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REMARKS ON THE SYMBOLIC METHOD IN INVARIANT THEORY OF BINARY FORMS.

by

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This paper is about work in progress. It is mainly concerned with the notions of *umbral space* and *umbral operator* as they are found, for instance, in [1].

These concepts, though useful, nevertheless maintain a certain air of charming and mysterious witchcraft. Our main concern in this research has been to dissolve this magical atmosphere by giving them a natural justification.

The course we shall follow seems likely to succeed. In fact, the interpretation we shall describe gives quite trivial proof to some theorems and allows a generalization of the results in the case of m-ary forms. Moreover, this interpretation enables us to separate the algebraic from the combinatorial aspects of the problem.

1.- Let us consider a binary form of degree n in the variables x^1, x^2 :

$$f(x^1,x^2) = \sum_{p=o}^{n} \binom{n}{p} a_p (x^1)^p (x^2)^{n-p} \tag{1}$$

over a field K of characteristic zero.

Under a linear change of variables

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{pmatrix} \begin{pmatrix} \bar{x}^1 \\ \bar{x}^2 \end{pmatrix}, \qquad \Delta = c_1^1 c_2^2 - c_2^1 c_1^2 \neq 0, \tag{2}$$

(1) is transformed into

$$(\bar{1}) \qquad \bar{f}(\bar{x}^1,\bar{x}^2) = \sum_{p=o}^{n} \binom{n}{p} \bar{a}_p (\bar{x}^1)^p (\bar{x}^2)^{n-p}.$$

One may easily check that the coefficients $\bar{a}_p$ are connected with the a_p by the linear invertible transformation

$$(3) \qquad \bar{a}_p = \sum_{q=o}^{n} C_p^q a_q$$

where the matrix (C_p^q) is given by

$$(4) \qquad C_p^q = \sum_{i_1+\ldots+i_n=2n-q} c_1^{i_1} \ldots c_1^{i_p} c_2^{i_{p+1}} \ldots c_2^{i_n}.$$

Let us denote with $\begin{pmatrix} d_1^1 & d_2^1 \\ d_1^2 & d_2^2 \end{pmatrix}$ the inverse matrix of $\begin{vmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{vmatrix}$.

Then

$$(5) \qquad a_p = \sum_{q=o}^{n} D_p^q \bar{a}_q$$

with

$$(6) \qquad D_p^q = \sum_{i_1+\ldots+i_n=2n-q} d_1^{i_1} \ldots d_1^{i_p} d_2^{i_{p+1}} \ldots d_2^{i_n}.$$

Consider the polynomial algebra $\mathcal{P} := K[A_o,\ldots,A_n,u^1,u^2]$. A *covariant of index g* is a polynomial

$$I(A_o,\ldots,A_n,u^1,u^2) \in \mathcal{P}$$

such that

$$(7) \qquad I(\bar{a}_o,\ldots,\bar{a}_n,\bar{x}^1,\bar{x}^2) = \Delta^g \, I(a_o,\ldots,a_n,x^1,x^2)$$

for every form (1) and every change of variables (2). A covariant in which the variables u^1,u^2 do not occur is said to be an *invariant*.

The *umbral space* $\mathcal{U}$ is the algebra of polynomials $K[\alpha_1,\alpha_2,\ldots,\lambda_1,\lambda_2,u^1,u^2]$ in d pairs of Greek variables and in a pair of Roman variables.

The *umbral operator* U is defined as the linear operator

$$
(8)\quad \begin{array}{rcl} U: \quad \mathcal{U} & \longrightarrow & \mathcal{P} \\ Q(\alpha_1,\alpha_2,\dots,u^1,u^2) & \longmapsto & P(A_0,\dots,A_n,u^1,u^2) = \\ & & = \langle U | Q(\alpha_1,\alpha_2,\dots,u^1,u^2)\rangle \end{array}
$$

which satisfies the following conditions:

(i) $\langle U|\alpha_1^i\alpha_2^j\rangle = A_i\delta_n^{i+j}$ for every Greek letter α

(ii) $\langle U|(u^s)^i\rangle = (u^s)^i, \quad s=1,2$

(iii) $\langle U|\alpha_1^k\alpha_2^h \dots \lambda_1^p\lambda_2^q(u^1)^i(u^2)^j\rangle = \langle U|\alpha_1^k\alpha_2^h\rangle \dots \langle U|\lambda_1^p\lambda_2^q\rangle \cdot$
$\cdot \langle U|(u^1)^i\rangle \cdot \langle U|(u^2)^j\rangle$.

