

On Hales-Jewett's theorem

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Summary. We prove that, for every finite semigroup S , there exist elements $a_1, a_2, \dots, a_k, a_{k+1}$ of S and integers i_1, \dots, i_k such that

$$a_1 \cdot x^{i_1} \cdot a_2 \cdot x^{i_2} \cdot \dots \cdot a_k \cdot x^{i_k} \cdot a_{k+1} = a_1 \cdot y^{i_1} \cdot a_2 \cdot \dots \cdot a_k \cdot y^{i_k} \cdot a_{k+1}$$

for each x, y of S .

We refer to [1] for the notion of (combinatorial) line as well as for the other combinatorial concepts we use in the sequel.

The following theorem of Hales-Jewett is well known.

Theorem ([2]). Given any finite set A and any integer r there exists an integer $N=N(A, r)$ such that for each $n \geq N(A, r)$, in any r -coloring of A^n there is always a monochromatic line.

In [1], Graham presented an interesting algebraic application of this theorem: for every finite commutative semigroup S , there exist an element a of S and an integer n such that

$$a \cdot x^n = a \cdot y^n$$

for each x, y of S , i.e. $a \cdot x^n$ is independent of x (shortly, we speak of the constant word $a \cdot x^n$ for S).

Trivially, there are finite non commutative semigroup without constant word of type $a \cdot x^n$ as for example, the semigroup D presented by the following Cayley table

	u	v
u	u	v
v	u	v

True, for this semigroup the word $x \cdot u$ is constant, i.e. for this semigroup a particular case of the word $x^n \cdot b$ ("dual" of $a \cdot x^n$) is constant. But the semigroup D' presented by the following Cayley table

	u	v
u	u	u
v	v	v

shows that $x^n \cdot b$ is not a constant word for each finite semigroup.

Again, the direct product $D \times D'$, where D and D' are as previously described, is a semigroup without a constant word of type $a \cdot x^n$ and without a constant word of type $x^n \cdot b$.

After some other considerations like the previous ones, the following question naturally arises: given any finite semigroup does there exist a sort of constant word for it?

The following proposition gives us an answer.

Proposition. For any finite semigroup S there exist an integer k , a $(k+1)$ -tuple $a_1, a_2, \dots, a_k, a_{k+1}$ of elements of S , and a k -tuple i_1, \dots, i_k of integers such that

$$a_1 \cdot x^{i_1} \cdot a_2 \cdot x^{i_2} \cdot \dots \cdot a_k \cdot x^{i_k} \cdot a_{k+1}$$

is independent of x , i.e. it is a constant word for S .

Proof. Consider S both as an alphabet and as a set of colors. From Hales-Jewett's theorem there exists an integer $N(S, |S|)$ such that for each integer $n \geq N(S, |S|)$ and each $|S|$ -coloring of S^n , a line of S^n is monochromatic.

This is true, in particular, when each element

$$(x_1, x_2, \dots, x_n)$$

of S^n is colored by

$$x_1 \cdot x_2 \cdot \dots \cdot x_n,$$

i.e. the product in S of x_1, x_2, \dots, x_n .

Now, if the monochromatic line is the diagonal then the conclusion follows immediately.

If the monochromatic line is not the diagonal then the lengths of the "lakes" of the non-fixed coordinates (there is always such a coordinate) give us the integer i_j and the elements a_j are easily obtained looking at the fixed coordinates.

References.

- [1] Graham, R.L., "Rudiments of Ramsey Theory", 1981 (1983), American Math. Soc., Conference Board of Mathematical Sciences, Regional Conference Series in Mathematics n. 45 (St. Olaf College, June 1979).
- [2] Hales, A.W./Jewett, R.I., "Regulartiy and positional games", Trans. Amer. Math. Soc., 106 (1963), 222-229.

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