Combinatorics of Jacobi-Configurations II :

A Rational Approximation via Matching Polynomials

Volker Strehl

Institut für Mathematische Maschinen und Datenverarbeitung I

Universität Erlangen-Nürnberg

D-8520 Erlangen

Fed.Rep.Germany

Abstract: The notion of 'order' for Jacobi-configurations has been introduced in part I of this article. In this second part the exponential generating function for Jacobi-configurations of bounded order, i.e. the analog of the classical generating function of Jacobi's for his polynomials, is derived. This result (and its proof) make use of certain matching polynomials and their combinatorial properties. These matching polynomials are close relatives of the Tchebycheff-polynomials.

0 Introduction

In [FL] D.Foata and P.Leroux presented the first combinatorial model - "Jacobi-endofunctions" - for the Jacobi-polynomials, and using this model they succeeded in giving a completely combinatorial proof of Jacobi's generating function for his polynomials. This model has been used (and extended) in [LS] and [ST] for further combinatorial studies along these lines. In particular, in [ST] a concept of 'order' for the Jacobi-endofunctions was defined, which was used in that article in order to establish the equivalence between the Foata-Leroux-model and another model introduced by the present author. It is the purpose of this article to present the exponential generating function for Jacobi-endofunctions of bounded order in terms of matching polynomials. The matching polynomials which show up here are in fact close relatives of the Tchebycheff-polynomials (of either kind), and the main result of this article may be looked at as a kind of rational approximation of the Jacobi generating function in terms of Tchebycheff-polynomials. Though converging rapidly and being easy to calculate, the author feels that the main interest for this approximation comes from its strong combinatorial motivation.

This article is organized as follows: in sec.1 we recall briefly the Foata-Leroux model for the Jacobi-polynomials. In the second section the operations of 'reduction' and 'contraction' on Jacobi-endofunctions are reviewed - these lead in a natural way to the concept of 'order'. The generating functions of interest are defined in section 3, where the basic recurrence is proved by inverting the reduction-contraction procedure. The main result is stated in sec.4, where the relevant matching polynomial will be defined. The main combinatorial trick - the duplication formula for the matching polynomials - is presented in the fifth section. The proof of the main result will then be completed in sec.6 . Some additional remarks conclude the article.

The present article can be read independently from part I - all the relevant notions from [ST] are reviewed here. Nevertheless, it should be noted that the major motivation for the work presented here is to be found in the predecessor of this article.

1 Jacobi-endofunctions: definition and notation

To begin with, let us briefly recall the model of Jacobi-endofunctions as introduced by Foata/Leroux in [FL]. (Indeed, we do not use exactly the same notations).
For any pair (A,B) of disjoint, finite sets JAC(A,B) denotes the set of all endofunctions f of $S:=A\cup B$ such that the restrictions

$$f|A : A \longrightarrow S \quad \text{and} \quad f|B : B \longrightarrow S$$

are both injective. For any finite set S, JAC(S) denotes the set of all $\varphi=((A,B),f)$, where (A,B) is an ordered bipartition of S and $f\in JAC(A,B)$. We will write JAC[n] when S = [n] = {1,2,...,n}.
For $f\in JAC(A,B)$ let R(f) denote the set of recurrent elements of f. Then f|R(f) is a permutation of R(f) and we denote by cyc(f|A) the number of cycles of f|R(f) which are contained in A, and similarly for cyc(f|B).
Each $\varphi=((A,B),f)$ is the given the weight

$$w(\varphi) := (1+\alpha)^{cyc(f|A)}(1+\beta)^{cyc(f|B)}X^{|A|}Y^{|B|} ,$$

and it is shown in [FL] that

$$\mathcal{P}_n^{(\alpha,\beta)}(X,Y) = \sum \{ w(\varphi) \; ; \; \varphi\in JAC[n] \}$$

where $\mathcal{P}_n^{(\alpha,\beta)}(X,Y)$ denotes the n-th homogeneous Jacobi-polynomial:

$$\mathcal{P}_n^{(\alpha,\beta)}(X,Y) = \sum_{i+j=n} \binom{n}{i}(1+\alpha+j)_i(1+\beta+i)_j X^i Y^j ,$$

i.e. these polynomials are related to the Jacobi-polynomials in their standard notation $P_n^{(\alpha,\beta)}(x)$ (c.f. [AB],[AS],[ER],[RA]) by

$$\mathcal{P}_n^{(\alpha,\beta)}(X,Y) = n!\cdot(X-Y)^n\cdot P_n^{(\alpha,\beta)}((X+Y)/(X-Y)) ,$$

$$n!\cdot P_n^{(\alpha,\beta)}(x) = \mathcal{P}_n^{(\alpha,\beta)}((x+1)/2,(x-1)/2) .$$

Jacobi's generating function, when written in homogeneous form, states that

$$\begin{aligned}\mathbb{P}^{(\alpha,\beta)}(X,Y) &= \sum_n \mathcal{P}_n^{(\alpha,\beta)}(X,Y) / n! \\ &= \sum_n (1/n!) \sum \{ w(\varphi) \; ; \; \varphi\in JAC[n] \} \\ &= 2^{\alpha+\beta}\,\mathcal{R}^{-1}(1-(X-Y)+\mathcal{R})^{-\alpha}(1-(Y-X)+\mathcal{R})^{-\beta} ,\end{aligned}$$

where $\mathcal{R} = \mathcal{R}(X,Y) = (1-2(X+Y)+(X-Y)^2)^{1/2}$.

As indicated in the introduction, this is exactly what has been demonstrated in [FL] using the model of Jacobi-endofunctions. We will not go into the details of their proof, but we mention that the expression on the r.h.s. reflects the fact that in the combinatorial picture of Jacobi-endofunctions ((A,B),f) three types of connected (w.r.t. f) components show up:

- components where all the recurrent elements belong to A: type-a-components in the terminology of [FL],
- components where all the recurrent elements belong to B: type-b-components,
- mixed components, where both recurrent elements belonging to A and to B are present: type-m-components.

According to the general principles of enumeration of exponential structures (see e.g. [FO],[JO]), the exponential generating function in question must have the form

$$V_a^{1+\alpha} \cdot V_b^{1+\beta} \cdot V_m = V_a^{\alpha} \cdot V_b^{\beta} \cdot V_t$$

where $V_i = V_i(X,Y) = \sum_n (1/n!) \sum \{ v(\varphi) \ ; \ \varphi \in JAC_i[n] \}$ for $i = a$ (b, m, t, resp.) is the exponential generating function for type-a (type-b, type-m, all, resp.) configurations under the valuation

$$v(\varphi) = X^{|A|} Y^{|B|} \quad , \text{ for } \varphi = ((A,B),f) \ .$$

The same pattern will emerge below.