The polynomial $P(A_o,\dots,A_n,u^1,u^2) = \langle U|Q(\alpha_1,\alpha_2,\dots,u^1,u^2)\rangle$ is said to be the *umbral evaluation* of Q and Q is called the *umbral representation* of P.

We shall also use the following notation:

$\langle U(f)|Q(\alpha_1,\alpha_2,\dots,u^1,u^2)\rangle := P(a_o,\dots,a_n,x^1,x^2)$

$\langle U(\bar{f})|Q(\alpha_1,\alpha_2,\dots,u^1,u^2)\rangle := P(\bar{a}_o,\dots,\bar{a}_n,\bar{x}^1,\bar{x}^2)$.

<u>Remark</u>. Owing to the foregoing definition of umbral space, any umbral evaluation is a polynomial of degree lesser than or equal to d with respect to the variables $A_o,\dots,A_n$. Usually (viz [1]) this fact is avoided by considering infinite pairs of Greek letters. It does not, however, cause any serious restriction. On the contrary, the definition adopted here allows us to avoid more troublesome algebraic machinery.

The change of variables $(x^1,x^2) \mapsto (\bar{x}^1,\bar{x}^2)$ induces an isomorphism

$$
(9)\quad \begin{array}{rcl} \phi: \quad \mathcal{P} & \longrightarrow & \mathcal{P} \\ P(A_o,\dots,A_n,u^1,u^2) & \longmapsto & \bar{P}(A_o,\dots,A_n,u^1,u^2) \end{array}
$$

by which:

i) $\phi(u^i) = \sum c_j^i u^j, \quad \phi(A_p) = \sum D_p^q A_q$;

ii) for every binary form $f(x^1,x^2) = \bar{f}(\bar{x}^1,\bar{x}^2)$, we have

$$P(a_o,\ldots,a_n,x^1,x^2) = \bar{P}(\bar{a}_o,\ldots,\bar{a}_n,\bar{x}^1,\bar{x}^2);$$

iii) P is a covariant of index g iff

$$P(A_o,\ldots,A_n,u^1,u^2) = \Delta^g \bar{P}(A_o,\ldots,A_n,u^1,u^2).$$

The following proposition enable us to compute $\phi(P)$ in terms of its umbral representation. At the end of section 3 we shall give it a straightforward proof.

<u>Prop. 1.</u> If $P(A_o,\ldots,A_n,u^1,u^2) = \langle U|Q(\alpha_1,\alpha_2,\ldots,u^1,u^2)\rangle$ then $\bar{P} = \Phi(P) = \langle U|Q(d_1^1\alpha_1+d_1^2\alpha_2,d_2^1\alpha_1+d_2^2\alpha_2,\ldots,c_1^1u^1+c_2^1u^2,c_1^2u^1+c_2^2u^2)\rangle$.
Equivalently, if $\bar{P} = \langle U|\bar{Q}(\alpha_1,\alpha_2,\ldots,u^1,u^2)\rangle$, we have

$$\bar{P}(\bar{a}_o,\ldots,\bar{a}_n,\bar{x}^1,\bar{x}^2) = \langle U(\bar{f})|\bar{Q}(\alpha_1,\alpha_2,\ldots,u^1,u^2)\rangle =$$
$$= \langle U(f)|\bar{Q}(c_1^1\alpha_1+c_1^2\alpha_2,c_2^1\alpha_1+c_2^2\alpha_2,\ldots,d_1^1u^1+d_2^1u^2,d_1^2u^1+d_2^2u^2)\rangle.$$

2.- We shall need the notion of symmetric power $V^{\odot n}$ of a vector space V.

