# OPERATIONS ON MAPS AND HYPERMAPS

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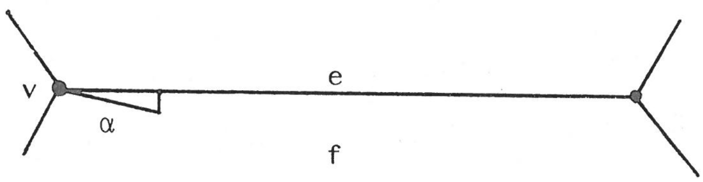
## 1. Introduction

The dual map of a cube is an octahedron : they are combinatorially distinct , but they have the same 'size' (e.g. number of edges) and the same automorphism group . Our aim here is to use group theory to construct operations on maps, hypermaps and higher-dimensional structures , which are similar to duality in that they preserve size and symmetry properties . The basic theme is that these combinatorial structures can be represented as transitive permutation representations of appropriate groups , whose outer automorphisms then induce the relevant operations . The ideas outlined here are the result of joint work with David Singerman , John Thornton and Lynne James .

## 2. An algebraic description of maps

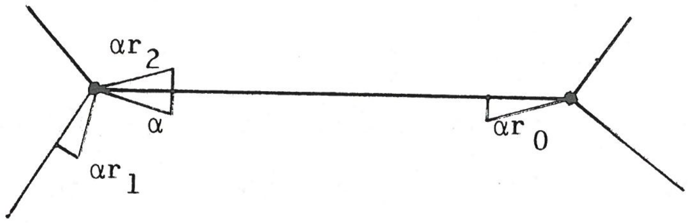
By a <u>map</u> $\mathcal{M}$ we mean a graph imbedded in a connected surface , each face homeomorphic to a disc . A <u>blade</u> of $\mathcal{M}$ is a flag $\alpha = (v,e,f)$ where $v$ is a vertex , $e$ an edge , and $f$ a face , all mutually incident ( Fig. 1 ) .

Fig. 1



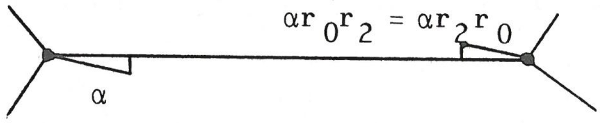

We define permutations $r_i$ ( $i = 0,1,2$ ) of the set $\Omega$ of all blades of $\mathcal{M}$ as follows : $r_i$ changes ( in the only possible way ) the i-dimensional component ( v , e or f ) of each blade , while fixing its other two components (see Fig. 2 ).

Fig. 2




We have $r_i^2 = 1$ ( $i = 0, 1, 2$ ) and $(r_0 r_2)^2 = 1$ (equivalently , $r_0 r_2 = r_2 r_0$ ), as shown in Fig. 3 .

Fig. 3




Let G be the subgroup of the symmetric group $S^{\Omega}$ generated by $r_0, r_1$ and $r_2$ . ( N.B. the elements of G are <u>not</u> generally automorphisms of $\mathcal{M}$ .) Since the faces are simply connected , the graph is connected and so G acts transitively on $\Omega$ . Thus $\mathcal{M}$ determines a transitive permutation representation of the group

$$\Gamma = \langle R_0 , R_1 , R_2 \mid R_0{}^2 = R_1{}^2 = R_2{}^2 = (R_0 R_2)^2 = 1 \rangle ,$$

by means of the obvious epimorphism $\Gamma \rightarrow G$, $R_i \mapsto r_i$ .

We can reverse this process , so that every transitive permutation representation of $\Gamma$ determines a map : we take the vertices , edges and faces to be the orbits of the subgroups $\langle R_1 , R_2 \rangle$ , $\langle R_0 , R_2 \rangle$ and $\langle R_0 , R_1 \rangle$ respectively, with incidence given by non-empty intersection . Then two maps are isomorphic if and only if they correspond to equivalent permutation representations , so we have bijections between the following three sets :

i) the isomorphism classes of maps $\mathcal{M}$ ,

ii) the transitive permutation representations of $\Gamma$ ,

iii) the conjugacy classes of subgroups $M \leqslant \Gamma$ ( the point-stabilisers in (ii) ) .