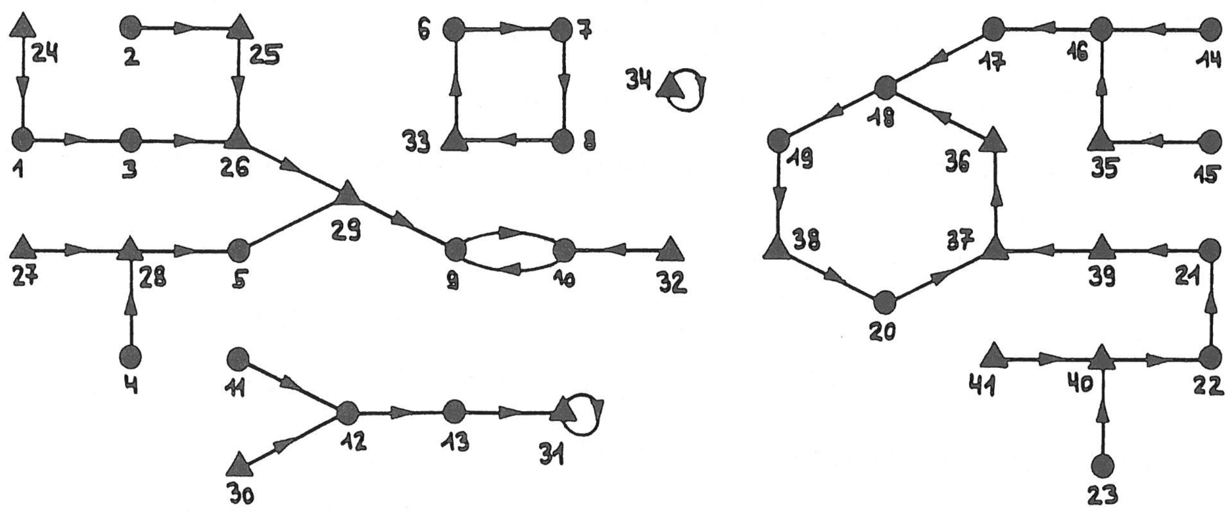


figure 1

2 Reduction, contraction, and order

The concept of 'order', as introduced in [ST], reflects further structural properties of Jacobi-endofunctions (and other types of discrete structures as e.g. the 'complete oriented matchings' which are equivalent to them) - it is a kind of complexity measure, closely related to the 'register number' of binary trees, see e.g. [FR],[PR], and the references cited there.

Let $\varphi=((A,B),f)\in JAC(S)$; an element $x\in S$ is said to be regular (singular resp.) w.r.t. f if $|f^{-1}(f(x))| = 2$ ($=1$ resp.).
φ is reduced if f has no singular points - S is then a set of even cardinality. $JAC_{red}(S)$ denotes the set of reduced Jacobi-endofunctions on S. The following facts were proved in [ST] :

Proposition: a) For any finite set S,

$$JAC(S) \simeq \bigoplus \{JAC_{red}(S)\times LAG_2(E,D)\ ;\ D\cup E=S,\ |D|\ \text{even}\}\ ;$$

b) For any finite set S of even cardinality,

$$JAC_{red}(S)\simeq \bigoplus \{JAC(A)\times BIJ(A,B)\ ;\ A\cup B=S,\ |A|=|B|\}\ .$$

Here "...$\simeq$..." is to be read as: "there exists a bijection between... and ...", $\bigoplus$ stands for a disjoint union. $LAG_2(E,D)$ denotes the set of all constructs $(((E_1,E_2),D),f)$, where (E_1,E_2) is an ordered bipartition of E, and E,D are disjoint, finite sets, and f is an injective mapping from E into $E\cup D$. Thus $((E,D),f)$ is a 'Laguerre-configuration' in the sense of [FS]. Finally BIJ(A,B) denotes the set of all bijective mappings from A into B, where it is understood that A,B are disjoint, finite sets of equal cardinality.

A proof of this proposition has been given in [ST]. In order to keep the present article reasonably self-contained, the proof will be illustrated by an example.

a) Let $A=\{1,2,...,23\}$, $B=\{24,25,...41\}$, and let $f:[41]\longrightarrow[41]$ be given via its graphical representation: (see figure 1.).
It is easy to check that $((A,B),f)\in JAC[41]$. Indeed, there is one type-a-component, there are two type-b-components, and there are two type-m-components. We have

$$D_f = \{\text{f-regular points}\} = \{3,4,5,9,10,11,13,14,17,20,23,25,26,27,29,30,31,32,35,36,39,41\}\ ,$$

$$E_f = \{\text{f-singular points}\} = \{1,2,6,7,8,12,15,16,18,19,21,22,24,28,33,34,37,38,40\}\ .$$

For $x \in D_f$ let $g(x) := f^m(x)$, where m is the least $i \geqslant 1$ s.th. $f^i(x) \in D_f$. Then $((A \cap D_f, B \cap D_f), g) \in JAC_{red}(D_f)$, which is visualized by figure 2. :

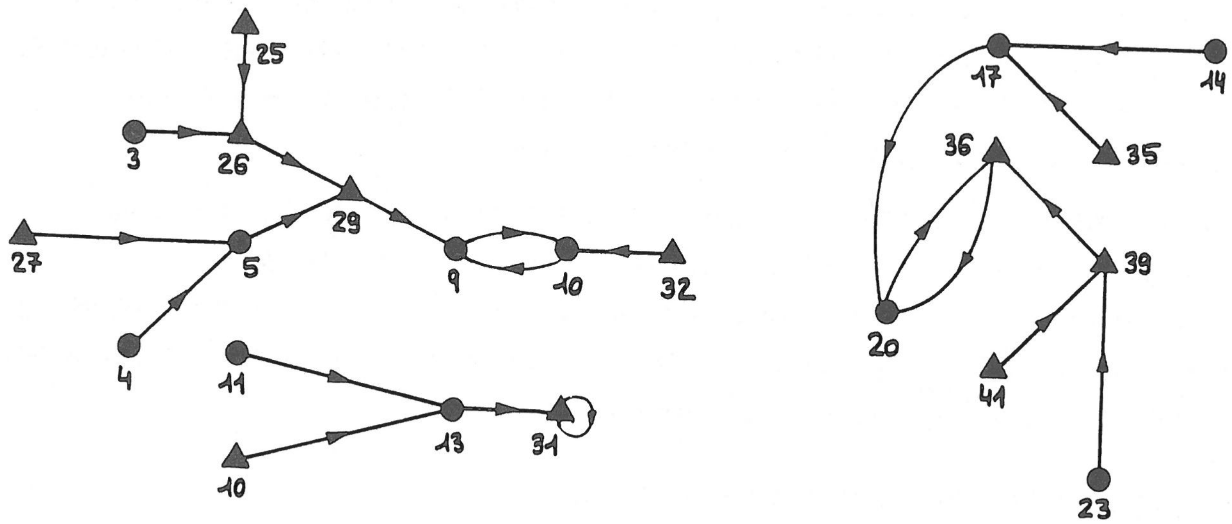


figure 2

For $x \in E$ simply let $h(x) := f(x)$, then $(((A \cap E_f, B \cap E_f), D_f), h) \in LAG_2(E_f, D_f)$; this part is visualized by figure 3. :

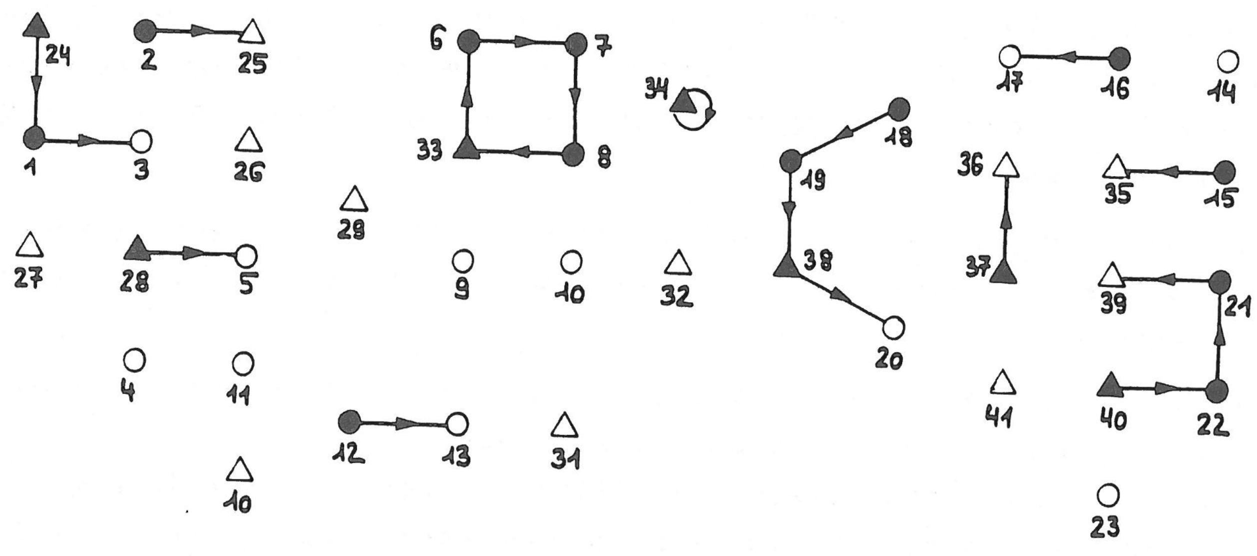


figure 3

The reader should check this decomposition ("reduction") of Jacobi--endofunctions into a 'regular part' and a 'singular part' is indeed perfectly reversible.

