A Combinatorial Interpretation of the Integral of the Product of Legendre Polynomials

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I The Legendre polynomials, $P_n(x)$, can be written in the form

$$P_n(x) = 2^{-n} \sum_{\alpha} \binom{n}{\alpha}^2 (x+1)^{\alpha}(x-1)^{n-\alpha} \tag{1}$$

From this it is easily deduced that

$$I_{n_1,n_2,\ldots,n_k} = \int_{-1}^{+1} P_{n_1}(x)P_{n_2}(x)\ldots P_{n_k}(x)dx$$

$$= 2(1+\Sigma n_i)^{-1} \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_k)} (-1)^{\Sigma\alpha_i} \{\prod_{i=1}^{k} \binom{n_i}{\alpha_i}^2\} \binom{\Sigma n_i}{\Sigma\alpha_i}^{-1} \tag{2}$$

II Consider now a set of elements of k different types, ordered by type number i $(i=1,2,\ldots,k)$ and within each type by a rank r $(r=1,2,\ldots,n_i)$. We represent these by points and form a directed graph by joining them so that each point has exactly one edge going into it and one coming out. We then colour the edges of this graph blue or yellow subject to the following <u>balance condition</u>:

(*) For each i, the number of points of type i at the beginning of blue edges, α_i (say), equals the number which are at the end of blue edges.

We call such a colored graph a <u>system</u>. For any graph Γ we denote by $\sigma(\Gamma)$ the set of all systems which can be formed by all possible colourings of Γ subject to the balance condition (*). If E is any collection of graphs then we write

$$\sigma(E) = \bigcup_{\Gamma \in E} \sigma(\Gamma) \ .$$

We call a system even or odd according to the parity of the number of its blue edges, $\Sigma\alpha_i$. If S is any collection of graphs, we denote by $\Pi(S)$ the difference between the total number of even and odd systems among all the systems comprising $\sigma(S)$. In the special case where S is the set of all graphs which can be formed according to the prescription at the beginning of this paragraph, we write

$$\Pi(S) = \Pi_{n_1, n_2, \ldots, n_k} \tag{3}$$

It is not difficult to show, using (2), that

$$I_{n_1, n_2, \ldots, n_k} = \frac{2}{(1+\Sigma n_i)!} \Pi_{n_1, n_2, \ldots, n_k} \tag{4}$$

III If a graph contains an edge joining two points of the same type, then we can set up a bijection between the odd and even systems that can be constructed on it. We simply take the lowest ranking such edge (defined by the type and rank of its starting point) and change its colour. The balance condition (*) will still be preserved. We deduce that if $n_1 > n_2+n_3 \ldots +n_k$ then

$$I_{n_1, n_2, \ldots, n_k} = 0 \tag{5}$$

The orthogonality of the Legendre polynomials is a special case of (5), while the Integral $I_{n,n}$ is also easily derived from (4).

IV Rather more elaborate arguments make it possible to exclude other graphs form consideration and thus to simplify the calculation of $\Pi_{n_1, n_2, \ldots, n_k}$. In particular, we develop combinatorial proof of (6) and (7) below. Formula (6) is a classical known result, but (7) appears to be new.

$$\int_{-1}^{+1} P_a(x)P_b(x)P_c(x)dx = \binom{2s-2a}{s-a}\binom{2s-2b}{s-b}\binom{2s-2c}{s-c}\binom{2s}{s}^{-1} \qquad (6)$$

provided that a+b+c is even and = 2s ,
and s > max (a,b,c),
and = 0 otherwise;

$$\int_{-1}^{+1} [P_a(x)]^4 dx = 2\sum_{\gamma=0}^{a} \frac{4\gamma+1}{(2a+2\gamma+1)^2}\left\{\binom{2\gamma}{\gamma}^4\binom{2a-2\gamma}{a-\gamma}^2\binom{2a+2\gamma}{a+\gamma}^4\right\} \qquad (7).$$

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