Thus combinatorial properties of $\mathcal{M}$ can be related to algebraic properties of $\Gamma$ , for instance :

i) the number $|\Omega|$ of blades of $\mathcal{M}$ is equal to the index $|\Gamma : M|$ of $M$ in $\Gamma$ ;

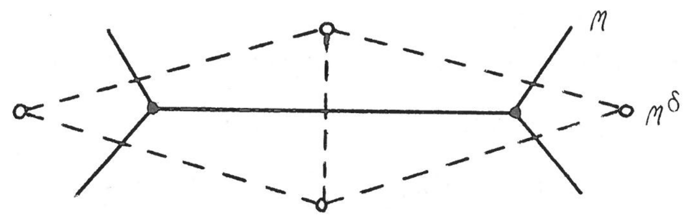
ii) $\mathrm{Aut}\mathcal{M} \cong N_{\Gamma}(M)/M$ , where $N_{\Gamma}$ denotes the normaliser in $\Gamma$ .

(These connections between maps and permutations have been explored recently by several authors , e.g. [5 , 7 , 9 , 10 , 16 , 17 , 18] ; for a full account , including surfaces with boundary , see [1] .)

## 3. Operations on maps

Wilson [20] and Lins [12] have described six operations on maps ( including duality and the identity operation ) which preserve numerical and algebraic properties such as $|\Omega|$ and $\mathrm{Aut}\,\mathcal{M}$ . We shall let $\mathcal{M}^{\delta}$ denote the dual map of $\mathcal{M}$ ( vertices and faces interchanged , see Fig. 4 ), and

Fig. 4




$\mathcal{M}^{\omega}$ the opposite map of $\mathcal{M}$ : this is formed by cutting $\mathcal{M}$ along each edge , and rejoining the corresponding pairs of faces with the reverse orientation (Fig. 5) . This has

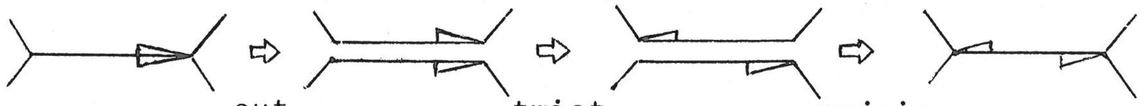
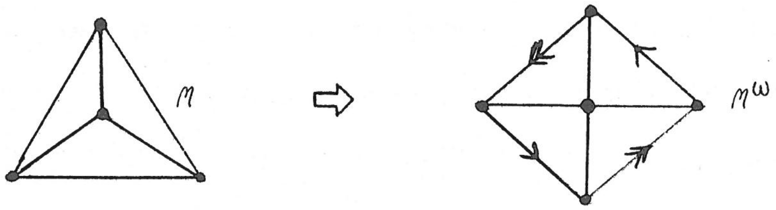



Fig. 5

the effect of interchanging vertices and Petrie polygons (zig-zag paths) , while preserving faces and edges . For example , if $\mathcal{M}$ is a tetrahedron , then $\mathcal{M} \cong \mathcal{M}^{\delta}$ , while it is not hard to see that $\mathcal{M}^{\omega}$ is the non-orientable map formed by projecting an octahedron antipodally from the sphere to the projective plane ( Fig. 6 ) .

Fig. 6




We have $\delta^2 = \omega^2 = (\delta\omega)^3 = 1$ , so $\delta$ and $\omega$ generate a group isomorphic to $S_3$ , inducing all $3! = 6$ permutations of vertices , faces and Petrie polygons . Now $\Gamma$ is the free product of a Klein 4-group $V = \langle R_0 , R_2 \rangle$ and a cyclic group $C = \langle R_1 \rangle$ of order 2 , and as shown by Jones and Thornton [11] these operations are induced by the outer automorphism group

$$\mathrm{Out}\,\Gamma = \mathrm{Aut}\,\Gamma / \mathrm{Inn}\,\Gamma \cong \mathrm{Aut}\,V \cong S_3$$

acting on permutation representations of $\Gamma$ ( or equivalently on conjugacy classes of subgroups of $\Gamma$ ) . For instance $\delta$ corresponds to the automorphism $R_i \longmapsto R_{2-i}$ , and $\omega$ to the automorphism $R_0 \longmapsto R_0$ , $R_1 \longmapsto R_1$ , $R_2 \longmapsto R_0R_2$ . (The algebraic interpretation of $\delta$ had already been given in [10] for orientable maps .)