b) Let $A = \{3,5,7,9,10,14,16,17,18,19,20,23,25,28\}$,

$B = \{1,2,4,6,8,11,12,13,15,21,22,24,26,27\}$;

$f:[28] \longrightarrow [28]$ is given graphically by figure 4. :

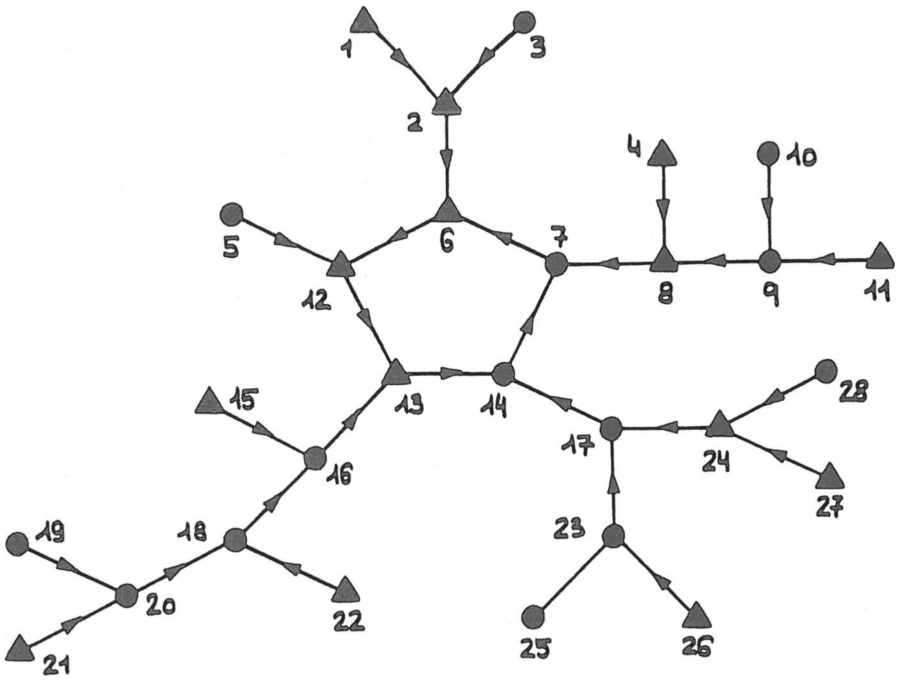


figure 4.

Obviously $((A,B),f) \in JAC_{red}[28]$. Let now

$A_f = A \cap f^{-1}(A) = \{10,14,17,18,19,20,23,25\}$,

$B_f = A \cap f^{-1}(B) = \{3,5,7,9,16,28\}$.

For $x \in A$ we define

$g(x)$:= the unique $y \in A$ s.th. $f(f(x)) = f(y)$,

so that $((A_f,B_f),g) \in JAC(A)$, as can be seen from figure 5. :

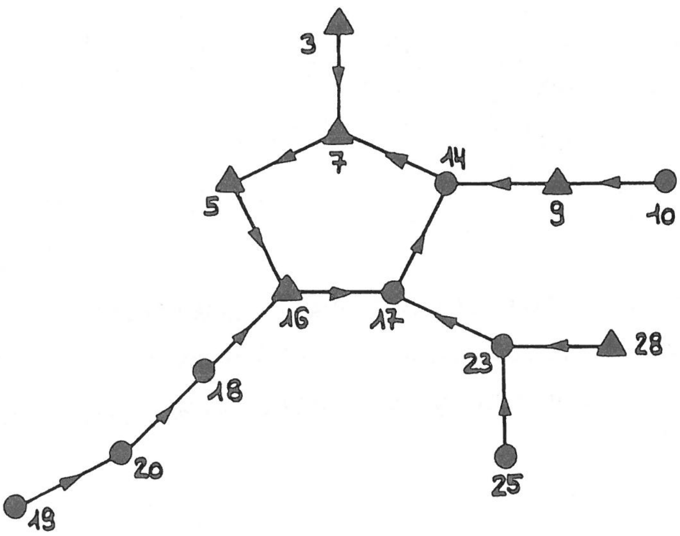


figure 5

On the other hand, for $x \in A$ we may define

$h(x) :=$ the unique $y \in B$ s.th. $f(x)=f(y)$;

then $h \in BIJ(A,B)$, in our case :

h:	A:	3	5	7	9	10	14	16	17	18	19	20	23	25	28
		↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
	B:	1	6	2	4	11	8	12	13	15	21	22	24	26	27

Again the reader should verify that the decomposition ("contraction") is reversible.

Let now $JAC := \bigoplus \{ JAC(S)$; S any finite set $\}$, and similarly for JAC_{red}. By the constructions indicated in the proof of the proposition we have mappings

$$R : JAC \longrightarrow JAC_{red} : ((A,B),f) \longmapsto ((A \cap D_f, B \cap D_f),g) \quad ,$$

$$C : JAC_{red} \longrightarrow JAC : ((A,B),f) \longmapsto ((A_f,B_f),g) \quad ,$$

called 'reduction' and 'contraction'.

Since each application of C reduces the size of a configuration by one half each configuration will eventually disappear when applying C-R-pairs repeatedly. We may thus define

$JAC^{(k)}$ ($JAC^{(k)}_{red}$ resp.), the set of Jacobi-configurations (reduced Jacobi configurations resp.) of order k,

inductively by:

$JAC^{(0)}_{red} := \{\emptyset\}$, where $\emptyset$ denotes the empty function ,

$JAC^{(0)} := R^{-1}(JAC^{(0)}_{red})$, which is the set of all $((A,B),f)$ with f a permutation of $A \cup B$;

$JAC^{(k)}_{red} := C^{-1}(JAC^{(k-1)})$, for $k > 0$,

$JAC^{(k)} := R^{-1}(JAC^{(k-1)}_{red})$, for $k > 0$.

We have

$$\begin{array}{ccccccccccc} JAC^{(0)}_{red} & \subseteq & JAC^{(1)}_{red} & \subseteq & \cdots\cdots & \subseteq & JAC^{(k)}_{red} & \subseteq & JAC^{(k+1)}_{red} & \subseteq & \cdots \\ \cap\!\!| & & \cap\!\!| & & & & \cap\!\!| & & \cap\!\!| & & \\ JAC^{(0)} & \subseteq & JAC^{(1)} & \subseteq & \cdots\cdots & \subseteq & JAC^{(k)} & \subseteq & JAC^{(k+1)} & \subseteq & \cdots \end{array}$$

and the limits of the horizontal $\subseteq$-chains are JAC_{red} and JAC, respectively.

As a side remark, we note that the functions appearing in $JAC^{(1)}$ are precisely the "pieuvres" introduced by F.Bergeron in [BE] as a combinatorial model for the study of orthogonal polynomials.

