to inject $k\cdot\binom{X}{n} \setminus k\cdot\binom{A}{n}$ into $\binom{X}{n+1} \setminus \binom{A}{n+1}$ **using only the means of ZF** .

Now, an element $p \in k\cdot\binom{X}{n} \setminus k\cdot\binom{A}{n}$ has the form $\langle i,\{\alpha_1,\ldots,\alpha_n\}\rangle$, where $i = 0,\ldots,k-1$ and $\alpha_\nu \in X$.

Let r be the number of the α_ν's belonging to A . Obviously $0 \leqslant r \leqslant n-1$. We transform p to an element of $\binom{X}{n+1} \setminus \binom{A}{n+1}$ as follows: We drop the first coordinate i of p . In the resulting set $\{\alpha_1,\ldots,\alpha_n\}$ we retain all $\alpha_\nu \in X \setminus A$, but replace the r A-elements by r+1 A-elements. The property (B) tells us that this transformation can be done injectively. In fact we must **choose** injections, but only a finite number of times, which is allowable in ZF.

The Lemma I is proved. However, it may be useful to illustrate the proof by a special case:
Let X be infinite. Show in ZF that

$$2\cdot\binom{X}{2} \leqslant \binom{X}{3} .$$

Take $8 = 2\cdot 3+2$ elements $A = \{a_1,\ldots,a_8\}$ from X .
Then $2\cdot\binom{A}{2} \leqslant \binom{A}{3}$, as one may verify just for fun in this special case.

So we have to inject $2\cdot\binom{X}{2}\setminus 2\binom{A}{2}$ into the set $\binom{X}{3}\setminus\binom{A}{3}$. Let then $p\in 2\cdot\binom{X}{2}\setminus 2\cdot\binom{A}{2}$. p is of one of the following four forms:

$<0,\{x,y\}>, <0,\{x,a\}>$ where $x,y\in X\setminus A$,

$<1,\{x,y\}>, <1,\{x,a\}>$ $a\in A$.

Send

$<0,\{x,y\}>$ to $\{x,y,a_1\}$, and

$<1,\{x,y\}>$ to $\{x,y,a_2\}$.

This disposes of the first column.

Now we know that $2\cdot\binom{8}{1}\leqslant\binom{8}{2}$. Again we can use an injection $I : 2\cdot\binom{A}{1}\leqslant\binom{A}{2}$ which disposes of the second column, by transforming $\underset{i=0,1}{<i,\{x,a\}>}$ into an $\{x,a',a''\}$, $a,a',a''\in A$. □

Remark

Lemma I is a sort of "Compactness argument" in so far as it jumps form **finitary** binomial coefficients to **infinitary** binomial coefficients. Though the idea is quite simple, namely the breaking of a task into two subtasks, it is remarkable that this can be done **constructively**, i.e. in ZF, whereas other compactness arguments like König's Lemma (for **arbitrary** binary trees) or the compactness theorem of first order logic (for not necessarily well-ordered languages) cannot be carried out in ZF, without AC .

There are many theorems of the same type as Lemma I . For the purpose of illustration, define $F(X;n)$ to be the set $\{f : f$ is a function from an n-element subset of X into $X\}$. Adapting the main ideas of the proof of Lemma I we can show:

$ZF\vdash$ For any $k, n\in\omega$ and any infinite set $X : k\cdot F(X;n)\lessdot F(X;n+1)$.

We now show how Lemma I yields the first part of the Main Theorem, and how Lemma I together with Fact 1 yields the second part of the Main Theorem.

Let $E(n) < F(n)$ for sufficiently large n .

Then we have two cases:

Case 1: degree of $F >$ degree of E .

Case 2: degree of F = degree of $E = d$, and for some $n \leqslant d$ the coefficient of $\binom{X}{n}$ in F is greater than the coefficient of $\binom{X}{n}$ in E . Let N be the largest such n .

We prove first 8.1.
Suppose we have Case 1. Let d_1 = degree of E , d_2 = degree of F ; $d_1 < d_2$. Repeated application of Lemma I yields 8.1. Again, in **Case 2**, Lemma I, repeatedly applied yields the desired result.

