An Erdös-Rado Theorem for Baire-mappings $\Delta:[\mathbb{R}]^n \to M$

Hanno Lefmann
Bielefeld

Let $\mathbb{R}$ be the set of reals endowed with the usual topology. Already known to Sierpiński is the following

Theorem A: Let M be a metric space and let $\Delta:\mathbb{R}\to M$ be a Baire-mapping.
Then there is a perfect subset $P \subseteq \mathbb{R}$ such that

either $\Delta \upharpoonright P$ is constant
or $\Delta \upharpoonright P$ is one-to-one.

Here a mapping $\Delta:\mathbb{R}\to M$ is a Baire-mapping iff the preimage $\Delta^{-1}(O)$ of every open set $O \subseteq M$ is a Baire-set, i.e. is the symmetric difference of an open and a meager set. A subset $P \subseteq \mathbb{R}$ is perfect iff it is nonempty, closed and has no isolated points.

Theorem A fails, if arbitrary mappings are allowed. This can be seen by use of the axiom of choice.

Blass considered in [Bl 81] Baire-mappings $\Delta:[\mathbb{R}]^n \to \{0,...,r-1\}$, where n,r are positive integers. Here $[\mathbb{R}]^n$ denotes the set of n-element subsets of $\mathbb{R}$, which inherits a topology from $\mathbb{R}^n$ endowed with the product topology. It turns out that certain subsets of $[\mathbb{R}]^n$ play an important role:

Notation: Let n be a positive integer. Let $(\{1,\ldots,n-1\};\preccurlyeq)$ be a total order and let $P \subseteq \mathbb{R}$. Then

$$[P]^n_{\prec} = \{\{p_0,\ldots,p_{n-1}\}_< \in [P]^n \mid p_i - p_{i-1} < p_j - p_{j-1} \text{ iff } i \prec j \text{ for all } 1 \le i < j < n\},$$

where $<$ denotes the usual order on $\mathbb{R}$.

So these subsets are determined by the relative distances between consecutive elements. Clearly, there are $(n-1)!$ such subsets of $[P]^n$.

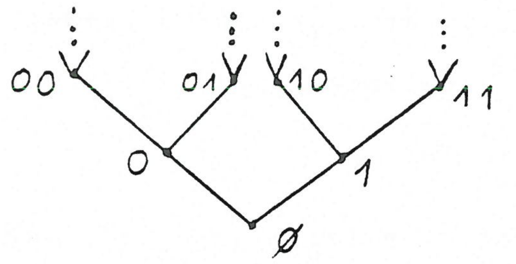
Blass proved in [Bl 81] the following

Theorem B: Let n,r be positive integers. Then for every Baire-mapping $\Delta:[\mathbb{R}]^n \to \{0,\ldots,r-1\}$ there exists a perfect subset $P \subseteq \mathbb{R}$ such that $\Delta \restriction [P]^n_{\prec} = \text{const.}$ for every total order $(\{1,\ldots,n-1\};\preccurlyeq)$.

Thus the image of n-element subsets of P only depend on their ordertype, and this result is optimal, since n-element subsets of $\mathbb{R}$ can be partitioned according to their ordertype.

Next we consider mappings with an arbitrary range, not only with a finite one. Let $\Delta:[\mathbb{R}]^n \to M$ be a Baire-mapping, where M is a metric space. The aim is to find a perfect subset $P \subseteq \mathbb{R}$ such that the mapping Δ restricted to $[P]^n$ behaves nice. By topological means there is a perfect subset $P \subseteq \mathbb{R}$ such that the restriction $\Delta \restriction [P]^n$ is continuous. Since every perfect subset of $\mathbb{R}$ contains a Cantor set, we restrict in the following to continuous mappings $\Delta:[2^\omega]^n \to M$.

To visualize 2^ω, consider the tree $2^{<\omega}$ whose points are finite 0-1 sequences ordered by the initial segment relation:



Clearly, every infinite path in $2^{<\omega}$ determines an element of 2^ω.

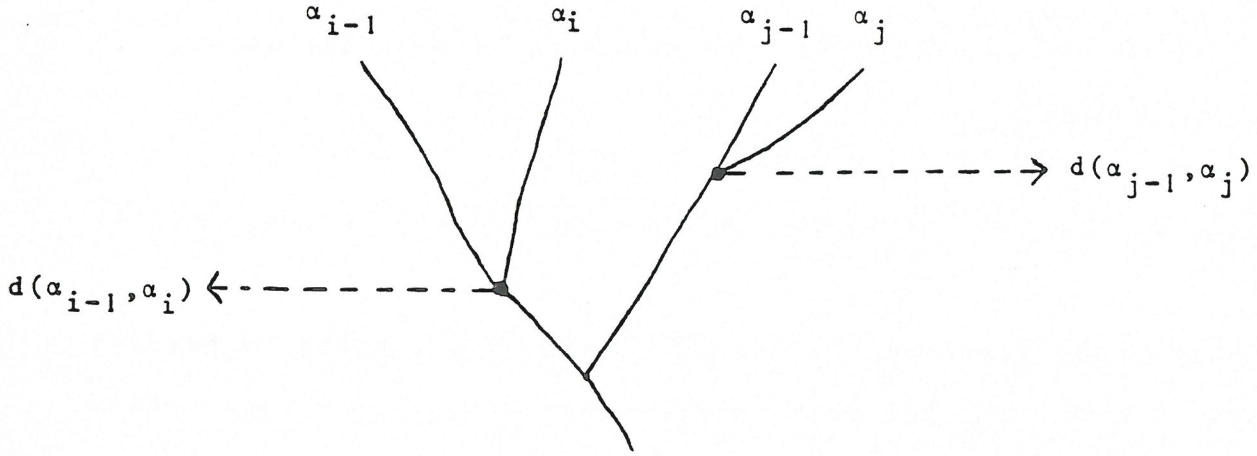
For the following it can be shown, that one need not consider the whole tree $2^{<\omega}$. It suffices to look at perfect, skew subtrees of $2^{<\omega}$. These subtrees satisfy: (1) above every node there is at least one ramification node and (2) on every level there is at most one ramification node.