3 Generating functions - the basic recurrence

What we are interested in is an explicit expression for the exponential generating function

$$\mathbb{P}_k^{(\alpha,\beta)}(X,Y) := \sum_n (1/n!) \sum \{ w(\varphi) \; ; \; \varphi \in JAC^{(k-1)}[n] \} \quad ,$$

for $k>0$. According to the general principles mentioned at the end of section 1 we will have

$$\mathbb{P}_k^{(\alpha,\beta)}(X,Y) = \Big(\sum_n (1/n!) \sum \{ v(\varphi) \; ; \; \varphi \in JAC_a^{(k-1)}[n] \} \Big)^{1+\alpha} \cdot \Big(\sum_n (1/n!) \sum \{ v(\varphi) \; ; \; \varphi \in JAC_b^{(k-1)}[n] \} \Big)^{1+\beta} \cdot \Big(\sum_n (1/n!) \sum \{ v(\varphi) \; ; \; \varphi \in JAC_m^{(k-1)}[n] \} \Big) \quad .$$

If we define

$$F_k(X_1,X_2,Y_1,Y_2) := \sum_n (1/n!) \sum \{ v(\varphi) \; ; \; \varphi \in JAC^{(k-1)}[n] \} \quad ,$$

where for $\varphi = ((A,B),f) \in JAC$ we put

$$A_1 := A \cap R(f) \;,\quad A_2 := A \setminus A_1 \;,\quad B_1 := B \cap R(f) \;,\quad B_2 := B \setminus B_1 \;,$$

$$v(\varphi) := X_1^{|A_1|} X_2^{|A_2|} Y_1^{|B_1|} Y_2^{|B_2|} \quad ,$$

then
$$\sum_n (1/n!) \sum \{ v(\varphi) \; ; \; \varphi \in JAC_a^{(k-1)}[n] \} = F_k(X,X,0,Y) \;,$$
$$\sum_n (1/n!) \sum \{ v(\varphi) \; ; \; \varphi \in JAC_b^{(k-1)}[n] \} = F_k(0,X,Y,Y) \;,$$
and
$$\sum_n (1/n!) \sum \{ v(\varphi) \; ; \; \varphi \in JAC^{(k-1)}[n] \} = F_k(X,X,Y,Y) \;,$$
so that
$$\sum_n (1/n!) \sum \{ v(\varphi) \; ; \; \varphi \in JAC_m^{(k-1)}[n] \} = \frac{F_k(X,X,Y,Y)}{F_k(X,X,0,Y) \cdot F_k(0,X,Y,Y)}$$

Thus our final result will be presented in the following form:

Proposition: For any $k>0$, the exponential generating function for Jacobi-endofunctions of order $k-1$ is given by

$$\mathbb{P}_k^{(\alpha,\beta)}(X,Y) = F_k(X,X,0,Y)^{\alpha} \cdot F_k(0,X,Y,Y)^{\beta} \cdot F_k(X,X,Y,Y) \quad .$$

The clue to the determination of $F_k(X_1,X_2,Y_1,Y_2)$ is contained in the following basic recurrence:

Proposition: For any $k>0$,

$$F_{k+1}(X_1,X_2,Y_1,Y_2) = F_1(X_1,X_2,Y_1,Y_2)\, F_k(\xi_1\eta_2, \xi_2\eta_2, \xi_2\eta_1, \xi_2\eta_2)$$

where $\xi_i = X_i/(1-X_i-Y_i)$, $\eta_i = Y_i/(1-X_i-Y_i)$, $(i=1,2)$.

Proof: First note that

$$F_1(X_1,X_2,Y_1,Y_2) = 1/(1-X_1-Y_1) \quad ,$$

which is just the exponential generating function for "bicolored permutations", i.e. pairs (permutation,bipartition). This factor takes care of those components which disappear when passing from $JAC^{(k)}$ to $JAC^{(k)}_{red}$ via reduction: these are the components which consist entirely of singular points, and these components together form a configuration of order 0. Its cofactor gives the contribution to $JAC^{(k)}$ which comes from those components which contain at least one (and thus at least two, because they always come in pairs) regular point(s). These components may be reconstructed in all possible ways from the objects counted by F_k in two steps - by simply reversing the reduction-contraction procedure:

1) replacing each point in some order-(k-1)-configuration by a pair of points, or, more specifically:
   - replacing each non-recurrent (i.e. X_2-or Y_2-valued) point by a pair consisting of one X_2-valued and one Y_2-valued point (i.e. both non-recurrent);
   - replacing each recurrent (i.e. X_1-or Y_1-valued) point by a pair consisting of one recurrent (X_1-or Y_1-valued) and one non-recurrent (Y_2- or X_2-valued) point.

   [all points existing at this stage will be regular].
2) adding to each of these (regular) points a (possibly empty) sequence of singular points, or, more precisely:
   - adding to each non-recurrent (i.e. X_2- or Y_2-valued) point a sequence of (non-recurrent) points which may carry X_2 or Y_2 as valuation;
   - adding to each recurrent (i.e. X_1-or Y_1-valued) point a sequence of (recurrent) points which may carry X_1 or Y_1 as valuation.

   [all the points introduced at this stage will be singular].

Apart from the fact that the functional parts of the Jacobi-configurations have to be assigned to the point sets thus constructed in a compatible way - this can be reconstructed from the description of reduction (i.e. the inverse of step 2) and contraction (i.e. the inverse of step 1) given in section 2 - it should be clear that on a quantitative basis this inverse procedure is then described by the substitutions given in the proposition.
(Note that Jacobi-configurations are actually labelled configurations. We have not considered labellings at all here - this is what the exponential generating functions automatically take care of).

To conclude this section, figure 6. gives an illustration how the substitution procedure of the proposition may be visualized (locally):