Now we prove 8.2.
By Fact 1 we have, consistently with ZF, an infinite set X , such that $\mathcal{P}(X)$ is $\mathcal{D}$-finite. It follows that $\binom{X}{n}$ is $\mathcal{D}$-finite for every n .
Take Case 1 first. Let w.l.o.g. $d+1$ be the degree of F and $d > 0$ the degree of E .

Suppose we had $F(X) \leqslant E(X)$.
A fortiori we would then have
$\binom{X}{d+1} \leqslant k_d \cdot \binom{X}{d} + \ldots [=E(X)]$.
We multiply this $\leqslant$-relation on both sides by 2. Then by Lemma I:

$2 \cdot \binom{X}{d+1} \leqslant \binom{X}{d+1}$. But this means that $\binom{X}{d+1}$ is $\mathcal{D}$-infinite.

Contradiction.

Finally we dispose of Case.2. (See the def. of N above). Let d be the common degree of E and F , and let **first $d = N$** .

Then if $F(X) \leqslant E(X)$ we have

$$k_N \binom{X}{N} \leqslant k'_N \binom{X}{N} + \ldots [=E(X)] \quad , \text{ where } k_N > k'_N .$$

Then $2 \cdot k_N \binom{X}{N} \leqslant 2 \cdot k'_N \binom{X}{N}$ by Lemma I .

But this means that $2 \cdot k'_N \binom{X}{N}$ and as one easily shows, hence $\binom{X}{N}$ is Dedekind infinite. Contradiction.
Secondly, the subcase $d > N$ is treated similarily.

This completes the proof of Theorem 8.

Remark

We may extend our Main Theorem to the case of **iterated** binomial coefficient, but refrain from that in order to avoid boredom. However, we propose the following Exercise

$$ZF \vdash \forall \text{infinite } X : \binom{X}{4} \leqslant \binom{\binom{X}{2}}{2} ,$$

$$ZF \nvdash \forall \text{infinite } X : \binom{\binom{X}{2}}{2} \leqslant \binom{X}{4} .$$

Infinite, $\mathcal{D}$-finite sets are clearly sufficient for such global independence results as in 8.2. However, there are "singular" independence results of the same type as in 8.2, which can be obtained concerning **Dedekind-infinite** sets.

9. Theorem

The following is consistent with ZF:
There is a Dedekind infinite set X such that **not** $\binom{X}{3} \leqslant \binom{X}{2}$.

Proof:

Define $X := A \cup \omega$, where $A \cap \omega = \emptyset$ and A is amorphous. Then proceed more or less the same way as in the proof of 3.2.

By AC(n) we mean that every family of n element sets possesses a choice function.

10. The Blowing Up Theorem

Let $n \geqslant 2$. Then ZF + AC(n+1) proves:
If X is n-fold infinite, then X is n+1-fold infinite.

Remark

We cannot take n = 1 in Thm 10. For then we would prove in ZF+AC(2) the full AC ; which is impossible.

Proof:

We show the Thm for $n = 2$. The general case follows the pattern exemplified in this case.

Let the 2-fold infinite set X be given. Then we have an injection $f : \binom{X}{2} \to X$. By AC (3) we have a choice function c on $\binom{X}{3}$.
Define a function $g : \binom{X}{3} \to \binom{\binom{X}{2}}{2}$ as follows:

$$\{x,y,z\} \xrightarrow{g} \{\{x,y\},\{y,z\}\} \text{ , where } y = c(\{x,y,z\}) \text{ .}$$

g is injective.

Then define a function $h : \binom{\binom{X}{2}}{2} \to \binom{X}{2}$ as follows:

$$\{\{x,y\},\{u,u\}\} \xrightarrow{h} \{f(\{x,y\}),f(\{z,u\})\} \text{ .}$$

h is also injective.

To inject $\binom{X}{3}$ into X we first apply g , then h , and finally f . □

Problem

Can we eliminate AC(n+1) in the above argument?

Perhaps there is a simple trick that I have overlooked!

Supplementary Remarks on the Main Theorem

First notice, that the unprovability part 8.2 is witnessed by one infinite set X such that $\mathcal{P}(X)$ is $\mathcal{D}$-finite.