Let m=dim(V). Let us consider the n-th tensor power $V^{\otimes n}$ of V. Let W be the subspace of $V^{\otimes n}$ generated by all the elements of the form $v_1\otimes\ldots\otimes v_n - v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(n)}$, $\sigma\in S_n$. Let us put

(10) $$V^{\odot n} := V^{\otimes n}/W$$

and

$$v_1\odot\ldots\odot v_n := [v_1\otimes\ldots\otimes v_n]_{\text{mod } W}$$

It may be useful to remark that W is the kernel of the linear application

$$s:\quad V^{\otimes n} \longrightarrow V^{\otimes n}$$
$$v_1\otimes\ldots\otimes v_n \longmapsto \frac{1}{n!}\sum_{\sigma\in S_n} v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(n)} .$$

Thus,

$$\mathrm{Im}(s) \cong V^{\odot n} .$$

We have:

a) $\dim(V^{\odot n}) = \binom{n+m-1}{n}$

b) if $b = (b_1,\ldots,b_m)$ is a basis of V, then the set of the elements of the form

(11) $B_{\hat{\imath}} := b_1\odot\ldots\odot b_1\odot\ldots\odot b_m\odot\ldots\odot b_m$, $\hat{\imath}=(1,\ldots,1,\ldots,m,\ldots,m)$

is a basis of $V^{\odot n}$.

Let us now consider the dual space V^* of V and let $b^1,\ldots,b^m$ be the dual basis of $b_1,\ldots,b_m$: $b^i(b_j) = \delta^i_j$.

By means of s, the isomorphism $(V^*)^{\otimes n} \cong (V^{\otimes n})^*$ induces an isomorphism between $(V^*)^{\odot n}$ and $(V^{\odot n})^*$. Thus, we may regard $X^1\odot\ldots\odot X^n$, $X^i\in V^*$ as a form on $V^{\odot n}$ and we have

$$(12)\quad (X^1\odot\ldots\odot X^n)(x_1\odot\ldots\odot x_n) = 1/n!\sum_{\sigma\in S_n} X^{\sigma(1)}(x_1)\ldots X^{\sigma(n)}(x_n) = 1/n!\sum_{\sigma\in S_n} X^1(x_{\sigma(1)})\ldots X^n(x_{\sigma(n)}).$$

In particular

$$(13)\quad B^{\hat{\imath}}(B_{\hat{\jmath}}) = \binom{n}{p_1,\ldots,p_m}^{-1}\delta^{\hat{\imath}}_{\hat{\jmath}},\quad \hat{\imath}=(\underbrace{1,\ldots,1}_{p_1},\ldots,\underbrace{m,\ldots,m}_{p_m}).$$

3.- For the sake of simplicity, let us now suppose that:

1) $\dim(V) = 2$

2) $\mathbf{b} = (b_1,b_2)$ and $\bar{\mathbf{b}} = (\bar{b}_1,\bar{b}_2)$ are two basis on V such that

$$\bar{b}_1=c^1_1 b_1+c^2_1 b_2 \qquad b_1=d^1_1\bar{b}_1+d^2_1\bar{b}_2$$
$$\bar{b}_2=c^1_2 b_1+c^2_2 b_2 \qquad b_2=d^1_2\bar{b}_1+d^2_2\bar{b}_2$$

3) $x = x^1 b_1+x^2 b_2 = \bar{x}^1\bar{b}_1+\bar{x}^2\bar{b}_2\in V$.

If K is an algebraically closed field, it is possible to find - in ∞^{n-1} different ways - n linear forms $X^i=X^i_1 b^1+X^i_2 b^2=\bar{X}^i_1\bar{b}^1+\bar{X}^i_2\bar{b}^2\in V^*$, i=1,...,n, such that

$$(14)\quad f(x^1,x^2)=(X^1_1x^1+X^1_2x^2)\ldots(X^n_1x^1+X^n_2x^2)=(\bar{X}^1_1\bar{x}^1+\bar{X}^1_2\bar{x}^2)\ldots(\bar{X}^n_1\bar{x}^1+\bar{X}^n_2\bar{x}^2)=\bar{f}(\bar{x}^1,\bar{x}^2);$$

that is

(15) $$f(x^1,x^2) = \bar{f}(\bar{x}^1,\bar{x}^2) = (X^1 \otimes \ldots \otimes X^n)(x^{\otimes n}).$$