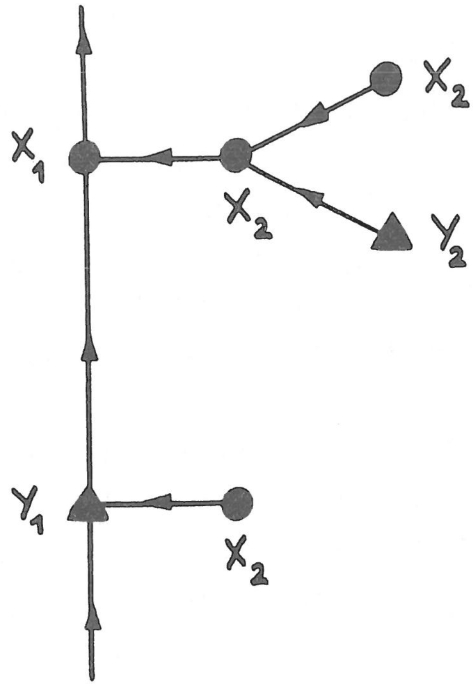
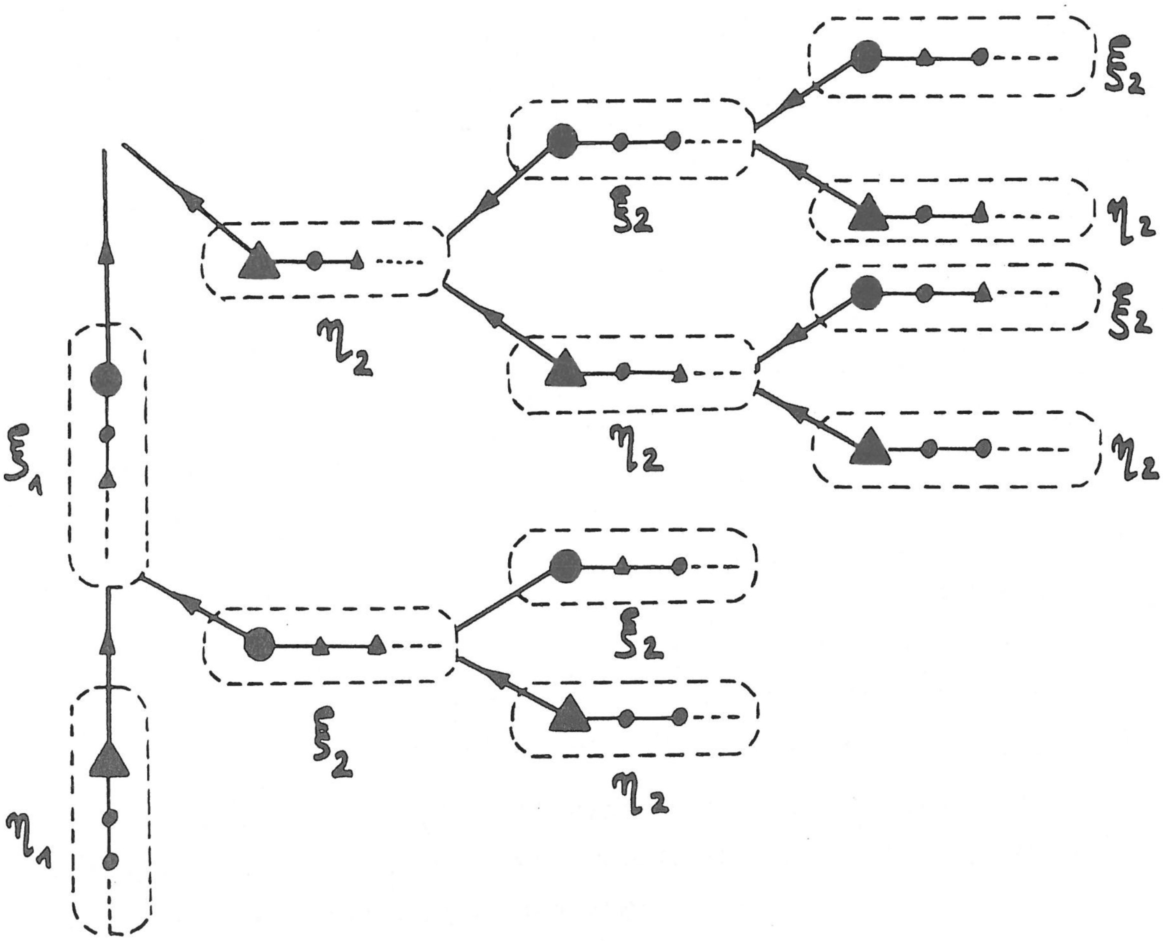


figure 6. :

the substitution viewed locally



4 Statement of the main result

In order to state the main result we introduce a kind of matching polynom[ial] for graphs of type 'line' and of type 'cycle'.

Let L_n (C_n resp.) denote the line (cycle resp.) on n vertices. We may choo[se] $[n]=\{1,2,...,n\}$ as vertex set for both L_n and C_n ; the edge sets are

$$E(L_n) = \{(i,i+1);\ 1\leq i<n\} \quad \text{and} \quad E(C_n) = E(L_n)\cup\{(n,1)\}\ .$$

A matching is then a subset $\mu \subseteq E(L_n)$ (or $\subseteq E(C_n)$) such that no two edges in μ have a vertex in common. $\mathcal{M}(L_n)$ ($\mathcal{M}(C_n)$ resp.) will denote the set of all matchings on L_n (C_n resp.). Each element of $E(L_n)$ will be given a weight as follows:

- the edges $(2i-1,2i)$, where $1\leq i<\lceil n/2\rceil$, are given weight $-x$,
- the edges $(2i,2i+1)$, where $1\leq i<\lceil n/2\rceil$, are given weight $-y$,
- if n is even, the edge $(n-1,n)$ is given weight $-z$.

For $E(C_n)$ we proceed similarly, and we complete the definition by:

- the edge $(n,1)$ is given weight $-w$.

To each matching $\mu\in\mathcal{M}(L_n)$ (or $\mu\in\mathcal{M}(C_n)$) we associate a monomial $M(\mu)$ in the variables x,y,z,w (where z and w do not necessarily show up) : this is the product of the weights of all the edges belonging to μ .

The matching polynomials associated to the graphs L_n and C_n are then

$$L_n(x,y,z) := \sum \{ M(\mu)\ ;\mu\in\mathcal{M}(L_n) \} \ ,$$
$$C_n(x,y,z,w) := \sum \{ M(\mu)\ ;\mu\in\mathcal{M}(C_n) \} \ .$$

The reader may check that the following are the first few values:

$$L_1(x,y,z) = 1$$
$$L_2(x,y,z) = 1-z$$
$$L_3(x,y,z) = 1-x-y$$
$$L_4(x,y,z) = 1-x-y-z+xz$$
$$L_5(x,y,z) = 1-2x-2y+x^2+xy+y^2$$
$$L_6(x,y,z) = 1-2x-2y-z+x^2+xy+2xz+y^2+yz$$

$$C_1(x,y,z,w) = 1$$
$$C_2(x,y,z,w) = 1-z-w$$
$$C_3(x,y,z,w) = 1-x-y-w$$
$$C_4(x,y,z,w) = 1-x-y-z-w+xz+yw$$
$$C_5(x,y,z,w) = 1-2x-2y-w+x^2+xy+y^2+yw+xw$$
$$C_6(x,y,z,w) = 1-2x-2y-z-w+x^2+xy+2xz+xw+y^2+2yw+yz$$

It is clear that for n odd the polynomials $L_n(x,y,z)$ and $C_n(x,y,z,w)$ do not contain the variable z (in these cases we may simply write $L_n(x,y)$ and $C_n(x,y,w)$), and that for n even these polynomials are linear in z. The polynomials $C_n(x,y,z,w)$ are always linear in w.

Indeed, all these polynomials are easily represented in terms of the two-variable polynomials $L_n(x,y)$ (for n odd), $L_n(x,y,x)$ and $L_n(y,x,y)$ (for n even):

(1a) $L_{2k}(x,y,z) = L_{2k-1}(x,y) - zL_{2k-2}(x,y,x)$,

(1b) $C_{2k}(x,y,z,w) = L_{2k-1}(x,y) - wL_{2k-2}(y,x,y) - zL_{2k-2}(x,y,x)$,

(1c) $C_{2k+1}(x,y,w) = L_{2k+1}(x,y) - wL_{2k-1}(x,y)$.

These identities (and many more of the same type) are simple consequences from the underlying matching model. We mention as examples

(2a) $L_{2k-1}(x,y) = L_{2k-2}(x,y,x) - yL_{2k-3}(x,y)$,

(2b) $L_{2k-1}(x,y) = L_{2k-2}(y,x,y) - xL_{2k-3}(x,y)$,

which can be combined to give

(2c) $zL_{2k-2}(x,y,x) + wL_{2k-2}(y,x,y) = -(z+w)L_{2k-1}(x,y) + (yz+xw)L_{2k-3}(x,y)$,

and, together with (1b) :

(3) $C_{2k}(x,y,z,w) = (1-z-w)L_{2k-1}(x,y) - (yz+xw)L_{2k-3}(x,y)$.

This will be used below.