During the proof of Thm 6 we encountered a $\mathcal{D}$-finite set X such that $\binom{X}{2}$ and hence $\mathcal{P}(X)$ is $\mathcal{D}$-infinite. This set therefore cannot be a witness for 8.2 $\left[\text{since } \binom{X}{2} + 1 \preccurlyeq \binom{X}{2}\right]$.

On the other hand, if we consider ordinary polynomials over ω , i.e. polynomials whose monomials are of the form $k \cdot X^n$, then we can prove the following:

Let P and Q be ordinary polynomials such that $P(n) < Q(n)$ for sufficiently large n . Then

8.1* $ZF \vdash \forall$infinite X : $P(X) \leq Q(X)$, and the fact

8.2* $ZF \nvdash \forall$infinite X : $Q(X) \leq P(X)$ can be witnessed by an
infinite $\mathcal{D}$-finite set X .

Since $\binom{n}{2} < n^2$ for sufficiently large n , and $ZF \nvdash \forall$infinite X : $\binom{X}{2} \leq X^2$, the first part of our Main Theorem fails for "mixed polynomials". Observe that the mixed polynomial identity $\forall$ set X : $2 \cdot \binom{X}{2} + X \sim X^2$, is ZF-unprovable.

After all these curiosities about binomial coefficients in ZF, it comes as a big relief to learn that ZF can prove the **Binomial Theorem for Sets** in the following version:

Let X and Y be any two disjoint sets, and $n \in \omega$. Then

$$(X+Y)^n \sim \sum_{i=0}^{n} \binom{n}{i} \cdot X^{n-i} \cdot Y^i \quad ,$$

Where all of the notation is to be understood in the set-theoretic sense.

The proof is left as an exercise. $\square$

And also the fundamental recursive equation

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

can be proved in ZF in the following generalization.
Let $k > 0$, X any set, and $a \notin X$. Then

$$\binom{X+\{a\}}{k} \sim \binom{X}{k} + \binom{X}{k-1} \quad .$$

The essential content of the paper may be recovered from the accompanying figure, which depicts a transfinite extension of Pascal's triangle.

If Zermelo's Ac is to be satisfied, then no universe of set theory with such a triangle can exist. I think that combinatorialists owe such a great debt to Pascal, that they should help to save his Transfinite Triangle against the onslaught of arbitrary choice.

Reference: Jech, "The axiom of choice", 1973

Extending Pascal's Triangle to the Transfinite

Here A is an infinite set such that $\mathcal{P}(A)$ is Dedekind finite. The arrow $X \rightarrow Y$ in the figure means that $X \leqslant Y \wedge \neg\, Y \leqslant X$. Observe that $\binom{A}{i} \sim \binom{A}{A-i}$ for all $i \in \omega$, where $A-i$ denotes a set which arises from A by subtracting i elements. Thus the transfinite levels of this triangle, starting with A, is completely symmetric – without an axis of symmetry. $\rightarrow \cdots \leftarrow$ will meet in chaos.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

. . . .

and so on...

$$1 = \binom{A}{0} \rightarrow \binom{A}{1} \rightarrow \binom{A}{2} \rightarrow \binom{A}{3} \rightarrow \cdots \qquad \cdots \leftarrow \binom{A}{A-3} \leftarrow \binom{A}{A-2} \leftarrow \binom{A}{A-1} \leftarrow \binom{A}{A} = 1$$

$$1 \rightarrow 1+\binom{A}{1} \rightarrow \binom{A}{1}+\binom{A}{2} \rightarrow \binom{A}{2}+\binom{A}{3} \rightarrow \cdots \qquad \cdots \leftarrow \binom{A}{A-3}+\binom{A}{A-2} \leftarrow \binom{A}{A-2}+\binom{A}{A-1} \leftarrow \binom{A}{A-1}+1 \leftarrow 1$$

$$1 \rightarrow 2+\binom{A}{1} \rightarrow 1+2\binom{A}{1}+\binom{A}{2} \rightarrow \binom{A}{1}+2\binom{A}{2}+\binom{A}{3} \rightarrow \cdots \qquad \cdots \leftarrow \binom{A}{A-3}+2\binom{A}{A-2}+\binom{A}{A-1} \leftarrow \binom{A}{A-2}+2\binom{A}{A-1}+1 \leftarrow \binom{A}{A-1}+2 \leftarrow 1$$

AND SO ON...