Because of $(X^1 \otimes \ldots \otimes X^n)(x^{\otimes n}) = (X^{\sigma(1)} \otimes \ldots \otimes X^{\sigma(n)})(x^{\otimes n}), \sigma \in S_n$, we may also put

(16) $$f(x^1,x^2) = (X^1 \odot \ldots \odot X^n)(x^{\odot n}).$$

If $\hat{\imath} = (\underbrace{1,\ldots,1}_{p},\underbrace{2,\ldots,2}_{n-p})$, instead of $B^{\hat{\imath}}$ we may use B^p:

$$B^p := (b^1)^{\odot p} \odot (b^2)^{\odot(n-p)}.$$

We have $B^p(x^{\odot n}) = (x^1)^p (x^2)^{n-p}$. Thus,

$$f(x^1,x^2) = \sum_{p=0}^{n} \binom{n}{p} a_p (x^1)^p (x^2)^{n-p} = \left(\sum_{p=0}^{n} \binom{n}{p} a_p B^p\right)(x^{\odot n})$$

with $X^1 \odot \ldots \odot X^n = \sum_{p=0}^{n} \binom{n}{p} a_p B^p$.

All this suggests that we can associate the linear form

(17) $$F := \sum_{p=0}^{n} \binom{n}{p} a_p B^p \in (V^{\odot n})^*$$

with the binary form $f(x^1,x^2) = \sum_{p=0}^{n} \binom{n}{p} a_p (x^1)^p (x^2)^{n-p}$ even when K is not algebraically closed.

Of course, with reference to a new basis $\bar{b}$ in V, we also have

(17') $$F = \sum_{p=0}^{n} \binom{n}{p} \bar{a}_p \bar{B}^p.$$

Applying B_p, $\bar{B}_p \in (V^{\odot n})^{**} \cong V^{\odot n}$ to (17), (17') respectively, we have

(18) $$a_p = B^p(F), \qquad \bar{a}_p = \bar{B}^p(F).$$

By a similar argument, more generally we get

(19) $$a_{p_1} a_{p_2} \ldots a_{p_d} (x^1)^s (x^2)^{t-s} = [(B_{p_1} \odot \ldots \odot B_{p_d}) \otimes B^s](F^{\odot d} \otimes x^{\odot t})$$

as well as

$$(19') \qquad \bar{a}_{p_1}\cdot\bar{a}_{p_2}\cdot\ldots\bar{a}_{p_d}\cdot(\bar{x}^1)^s(\bar{x}^2)^{t-s} = [(\bar{B}_{p_1}\odot\ldots\odot\bar{B}_{p_d})\otimes\bar{B}^s](F^{\odot d}\otimes x^{\odot t})$$

To sum up, we associate with each polynomial

$$P(A_o,\ldots,A_n,u^1,u^2) = \sum \Gamma_s^{p_1\ldots p_d}\ A_{p_1}\ldots A_{p_d}(u^1)^s(u^2)^{t-s}$$

(homogeneous both as a polynomial in $A_o,\ldots,A_n$ and as a polynomialin u^1, u^2) the element

$$v = \sum \Gamma_s^{p_1\ldots p_d}(B_{p_1}\odot\ldots\odot B_{p_d})\otimes B^s\ \varepsilon\,(V^{\odot n})^{\odot d}\otimes(V^*)^{\odot t}.$$

If we express the same element v in the basis induced by $\bar{\mathfrak{b}}$:

$$v = \sum \bar{\Gamma}_s^{p_1\ldots p_d}(\bar{B}_{p_1}\odot\ldots\odot\bar{B}_{p_d})\otimes\bar{B}^s$$

we obtain the polynomial

$$\phi(P) = \bar{P}(A_o,\ldots,A_n,u^1,u^2) = \sum \bar{\Gamma}_s^{p_1\ldots p_d}A_{p_1}\ldots A_{p_d}(u^1)^s(u^2)^{t-s}$$