Our main result can now be stated:

Theorem: The exponential generating function $F_k(X_1,X_2,Y_1,Y_2)$ for Jacobi-endofunctions of order k-1 can be written in terms of matching polynomials as:

$$F_k(X_1,X_2,Y_1,Y_2) = \frac{L_{2^k-1}(X_2,Y_2)}{C_{2^k}(X_2,Y_2,X_1,Y_1)} .$$

Since for n even we obviously have

$$C_n(x,y,z,0) = L_n(x,y,z) ,$$
$$C_n(x,y,0,w) = L_n(y,x,w) ,$$

we find:

Corollary: The exponential generating function $\mathbb{P}_k^{(\alpha,\beta)}(X,Y)$ for Jacobi-endofunctions of order k-1 can be written in terms of matching polynomials as

$$\mathbb{P}_k^{(\alpha,\beta)}(X,Y) = \left[\frac{L_{2^k-1}(X,Y)}{L_{2^k}(X,Y,X)}\right]^{\alpha} \cdot \left[\frac{L_{2^k-1}(X,Y)}{L_{2^k}(Y,X,Y)}\right]^{\beta} \cdot \left[\frac{L_{2^k-1}(X,Y)}{C_{2^k}(X,Y,X,Y)}\right] .$$

This result can be written in a more attractive way. We make use of the combinatorially obvious identities

(4) $C_{2n}(X,Y,X,Y) = L_{2n}(X,Y,X) + L_{2n}(Y,X,Y) - L_{2n-1}(X,Y)$, and

(5) $L_{2n}(X,Y,X) - XL_{2n-1}(X,Y) = L_{2n+1}(X,Y) = L_{2n}(Y,X,Y) - YL_{2n-1}(X,Y)$.

By substituting (5) into (4) we find

$$(6)\quad (1-X+Y)L_{2n-1}(X,Y) + C_{2n}(X,Y,X,Y) = 2L_{2n}(X,Y,X) \;.$$

Putting now

$$\mathcal{R}_k(X,Y) := \frac{C_{2^k}(X,Y,X,Y)}{L_{2^k-1}(X,Y)} \quad ,$$

we finally arrive at:

$$\mathbb{P}_k^{(\alpha,\beta)}(X,Y) = [\frac{2}{1-(X-Y)+\mathcal{R}_k}]^{\alpha}\cdot[\frac{2}{1-(Y-X)+\mathcal{R}_k}]^{\beta}\cdot \mathcal{R}_k^{-1}$$

which exactly matches the form of the classical Jacobi-generating function

The duplication formulas for the matching polynomials

he proof of the theorem stated in the previous section relies on a simple roperty of matching polynomials which is stated and proved combinatorially elow. Though we will only need the result for 'lines', the corresponding esult for 'cycles' will be stated, without going into the details of the similar) proof.

et us again consider the graphs L_n and C_n, as in section 4. Now we put weight v on each vertex and a weight -u on each edge. To each matching $\mu \in \mathcal{M}(L_n)$ (or $\mathcal{M}(C_n)$) we now associate the monomial $m(\mu)$ (in the variables u and v) : the product of the weights of the edges in times the product of the weights of the vertices not covered by an edge of μ, i.e. $m(\mu) = (-u)^{|\mu|}v^{n-2|\mu|}$, where $|\mu|$ denotes the number of edges in μ. The matching polynomials l_n, c_n are then defined by

$$l_n(u,v) = \sum \{ m(\mu) ; \mu \in \mathcal{M}(L_n) \} ,$$
$$c_n(u,v) = \sum \{ m(\mu) ; \mu \in \mathcal{M}(C_n) \} .$$

For convenience , we give a table of the first few values:

$l_1(u,v) = v$	$c_1(u,v) = v$
$l_2(u,v) = v^2-u$	$c_2(u,v) = v^2-2u$
$l_3(u,v) = v^3-2uv$	$c_3(u,v) = v^3-3uv$
$l_4(u,v) = v^4-3uv^2+u^2$	$c_4(u,v) = v^4-4uv^2+2u^2$
$l_5(u,v) = v^5-4uv^3+2u^2v$	$c_5(u,v) = v^5-5uv^3+6u^2v$
$l_6(u,v) = v^6-5uv^4+6u^2v^2-u^3$	$c_6(u,v) = v^6-6uv^4+9u^2v^2-2u^3$

The following identities are obvious from the definition

$$l_n(u,v) = u^{n/2}l_n(1,v/u^{1/2}) = v^n l_n(u/v^2,1) \quad ,$$
$$c_n(u,v) = u^{n/2}c_n(1,v/u^{1/2}) = v^n c_n(u/v^2,1) \quad .$$

It is also clear that by unifying variables in L_n, C_n we can pass to l_n,c_n :

$$L_n(x,x,x) = l_n(x,1) \quad ,$$
$$C_n(x,x,x,x) = c_n(x,1) \quad .$$

Combining these observations we get

$$\text{(7a)} \quad L_n(x/y^2,x/y^2,x/y^2) = y^{-n}l_n(x,y) \quad ,$$
$$\text{(7b)} \quad C_n(x/y^2,x/y^2,x/y^2,x/y^2) = y^{-n}c_n(x,y) \quad .$$

A more interesting relation is given in the following

Lemma (the duplication formulas) :

$$\text{(8a)} \quad l_n(xy,1-x-y) = L_{2n+1}(x,y) \quad ,$$
$$\text{(8b)} \quad c_n(xy,1-x-y) = C_{2n}(x,y,x,y) \quad .$$

Proof: Only (8a) will be proved - a proof of (8b) can be given along the same lines.

For the purpose of proof we will introduce the notion of 'pseudomatching' (p.m.). A p.m. on L_{2n+1} is a subset $\tilde{\pi}$ of $E(L_{2n+1})$ such that none of the vertices 2i ($1 \leq i \leq n$) is covered by two edges in $\tilde{\pi}$. It is thus permitted that both (2i-2,2i-1) and (2i-1,2i) belong to $\tilde{\pi}$, and we will speak of a 'collision at vertex 2i-1 in $\tilde{\pi}$' in this case. (Note that if there is a collision at vertex 2i-1 in $\tilde{\pi}$ then there can't be a collision at vertex 2i+1 in $\tilde{\pi}$ (and vice versa), because otherwise we would also have a collisic at vertex 2i in $\tilde{\pi}$). By $<\tilde{\pi}>$ we will denote the number of collisions in $\tilde{\pi}$; the pseudomatchings $\tilde{\pi}$ with $<\tilde{\pi}> = 0$ are then exactly the matchings.

$\tilde{\mathfrak{M}}(L_{2n+1})$ will then denote the set of all p.m.'s on L_{2n+1}.

We will now define a relation $Q_n \subseteq \mathfrak{M}(L_n) \times \tilde{\mathfrak{M}}(L_{2n+1})$:

$(\mu, \tilde{\pi}) \in Q_n$ iff for all $1 \leq i < n$: if $(i,i+1) \in \mu$, then there is a collision at vertex 2i+1 in $\tilde{\pi}$.

Thus if $\mu \in \mathfrak{M}(L_n)$, an edge $(i,i+1) \in \mu$ determines the situation for $\tilde{\pi}$ between 2i-1 and 2i+3 (if $(\mu, \tilde{\pi}) \in Q_n$): edges (2i,2i+1) and (2i+1,2i+2) are present in $\tilde{\pi}$, wheras (2i-1,2i) and (2i+2,2i+3) aren't.