## 4. Higher dimensions

The n-dimensional analogue of a map is a suitably decomposed n-manifold ; for topological details , see [5 , 17] . As in the case $n = 2$ , we can define permutations $r_0 , \ldots , r_n$ of the set of flags , and the analogue of $\Gamma$ is the Coxeter group

$$\Gamma = \langle R_0 , \ldots , R_n \mid R_i^{\,2} = 1 , (R_i R_j)^2 = 1 \text{ for } j > i+1 \rangle$$

corresponding to the Dynkin diagram


As shown by James [8] , if $n \geqslant 3$ then $\mathrm{Out}\,\Gamma_n \cong D_4$ , a dihedral group of order 8 generated by the automorphisms

$$\delta : R_i \longmapsto R_{n-i}$$

and

$$\omega : \begin{cases} R_2 \longmapsto R_0 R_2 \\ R_i \longmapsto R_i \quad (i \neq 2) \end{cases}$$

satisfying $\delta^2 = \omega^2 = (\delta\omega)^4 = 1$ . This induces a group of eight operations on n-dimensional maps , generated by the duality operation $\delta$ (interchanging i-dimensional and (n-i)-dimensional components of flags) and the 'opposite' operation $\omega$ (the analogue of the corresponding operation for maps on surfaces) .

The calculation of $\mathrm{Out}\,\Gamma_n$ relies on various decompositions of $\Gamma_n$ as a free product with amalgamation (cf. the corresponding decomposition for $\Gamma = \Gamma_2$ in § 3) . More generally , Tits [15] has a method for determining the automorphism group of <u>any</u> finitely-generated Coxeter group with defining relations $R_i^{\,2} = (R_i R_j)^2 = 1$ for various $i , j$ .

## 5. An algebraic description of hypermaps

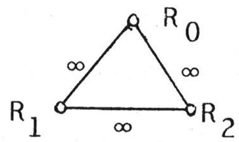
A hypermap $\mathcal{H}$ is an imbedding of a hypergraph in a connected surface, the faces being simply connected. The only essential difference between hypermaps and maps is that in a hypermap an edge may be incident with more than two vertices. As before, we let $\Omega$ denote the set of flags $(v,e,f)$, where $v$, $e$ and $f$ are a vertex, edge and face, all mutually incident. The permutations $r_i$ of $\Omega$ (changing the i-dimensional component of each flag while fixing the others) satisfy

$$r_i^2 = 1 \quad (i = 0, 1, 2),$$

(but not necessarily $(r_0 r_2)^2 = 1$), so in place of $\Gamma$ we use the Coxeter group

$$\Delta = \langle R_0, R_1, R_2 \mid R_0^2 = R_1^2 = R_2^2 = 1 \rangle$$
$$\cong C_2 * C_2 * C_2,$$

given by the Dynkin diagram




Then $\mathcal{H}$ determines a transitive permutation representation of $\Delta$, and conversely, given such a representation we can reconstruct $\mathcal{H}$ by defining the vertices, edges and faces to be the orbits of $\langle R_1, R_2 \rangle$, $\langle R_0, R_2 \rangle$ and $\langle R_0, R_1 \rangle$ as in the case of maps. Figure 7 shows a Schreier coset diagram for $\Delta$, in which the vertices represent flags of $\mathcal{H}$, the edges the permutations $r_i$, and the regions (labelled $v$, $e$, $f$) the vertices, edges and faces of $\mathcal{H}$.

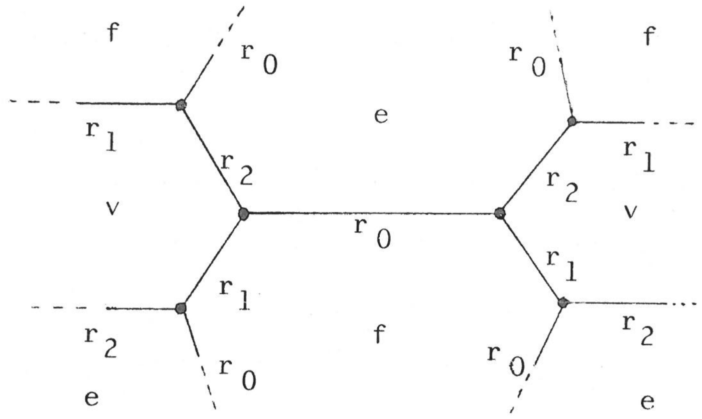



Fig. 7

( This is a generalisation of the algebraic theory of oriented hypermaps, developed by Cori, Machì and others [2, 3, 4, 14, 19], which is based on the 'even subgroup' $\Delta^+$ of $\Delta$ generated by the elements $\sigma = R_1R_2$ and $\alpha = R_2R_0$ representing rotations around vertices and edges; $\Delta^+$ is freely generated by $\sigma$ and $\alpha$, so Dress and Frank's parametrization [5] of the subgroups of finite index in the free group $F_2$ can be seen as a 'theorème de codage' for orientable finite hypermaps, cf. [2].)