Denoting the foregoing map by

$$(20) \qquad \begin{array}{rcl} \psi_b:\ (V^{\odot n})^{\odot d}\otimes(V^*)^{\odot s} & \longrightarrow & \mathcal{P} \\ v & \longmapsto & P \end{array}$$

we have proved that

Prop. 2.- The diagram

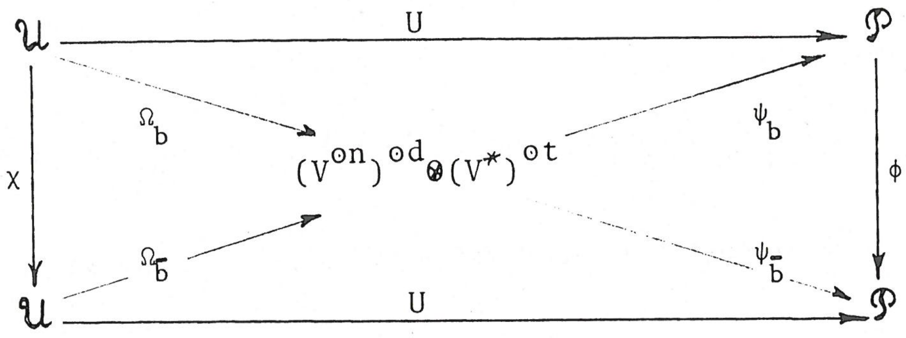
$$(21) \qquad \begin{array}{ccc} \mathcal{P} & \xrightarrow{\ \ \phi\ \ } & \mathcal{P} \\ & {}_{\psi_b}\nwarrow \quad \nearrow{}_{\psi_{\bar{b}}} & \\ & (V^{\odot n})^{\odot d}\otimes(V\ )^{\odot t} & \end{array}$$

is commutative. □

We are now able to prove Prop. 1.

Proof of Prop. 1. Consider the diagram

(22)



where Ω_b is the linear map by which

$$\Omega_b(M) = \begin{cases} B_p \odot \ldots \odot B_p \otimes B^s & \text{if } M \text{ is a monomial of the form } M=\alpha_1^{p_1}\alpha_2^{n-p_1}\ldots\lambda_1^{p_d}\lambda_2^{n-p_d}(u^1)^s(u^2)^{t-s} \\ 0 & \text{for any other monomial } M \end{cases}$$

and χ is the morphism of algebras defined by

$$\chi(\alpha_i) = d_i^1\alpha_1 + d_i^2\alpha_2 \qquad \text{(for any Greek letter } \alpha)$$

$$\chi(u^i) = c_1^i u^1 + c_2^i u^2 \ .$$

It is easy to see that every triangular diagram in (22) is commutative. Thus, the same is true for the rectangular one. This is precisely what we had to prove. □

What we have seen up to now seems to point out that the two spaces $(V^{\odot n})^{\otimes d} \otimes (V^*)^{\odot t}$ and $(V^{\odot n})^{\odot d} \otimes (V^*)^{\odot t}$, together with the canonical projection, supply a natural framework for studying homogeneous covariants of binary forms.

Indeed, going into slightly more detail in the calculations involved, it becomes clear, via the identification

$$B_{p_1} \otimes \ldots \otimes B_{p_d} \otimes B^s = \alpha_1^{p_1}\alpha_2^{n-p_1}\ldots\lambda_1^{p_d}\lambda_2^{n-p_d}(u^1)^s(u^2)^{t-s}$$

that i) as a first approximation, the umbral space $\mathcal{U}$ is nothing but $(V^{\odot n})^{\otimes d} \otimes (V^*)^{\odot t}$; ii) the umbral operator U represents the canonical projection from $(V^{\odot n})^{\otimes d} \otimes (V^*)^{\odot t}$ to $(V^{\odot n})^{\odot d} \odot (V^*)^{\odot t}$.

4.- In what follows the graded algebra $\mathcal{V} := (S(V))^{\otimes d}$ - more precisely, the internally graded algebra $\Sigma\mathcal{V}$ induced by $\mathcal{V}$ - plays the role played by $(V^{\odot n})^{\otimes d}$ in the foregoing treatment. In such a way, we shall gain a twofold advantage. On the one hand, it allows us to represent every polynomial $P \in \mathcal{P}$ (not necessarily homogeneous) of degree d. On the other hand - and this is the main point - $\mathcal{V}$ appears to be the suitable setting where such representations of invariants may be factorized. Let us sketch how this can be done.