On the other hand, if $i \in [n]$ is not covered by an edge of μ, then at most one of the edges (2i-1,2i) and (2i,2i+1) may belong to $\tilde{\pi}$ (if $(\mu, \tilde{\pi}) \in Q_n$), which gives three possibilities. We find that for each $\mu \in \mathfrak{M}(L_n)$ there are exactly $3^{n-2|\mu|}$ $\tilde{\pi} \in \tilde{\mathfrak{M}}(L_{2n+1})$ s.th. $(\mu, \tilde{\pi}) \in Q_n$. Looking at these possibilities and taking the valuations of the edges into account gives:

Fact 1: for each $\mu \in \mathfrak{M}(L_n)$:

$$m(\mu)\big|_{u \leftarrow xy,\, v \leftarrow 1-x-y} = (-1)^{|\mu|} \sum_{\tilde{\pi}} \{ M(\tilde{\pi}) \; ; \; \tilde{\pi} \in \tilde{\mathfrak{M}}(L_{2n+1}),\; (\mu, \tilde{\pi}) \in Q_n \}$$

Next we take $\tilde{\pi} \in \tilde{\mathfrak{M}}(L_{2n+1})$. If there is no collision at vertex 2i+1 in $\tilde{\pi}$, then vertex i cannot be covered by an edge of μ if $(\mu, \tilde{\pi}) \in Q_n$. But if there is a collision at vertex 2i+1 in $\tilde{\pi}$, then μ may or may not contain the edg (i,i+1). [Note that we are guaranteed that edges (i-1,i) and (i+1,i+2) are not in μ in this case!]. We find that for each $\tilde{\pi} \in \tilde{\mathfrak{M}}(L_{2n+1})$ there are $2^{<\tilde{\pi}>}$ $\mu \in \mathfrak{M}(L_n)$ s.th. $(\mu, \tilde{\pi}) \in Q_n$. More precisely: for each $0 \leq k \leq <\tilde{\pi}>$ there are $\binom{<\tilde{\pi}>}{k}$ $\mu \in \mathfrak{M}(L_n)$ s.th. $(\mu, \tilde{\pi}) \in Q_n$ and $|\mu| = k$.

As a consequence we get:

Fact 2: For each $\tilde{\pi} \in \tilde{\mathfrak{M}}(L_{2n+1})$:

$$\sum_{\mu} \{ (-1)^{|\mu|} \; ; \; \mu \in \mathfrak{M}(L_n),\; (\mu, \tilde{\pi}) \in Q_n \} = \begin{cases} 1 & \text{if } <\tilde{\pi}> = 0 , \\ 0 & \text{if } <\tilde{\pi}> > 0 . \end{cases}$$

Now the proof of the lemma is easily completed:

$$l_n(xy,1-x-y) = \sum_{\mu} \{ m(\mu) \; ; \mu \in \mathfrak{M}(L_n) \} \Big|_{u \leftarrow xy, v \leftarrow 1-x-y}$$

$$\overset{(F1)}{=} \sum_{\mu} \{ (-1)^{|\mu|} \sum_{\tilde{\pi}} \{ M(\tilde{\pi}) ; \tilde{\pi} \in \tilde{\mathfrak{M}}(L_{2n+1}), (\mu,\tilde{\pi}) \in Q_n \} ; \mu \in \mathfrak{M}(L_n) \}$$

$$= \sum_{\tilde{\pi}} \{ M(\tilde{\pi}) \sum_{\mu} \{ (-1)^{|\mu|} ; \mu \in \mathfrak{M}(L_n), \; (\mu,\tilde{\pi}) \quad Q_n \} ; \tilde{\pi} \in \tilde{\mathfrak{M}}(L_{2n+1})) \}$$

$$\overset{(F2)}{=} \sum_{\tilde{\pi}} \{ M(\tilde{\pi}) \; ; \tilde{\pi} \in \tilde{\mathfrak{M}}(L_{2n+1}), \; \langle \tilde{\pi} \rangle = 0 \}$$

$$= \quad L_{2n+1}(x,y) \quad .$$

To conclude this section we mention briefly the relation between the Tchebycheff-polynomials $U_n(x)$, $T_n(x)$ (in the standard terminology of [AB], [ER], [RA]) and our matching polynomials for L_n and C_n :

$$l_n(1,x) = U_n(x/2) \quad ,$$
$$c_n(1,x) = 2 \cdot T_n(x/2) \quad .$$

This follows from the fact that the familiar recurrence formulae for the Tchebycheff-polynomials:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \quad , \quad U_0(x) = 1, \; U_1(x) = 2x \; ,$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad , \quad T_0(x) = 1, \; T_1(x) = x \; ,$$

correspond to

$$l_{n+1}(1,x) = xl_n(1,x) - l_{n-1}(1,x) \; , \; l_0(1,x) = 1, \; l_1(1,x) = x \; ,$$

$$c_{n+1}(1,x) = xc_n(1,x) - c_{n-1}(1,x) \; , \; c_0(1,x) = 2, \; c_1(1,x) = x \; .$$

These two identities follow from the combinatorial definition of the polynomials $l_n(u,v)$, $c_n(u,v)$ just as the corresponding identities (1a),(1b), (1c) in section 4. One might now translate all the expressions involving the matching polynomials $L_{2n}(x,y,z)$, $L_{2n+1}(x,y)$, and $C_{2n}(x,y)$ into expressions involving the Tchebycheff-polynomials using the identities stated at the beginning of this section.

6 Proof of the main result

The theorem stated in section 4 can now be proved by combining the basic recurrence for the $F_k(X_1,X_2,Y_1,Y_2)$ (sec.3) and some facts about matching polynomials (sec.4 and sec.5).
Recall that

$$F_1(X_1,X_2,Y_1,Y_2) = \frac{1}{1-X_1-Y_1} = \frac{L_1(X_2,Y_2)}{C_2(X_2,Y_2,X_1,Y_1)}$$

From the basic recurrence we get via induction

$$F_{k+1}(X_1,X_2,Y_1,Y_2) = F_1(X_1,X_2,Y_1,Y_2)\cdot F_k(\xi_1\eta_2,\xi_2\eta_2,\xi_2\eta_1,\xi_2\eta_2)$$

$$= \frac{1}{1-X_1-Y_1}\cdot\frac{L_{2^k-1}(\xi_2\eta_2,\xi_2\eta_2)}{C_{2^k}(\xi_2\eta_2,\xi_2\eta_2,\xi_1\eta_2,\xi_2\eta_1)} ,$$

where $\xi_i = \frac{X_i}{1-X_i-Y_i}$, $\eta_i = \frac{Y_i}{1-X_i-Y_i}$, $i=1,2$.

Due to (1b) the denominator can be rewritten as

$$(1-X_1-Y_1)\left(L_{2^k-1}(\xi_2\eta_2,\xi_2\eta_2) - (\xi_1\eta_2+\xi_2\eta_1)L_{2^k-2}(\xi_2\eta_2,\xi_2\eta_2,\xi_2\eta_2)\right) .$$

Using (7a) in both numerator and denominator leads to

$$\frac{(1-X_2-Y_2)^{-2^k+1}\, l_{2^k-1}(X_2Y_2,1-X_2-Y_2)}{(1-X_1-Y_1)(1-X_2-Y_2)^{-2^k+1}\left[l_{2^k-1}(X_2Y_2,1-X_2-Y_2) - \frac{X_1Y_2+X_2Y_1}{1-X_1-Y_1}\cdot l_{2^k-2}(X_2Y_2,1-X_2-Y_2)\right]}$$

$$= \frac{l_{2^k-1}(X_2Y_2,1-X_2-Y_2)}{(1-X_1-Y_1)l_{2^k-1}(X_2Y_2,1-X_2-Y_2) - (X_1Y_2+X_2Y_1)l_{2^k-2}(X_2Y_2,1-X_2-Y_2)}$$

Now the duplication formula (8a) comes into play, which gives

$$\frac{L_{2^{k+1}-1}(X_2,Y_2)}{(1-X_1-Y_1)L_{2^{k+1}-1}(X_2,Y_2) - (X_1Y_2+X_2Y_1)L_{2^{k+1}-3}(X_2,Y_2)} .$$

Identity (3) allows to rewrite the denominator as

$$C_{2^{k+1}}(X_2,Y_2,X_1,Y_1) ,$$

so that we finally arrive at

$$F_{k+1}(X_1,X_2,Y_1,Y_2) = \frac{L_{2^{k+1}-1}(X_2,Y_2)}{C_{2^{k+1}}(X_2,Y_2,X_1,Y_1)} ,$$

as desired.