## 6. Operations on hypermaps

Machì [14] has described a group $S$ of six operations on oriented hypermaps; these can be extended to all hypermaps, and they correspond to permuting vertices, edges and faces, that is, to permuting the generators $R_0$, $R_1$ and $R_2$ of $\Delta$, so that $S \cong S_3$. The full group of operations on oriented hypermaps is

$$\mathrm{Out}\,\Delta^+ \cong \mathrm{Out}\,F_2 \cong GL_2(\mathbb{Z}),$$

represented faithfully on the abelianized group $F_2^{ab} \cong \mathbb{Z}^2$ [13, §I.4]. By contrast with the previous cases, this is an <u>infinite</u> group, generated by S and the following operation $\tau$ :

(1) shrink each face of $\mathcal{H}$ to a point, giving a 2-coloured map $\mathcal{M}$ in which the faces correspond to the vertices and edges of $\mathcal{H}$, and the vertices correspond to the faces of $\mathcal{H}$ ( this is the dual of Walsh's bipartite map $G(\mathcal{H})$ [<u>19</u>]);

(2) apply the 'opposite' operation $\omega$ to $\mathcal{M}$, giving a 2-coloured map $\mathcal{M}^\omega$, the faces corresponding to those of $\mathcal{M}$ ;

(3) form a hypermap $\mathcal{H}^\tau$ by reversing step (1), that is, by expanding the vertices of $\mathcal{M}^\omega$ into regions corresponding to faces of $\mathcal{H}^\tau$ .

Notice that $\tau$ preserves the vertices and edges of $\mathcal{H}$, together with their incidence, so that $\mathcal{H}$ and $\mathcal{H}^\tau$ imbed the same hypergraph ; $\tau$ corresponds to the automorphism $R_0 \longmapsto R_2R_0R_2$ of $\Delta$, and hence to the automorphism $\alpha\ (=R_2R_0) \longmapsto \alpha^{-1}\ (=R_0R_2)$ of $\Delta^+$, and thus to the matrix $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ of $GL_2(\mathbb{Z})$ . Similarly we can represent the elements of S as elements of $GL_2(\mathbb{Z})$ by calculating their effect on the free generators $\sigma$ and $\alpha$ of $\Delta^+$.

For the class of <u>all</u> hypermaps (not necessarily oriented), the group of operations is now

$$\mathrm{Out}\,\Delta \cong PGL_2(\mathbb{Z}) = GL_2(\mathbb{Z})/\{\pm I\} .$$

The matrix $-I \in GL_2(\mathbb{Z})$ corresponds to the automorphism $\sigma \longmapsto \sigma^{-1}$, $\alpha \longmapsto \alpha^{-1}$ of $\Delta^+$, or equivalently the inner automorphism of $\Delta$ induced by conjugation by $R_2$ . This amounts to taking mirror-images of hypermaps, but without a preferred orientation we cannot distinguish a hypermap from its mirror-image, so $-I$ corresponds to the identity operation on the class of all hypermaps. This is why we obtain $PGL_2(\mathbb{Z})$ rather than $GL_2(\mathbb{Z})$ when we allow non-orientable hypermaps.

## 7. Open problems

1) Given a hypermap $\mathcal{H}$, there will be a subgroup $H$ of $PGL_2(\mathbb{Z})$ corresponding to the operations preserving $\mathcal{H}$; which subgroups correspond to which hypermaps? Which hypermaps correspond to congruence subgroups? What can be said about the modular functions and forms related to $\mathcal{H}$ (via $H$)?

2) $PGL_2(\mathbb{Z})$ can be obtained from $\Gamma$ by imposing the extra defining relation $(R_1R_2)^3 = 1$, so conjugacy classes of subgroups $H \leqslant PGL_2(\mathbb{Z})$ correspond to trivalent maps, by the method of §2 (see also [11]); thus each hypermap $\mathcal{H}$ determines (via $H$) a trivalent map $\mathcal{T}$, so how are the combinatorial properties of $\mathcal{H}$ and $\mathcal{T}$ related?

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