The Cyclotomic Identity and the Cyclic Group.

(A.Dress and Ch.Siebeneicher , Bielefeld)

Metropolis and Rota discuss the cyclotomic identity

$$\frac{1}{1 - \alpha \cdot t} = \prod_{n \in \mathbb{N}} \left(\frac{1}{1 - t^n} \right)^{M(\alpha, n)}$$

Thereby they define the necklace-algebra Nr(A) for a commutative A with unit element 1 . We show that Nr($\mathbb{Z}$) is the Burside-ring $\hat{\Omega}(\mathbb{Z})$ of almost finite $\mathbb{Z}$-sets , where an almost finite $\mathbb{Z}$-set is a set with an operation of the infinite cyclic group $\mathbb{Z}$, such that every element lies in a finite orbit and every orbit type $\mathbb{Z}/n\mathbb{Z}$ occurs only with finite multiplicity. For every $n \in \mathbb{N}$ we have a homomorphism

$$\varphi_n : \hat{\Omega}(\mathbb{Z}) \longrightarrow \mathbb{Z} ,$$

which assigns to a $\mathbb{Z}$-set the number of elements invariant under the subgroup $n\mathbb{Z}$. The family φ_n with $n > 0$ defines an injective homomorphism

$$\varphi : \hat{\Omega}(\mathbb{Z}) \longrightarrow \mathbb{Z}^J$$

of $\hat{\Omega}(\mathbb{Z})$ into the product ring $\mathbb{Z}^J$, where J designes the set of posive integers.

By defining symmetric powers of almost finite $\mathbb{Z}$-sets we get a homomorphism

$$s_t : \hat{\Omega}(\mathbb{Z}) \longrightarrow 1 + t\,\Omega(\mathbb{Z})[[t]]$$

of the additive group of $\hat{\Omega}(\mathbb{Z})$ into the multiplicative group of formal power-series with constant term 1 and coefficients in the ring $\hat{\Omega}(\mathbb{Z})$. Combining this homomorphism with φ_1 we get an isomorphism

$$\varphi_1 \circ s_t : \hat{\Omega}(\mathbb{Z}) \longrightarrow 1 + t\,\mathbb{Z}[[t]] .$$

Finally we show that $\hat{\Omega}(\mathbb{Z})$ is isomorphic to the (generalized) ring of Witt-vectors W($\mathbb{Z}$).

The Burnside-ring construction applies to every profinite group providing thereby a generalized cyclotomic(?) identity. The construction allows also to define for an arbitrary profinite group and given commutative ring A a generalized ring of Witt-vectors over A.

Literature: N. Metropolis and G-C. Rota
Witt Vectors and the Algebra of Necklaces
Advances in Mathematics, vol.50, 1983