Some additional remarks

) It is clear from the underlying combinatorial model that the generating unctions $F_k(X,Y,X,Y)$ converge to $\mathcal{R}^{-1}(X,Y)$ as $k \longrightarrow \infty$ in the usual (disrete) topology for formal power series. This can be stated more precisely y observing that Jacobi-endofunctions of order $> k$ have size 2^k (at least), ince contraction reduces the size of each configuration by one half. Thus he series $F_k(X,Y,X,Y)$ and $\mathcal{R}^{-1}(X,Y) = \mathbb{P}^{(0,0)}(X,Y)$ coincide for all terms f degree $< 2^k$. The same holds for $\mathbb{P}_k^{(\alpha,\beta)}(X,Y)$ converging to $\mathbb{P}^{(\alpha,\beta)}(X,Y)$ s $k \longrightarrow \infty$.

) It is worth noting that the polynomials $L_{2^n-1}(x,y)$ and $C_{2^n}(x,y)$ which ppear in the rational approximation can easily be calculated. Among the dentities of relevance we mention

9) $$c_{2^{n+1}}(u,v) = c^2_{2^n}(u,v) - 2u^{2^n} ,$$

10) $$l_{2^{n+1}-1}(u,v) = c_{2^n}(u,v)\, l_{2^n-1}(u,v) .$$

nstead of giving combinatorial proofs of these identities "from scratch" which is not difficult), we content ourselves to remark that (10) is equivalent to

$$l_{2^{n+1}-1}(u,v) = c_1(u,v)\cdot c_2(u,v)\cdot c_4(u,v)\cdot \ldots \cdot c_{2^n}(u,v)$$

and that (9) is nothing but the duplication formula (8b) in disguise. To justify this remark let us write

$$c_{2n}(u,v) = v^{2n}C_{2n}(u/v^2,u/v^2) = v^{2n}c_n(u^2/v^4,1-2u^2/v^4) = c_n(u^2,v^2-2u) .$$

If we define polynomials $\overline{c}_k(u,v)$, $\overline{b}_k(u,v)$ by

$$\overline{b}_k(u,v) := u^{2^k} ,$$
$$\overline{c}_k(u,v) := c_{2^k}(u,v) ,$$

for $k \geqslant 0$, then we have the simultaneous recursion

$$\overline{b}_{k+1}(u,v) = \overline{b}_k(\overline{b}_1(u,v),\overline{c}_1(u,v)) ,$$
$$\overline{c}_{k+1}(u,v) = \overline{c}_k(\overline{b}_1(u,v),\overline{c}_1(u,v)) .$$

This extends obviously to

$$\overline{b}_{k+n}(u,v) = \overline{b}_k(\overline{b}_n(u,v),\overline{c}_n(u,v)) ,$$

$$\overline{c}_{k+n}(u,v) = \overline{c}_k(\overline{b}_n(u,v),\overline{c}_n(u,v)) ,$$

for $k,n \geqslant 0$. Putting now k=1 in the second identity gives

$$c_{2^{n+1}}(u,v) = \overline{c}_{1+n}(u,v) = \overline{c}_1(\overline{b}_n(u,v),\overline{c}_n(u,v)) = c^2_{2^n}(u,v) - 2u^{2^n} .$$

Along the same lines one can derive a generalized basic recursion:

$$F_{k+n}(X_1,X_2,Y_1,Y_2) = F_k(X_1,X_2,Y_1,Y_2)\cdot F_n(\xi_1^{(k)}\eta_2^{(k)},\xi_2^{(k)}\eta_2^{(k)},\xi_2^{(k)}\eta_1^{(k)},\xi_2^{(k)}\eta_2^{(k)})$$

where

$$\xi_i^{(k)} = \frac{X_iY_2(X_2Y_2)^{2^{k-2}-1}}{C_{2^k}(X_2,Y_2,X_i,Y_i)} \quad , \quad \eta_i^{(k)} = \frac{Y_iX_2(X_2Y_2)^{2^{k-2}-1}}{C_{2^k}(X_2,Y_2,X_i,Y_i)} \ , \ i = 1,2 \ ,$$

for $k,n \geqslant 1$.

3) As a concluding remark, the present author would like to draw the reader's attention to work in the same direction done by G.X.Viennot and his bordelais school of bijective combinatorics, where similar results were obtained recently. Indeed, what has been presented here for the special case of Jacobi-polynomials should be seen in a broader context, and a considerable portion of the results could be synthesized from specializations of more general theories under development. The interested reader should consult in particular the work of Viennot[VI], de Sainte--Catherine [SC], Viennot and Vauchaussade [VV], and Vauchaussade [VA]. Let us mention finally that the idea of combinatorially interpreting quotients of matching polynomials as generating functions for treelike structures is due to Godsil [GO].

References

[AB] Abramowitz,M. and I.Stegun, Handbook of Mathematical Functions, Dover 1965.

[AS] Askey,R. Jacobi's generating function for the Jacobi polynomials, Proc.Amer.Math.Soc. 71 (1978), 243-246.

[ER] Erdélyi et al.(eds.) Higher Transcendental Functions, 3 vols. McGraw-Hill, 1953, (reprint: Krieger Publ.Comp. 1981).

[FO] Foata,D. La série génératrice exponentielle das les problèmes d'énumeration, Séminaire de Mathématiques Supérieures - été 1971, Les Preses de l'Université de Montréal, 1974.

[FL] Foata,D. and P.Leroux Polynômes de Jacobi, interpretation combinatoire et fonction génératrice, Proc.Amer.Math.Soc. 87 (1983), 47-53.

[FS] Foata,D. and V.Strehl Combinatorics of Laguerre Polynomials, Proc. Waterloo Silver Jubilee Conf.,in: Enumeration and Design (D.M.Jackson and S.A.Vanstone eds.) Academic Press, Toronto 1984.

[FR] Francon,J. Sur le nombre de registres nécessaires a l'évaluation d'une expression arithmetique, R.A.I.R.O. Informatique Théorique 18 (1984), 355-364.

[GO] Godsil,C. Matchings and walks in graphs, J.Graph Theory 5 (1981), 285-291.

[JO] Joyal,A. Une théorie combinatoire des séries formelles, Adv.in Math. 42 (1981),1-82.

[LS] Leroux,P. and V.Strehl Jacobi-Polynomials: Combinatorics of the Basic Identities, to appear in Discrete Mathematics.

[PR] Prodinger,H. Abzaehlprobleme bei Baeumen, Seminaire Lotheringien de Combinatoire IX, Hollabrunn (Austria) 1983, Actes du Seminaire (G.Baron and P.Kirschenhofer eds.), 167-173.

[RA] Rainville,E. Special Functions, Chelsea, Bronx, N.Y.1960.

[SC] de Sainte-Catherine,M. Couplages et Pfaffiens en Combinatoire, Physique et Informatique, Thèse 3me cycle, Université de Bordeaux, 1983.

[ST] Strehl,V. Combinatorics of Jacobi-configurations I : complete oriented matchings, to appear in: Proc. "Colloque de Combinatoire Enumerative", Université du Québec à Montréal, may 28 - june 1st 1985.

[VA] Vauchaussade de Chaumont,M. Nombres de Strahler des arbres,langages algébriques et denombrement de structures secondaires en biologie moleculaire, These 3me cycle, Universite de Bordeaux, 1985.

[VV] Vauchaussade de Chaumont,M. and G.X.Viennot Enumeration of RNA's secondary structure by complexity Proc.Intern.Conf."Mathematics in Medecine and Biology", Bari, Italy, july 1983.

[VI] Viennot,G.X. Une théorie combinatoire des polynômes orthogonaux généraux, Notes de conferences données a l'Université du Québec à Montréal, 1983, 217p.