We shall need the symmetric graded algebra

(23) $$S(V) = (K, V, V^{\odot 2}, \ldots, V^{\odot p}, \ldots)$$

as well as its d-th tensor power

(24) $$(S(V))^{\otimes d} = \mathcal{V} = (\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_p, \ldots)$$

where

(25) $$\mathcal{V}_p := \bigoplus_{i_1+\ldots+i_d=p} V^{\odot i_1} \otimes \ldots \otimes V^{\odot i_d}.$$

The multiplication on $S(V)$ is defined as follows:

(26) $$\begin{array}{ccc} V^{\odot p} \times V^{\odot q} & \longrightarrow & V^{\odot (p+q)} \\ (v_1 \odot \ldots \odot v_p, w_1 \odot \ldots \odot w_q) & \longmapsto & v_1 \odot \ldots \odot v_p \odot w_1 \odot \ldots \odot w_q \end{array}$$

Therefore, the multiplication on $\mathcal{V} = (S(V))^{\otimes d}$ is given through the bilinear maps defined by

(27) $$\begin{array}{ccc} \mathcal{V}_p \times \mathcal{V}_q & \longrightarrow & \mathcal{V}_{p+q} \\ (v_1 \otimes \ldots \otimes v_d, w_1 \otimes \ldots \otimes w_d) & \longmapsto & (v_1 w_1) \otimes \ldots \otimes (v_d w_d) \end{array}$$

where the product $v_i w_i$ is as in (26).

Observe that both $S(V)$ and $\mathcal{V}$ are generated, as graded algebras, by their elements of the first degree.

Let us factorize the space $\mathcal{V}_p$ by means of the subspace $\mathcal{W}_p$ defined in the following way. If either $p>nd$ or $p\neq nh$, $\mathcal{W}_p$ is $\mathcal{V}_p$ itself. If instead $p=nh$, $h\leq d$, first observe that we have

$$(28) \qquad \mathcal{V}_p = \mathcal{L}_p \oplus \mathcal{L}'_p \qquad (p=nh)$$

where

$$(29) \qquad \mathcal{L}_p := \bigoplus_{j_o+\ldots+j_h=d-h} K^{\otimes j_o}\otimes V^{\odot n}\otimes\ldots\otimes K^{\otimes j_{h-1}}\otimes V^{\odot n}\otimes K^{\otimes j_h} \qquad (p=nh).$$

The elements of the $\mathcal{L}_p$'s are said to be the irredundant elements of $\mathcal{V}$.

Denoting with $\mathcal{H}_p$ the subspace of $\mathcal{L}_p$ generated by all the elements of the form

$$v_1\otimes\ldots\otimes v_d - v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(d)}, \qquad \sigma\in S_d$$

let us put

$$(30) \qquad \mathcal{W}_p := \mathcal{H}_p \oplus \mathcal{L}'_p \qquad (p=nh).$$

We have

$$(31) \qquad \frac{\mathcal{V}_p}{\mathcal{W}_p} \cong \begin{cases} 0 & \text{if } p\neq nh \text{ or } p>nd \\ (V^{\odot n})^{\odot h} & \text{if } p=nh \end{cases}$$

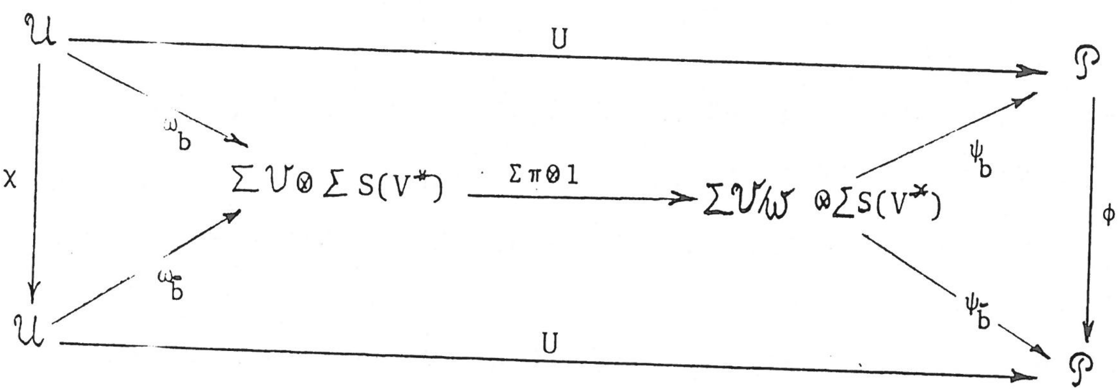
Consider the graded quotient module

$$(32) \qquad \mathcal{V}/\mathcal{W} := (\mathcal{V}_o/\mathcal{W}_o, \ \ldots \ , \mathcal{V}_p/\mathcal{W}_p, \ \ldots \ ) =$$