For different infinite paths $\alpha, \beta \in 2^\omega$ let $d(\alpha,\beta)$ denote the length of the maximal common initial segment of α and β.

Let 2^ω be endowed with the lexicographic order.

Let $T = \{\alpha_0, \ldots, \alpha_{n-1}\}_{<lex} \in [2^\omega]^n$. This set T induces a total order $\preccurlyeq$ on $\{1, \ldots, n-1\}$ by

$$i \preccurlyeq j \Longleftrightarrow d(\alpha_{i-1}, \alpha_i) \leq d(\alpha_{j-1}, \alpha_j)$$



By $V(T) = \{d(\alpha_{i-1},\alpha_i) \mid 1 \le i \le n-1\}$ we denote the set of the lengths of the ramification nodes of T.

For a total order $(\{1,\dots,n-1\};\preccurlyeq)$ and a subset $P \subseteq 2^\omega$ let $[P]^n_\preccurlyeq$ be the set of all n-element subsets of P, which have ordertype $(\{1,\dots,n-1\},\preccurlyeq)$.

For a totally ordered set $X = \{x_0,\dots,x_{n-1}\}_<$ and a subset $I \subseteq \{0,\dots,n-1\}$ let $X:I = \{x_i \mid i \in I\}$ be the I-subset of X.

We have the following

<u>Theorem C:</u> Let n be a positive integer and let M be a metric space. Then for every continuous mapping $\Delta:[2^\omega]^n \to M$ there exists a perfect subset $P \subseteq 2^\omega$ and for every total order $(\{1,\dots,n-1\};\preccurlyeq)$ there exist subsets $I_\preccurlyeq \subseteq \{0,\dots,n-1\}$ and $J_\preccurlyeq \subseteq \{1,\dots,n-1\}$ such that for every pair $(\preccurlyeq,\preccurlyeq^*)$ it is valid:

(i) $\Delta(A) \neq \Delta(B)$ for all $A \in [P]^n_\preccurlyeq$, $B \in [P]^n_{\preccurlyeq^*}$

or

(ii) $\Delta(A) = \Delta(B) \Longleftrightarrow A:I_\preccurlyeq = B:I_{\preccurlyeq^*}$ and $V(A):J_\preccurlyeq = V(B):J_{\preccurlyeq^*}$
for all $A \in [P]^n_\preccurlyeq$, $B \in [P]^n_{\preccurlyeq^*}$.

Thus the canonical patterns are determined by paths and ramification nodes.

With respect to partitioning n-element subsets of the set of reals this means:

<u>Theorem D:</u> Let n be a positive integer and let M be a metric space.
Then for every Baire-mapping $\Delta:[\mathbb{R}]^n \to M$ there exists a perfect subset $P \subseteq \mathbb{R}$ and for every total order $(\{1,\dots,n-1\};\preccurlyeq)$ there

exist subsets $I_{\prec} \subseteq \{0,\dots,n-1\}$ and $J_{\prec} \subseteq \{1,\dots,n-1\}$ such that for all pairs $(\prec,\prec^*)$ it is valid:

(i) $\Delta(A) \neq \Delta(B)$ for all $A \in [P]^n_{\prec}$ and $B \in [P]^n_{\prec^*}$

or

(ii) $\Delta(A) = \Delta(B)$ iff $A:I_{\prec} = B:I_{\prec^*}$ and

$\{k\in\mathbb{Z} \mid 2^{k-1} \leq a_j - a_{j-1} < 2^k$ for some $j \in J_{\prec}\}$

$= \{k\in\mathbb{Z} \mid 2^{k-1} \leq b_j - b_{j-1} < 2^k$ for some $j \in J_{\prec^*}\}$

for all $A \in [P]^n_{\prec}$ and $B \in [P]^n_{\prec^*}$ with

$A = \{a_0,\dots,a_{n-1}\}_<$ and $B = \{b_0,\dots,b_{n-1}\}_<$.

Proofs of these results will appear elsewhere.

Reference

[Bl 81] A. Blass, *A partition theorem for perfect sets*, Proc. Amer. Math. Soc. 82 (2), 1981, 271-277.

H. Lefmann
Fakultät für Mathematik
Universität Bielefeld
D-4800 Bielefeld