$$= (K,\underbrace{0,\ldots,0}_{(n-1)\text{times}},V^{\odot n},\underbrace{0,\ldots,0}_{(n-1)\text{times}},(V^{\odot n})^{\odot 2},0,\ldots,0,(V^{\odot n})^{\odot d},\underbrace{0,0,\ldots}_{\text{definitely}})$$

$\mathcal{V}/\mathcal{W}$ is also a graded algebra - namely, the symmetric graded algebra generated by its elements of degree n - but *it is not* a graded quotient algebra of $\mathcal{V}$. In other words, the submodule $\mathcal{W}=(\mathcal{W}_o,\ldots,\mathcal{W}_p,\ldots)$ is not an ideal of the graded algebra $\mathcal{V}$.

The relationship between the approach to invariant theory allowed by these algebraic structures and the symbolic method is visualized by the following commutative diagram (34):



(34)

where:

a) ω_b is the isomorphism of algebras defined by

$$\omega_b(z_i)=\begin{cases} \underbrace{1\otimes\ldots\otimes 1}_{(j-1)\text{times}}\otimes b_i\otimes\underbrace{1\otimes\ldots\otimes 1}_{(d-j+1)\text{times}} & \text{if } z \text{ is the j-th letter among } \{\alpha,\ldots,\lambda\} \\ \underbrace{1\otimes\ldots\otimes 1}_{d\text{ times}}\otimes b^i & \text{if } z=u\ ; \end{cases}$$

b) $\Sigma\pi : \Sigma\mathcal{V}\longrightarrow\Sigma\mathcal{V}/\mathcal{W}$ is the canonical projection;

c) ψ_b is the morphism of algebras such that

$$\psi_b:\ B_p\otimes 1=[\underbrace{b_1\odot\ldots\odot b_1}_{p\text{ times}}\odot\underbrace{b_2\odot\ldots\odot b_2}_{(n-p)}]\otimes 1\longmapsto A_p$$

and

$$\psi_b:\ 1\otimes b^i\longmapsto u^i.$$

Lastly, consider another morphism (of modules), the symmetrization map

(35) $\mathcal{S}=\Sigma(\mathcal{S}_o,\ldots,\mathcal{S}_p,\ldots)$:

Here $\mathcal{S}_p$ is the linear map defined by

$$\mathcal{S}_p : \mathcal{V}_p \longrightarrow \mathcal{V}_p$$

$$v_1 \otimes \ldots \otimes v_d \longmapsto 1/d! \sum_{\sigma \varepsilon S_d} \bar{v}_{\sigma(1)} \otimes \ldots \otimes \bar{v}_{\sigma(d)}$$

where

$$\bar{v}_i := \begin{cases} v_i & \text{if } v_i \varepsilon V^{\odot n} \text{ or } v_i \varepsilon K \\ 0 & \text{otherwise} \end{cases}$$

Lemma. Let w be an element of $\mathcal{L}_{nh} \subseteq \mathcal{V}_{nh}$ $(h \leqslant d)$; then $\mathcal{S}(w)=0$ if and only if $w \varepsilon H_{nh}$.

Proof. (Assume dimV=2; in the general case the same pattern may be followed).

Throughout the proof, we shall only consider d-ples $s_1, \ldots, s_d$ such that $-1 \leqslant s_i \leqslant n$ and $\#\{s_i | s_i = -1\} = d-h$.

The elements of the form

$$B_{s_1} \otimes \ldots \otimes B_{s_d}$$

with

$$B_{s_i} = \underbrace{b_1 \odot \ldots \odot b_1}_{s_i} \odot \underbrace{b_2 \odot \ldots \odot b_2}_{n-s_i} \quad \text{if } s_i \geqslant 0$$

and

$$B_{-1} = 1 \varepsilon K$$

provide a basis for $\mathcal{L}_{nh}$. We have:

$$\mathcal{S}(w) = \mathcal{S}(\sum \Lambda^{s_1, \ldots, s_d} B_{s_1} \otimes \ldots \otimes B_{s_d}) =$$

$$= \mathcal{S}\left[\sum_{-1 \leqslant s_1 \leqslant \ldots \leqslant s_d \leqslant n} (\sum \Lambda^{t_1, \ldots, t_d} B_{t_1} \otimes \ldots \otimes B_{t_d})\right] =$$

(where the sum into parentheses ranges over all the d-ples $t_1, \ldots, t_d$ such that $\#\{t_i | t_i = r\} = \#\{s_i | s_i = r\}$ for every $r \varepsilon \{-1, 0, 1, \ldots, n\}$)

$$= \sum_{-1 \leqslant s_1 \leqslant \ldots \leqslant s_d \leqslant n} (\sum \Lambda^{t_1, \ldots, t_d} \mathcal{S}(B_{t_1} \otimes \ldots \otimes B_{t_d})) =$$

$$= \sum_{-1 \leqslant s_1 \leqslant \ldots \leqslant s_d \leqslant n} (\sum \Lambda^{t_1, \ldots, t_d}) \mathcal{S}(B_{s_1} \otimes \ldots \otimes B_{s_d}) .$$

Since the vectors $\mathcal{S}(B_{s_1}\otimes\ldots\otimes B_{s_d})$, $-1\leqslant s_1\leqslant\ldots\leqslant s_d\leqslant n$, are linearly independent, $\mathcal{S}(w)=0$ implies $\sum\Lambda^{t_1,\ldots,t_d}=0$. It follows that, if $\mathcal{S}(w)=0$, then

$$w = \sum_{-1\leqslant s_1\leqslant\ldots\leqslant s_d\leqslant n}(\sum\Lambda^{t_1,\ldots,t_d}[B_{t_1}\otimes\ldots\otimes B_{t_d} - B_{s_1}\otimes\ldots\otimes B_{s_d}]),$$

that is, $w\in H_{nh}$ □

The proof of the following proposition presents now no special difficulties.

Prop. 3. $\mathrm{Ker}(\mathcal{S}_p) = \mathcal{W}_p$ $(= H_p \oplus \mathcal{L}'_p$ if $p=nh)$.

As a corollary we deduce the following Proposition 4.

Prop. 4. (The symmetrization condition) Let $v\in\sum\mathcal{V}$; then $(\sum\pi)(v)=0$ in $\sum\mathcal{V}/\mathcal{W}$ if and only if $\mathcal{S}(v)=0$ in $\sum\mathcal{V}$.

Referring to the notion of *symmetrization* $S(Q)$ of an *irredundant polynomial* $Q\in\mathcal{U}$ as it appears in [1], it is easy to prove that:

i) if $Q\in\mathcal{U}$ is an irredundant polynomial in h Greek letters $(h\leqslant d)$, then $\omega_b(Q)\in\mathcal{L}_{nh}$; conversely, if $\omega_b(Q)\in\mathcal{L}_{nh}$, then $Q = Q_1+\ldots+Q_p$, where Q_i is an irredundant polynomial in h Greek letters.

ii) $S(Q)=0$ if and only if $\mathcal{S}(\omega_b(Q))=0$. More precisely, we have

$$(\omega_b^{-1}\circ\mathcal{S}\circ\omega_b)(Q) = \binom{d}{h}^{-1}\sum_f S(Q_f)$$

where, denoting with $\xi_1,\ldots,\xi_h$ the Greek letters which occur in Q, f is any injective, increasing map from $\{\xi_1,\ldots,\xi_h\}$

to $\{\alpha,\beta,\ldots,\lambda\}$ and $Q_f := Q(f(\xi_1),\ldots,f(\xi_h))$.

As a consequence, Proposition 4 is equivalent to the usual form of the symmetrization condition, that is

$$\mathcal{E}(Q)=0 \Longleftrightarrow U(Q)=0.$$

R E F E R E N C E S

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