SOME OPERATIONS ON TOLERANCES

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ABSTRACT

Studied are conditions in order that the result of an operation (intersection, union, composition) on tolerances be a partition or a partial plane.

INTRODUCTION

We firstly examine definitions and elementary results about intersection, union and composition of tolerances (i.e. reflexive and symmetric binary relations). Some analogies between tolerances, (hyper)graphs and incidence structures are also pointed out, which allow us to reformulate results in the language of each category.

Then we list a series of results, whose proofs are deferred to subsequent papers.

Proposition (I) deals with the intersection of tolerances, while (U1) and (U2) give representations of the union. Necessary and sufficient conditions for the union of partitions to be still a partition are indicated by Propositions (UP1) and (UP2). An analogous investigation is accomplished for the union of two partial planes, and a condition on bicoloured quadrangles is found, see (UPP1) and (UPP2).

The study of the composition of two tolerances leads to the concept of commuting tolerances (see propositions (P1) and (P2)) thus generalizing that of commuting partitions (recently indagated by M. Haiman, O. Nava and G.C. Rota among others, see [Hai] and [NR]).

PRELIMINARY DEFINITIONS AND RESULTS

Let R be a binary relation on a set $X(\neq \phi)$. A *(relational) R-block* is a non-void subset B of X such that $B \times B \subseteq R$; the family of all R-blocks will be denoted by $\mathcal{M}^S(R)$.

Remark. $\mathcal{M}^S(R)$ is an example of a family $\mathcal{F}$ of non-void subsets of X with the property:

(S) $\phi \neq A \subseteq B \in \mathcal{F} \Rightarrow A \in \mathcal{F}$.

Clearly the union of a (possibly infinite) number of families, each of them verifying (S), again satisfies this property.

The existence of maximal R-blocks (i.e. R-blocks B such that: $B \subseteq C$, $C \times C \subseteq R$ implies B=C) is guaranteed by Zorn's Lemma, since $\mathcal{M}^S(R)$ is inductive. In the sequel let us denote by $\mathcal{F}^M$ the (possibly void) collection of maximal (for the inclusion relation $\subseteq$) elements taken from a family $\mathcal{F}$ of subsets of X, and by $\mathcal{M}^M(R)(=(\mathcal{M}^S(R))^M)$ the set of maximal R-blocks, or R-plot (see also [BC]).

Remark. The family $\mathcal{M}^S(R)$ is inductive and therefore enjoys the property:

(M) each of its subsets is contained in a maximal one.

Clearly the union of a finite number of families, each of them verifying (M), again satisfies this property.

A symmetric and reflexive binary relation T on X is a *tolerance* (see e.g. [CNZ]). It is immediately seen that tolerances on X form a complete boolean lattice, with partial order: $\subseteq$; absolute maximum (1): X^2; absolute minimum (0): $\Delta_X=\{(x,x)\mid x\in X\}$; infimum of a family $\{T_i\}_{i\in I}$: $\bigcap_{i\in I} T_i$; supremum of a family $\{T_i\}_{i\in I}$: $\bigcup_{i\in I} T_i$; complement of a tolerance T: $(X^2-T)\cup\Delta_X$ and that there is a closure operator $R\to\bar{R}^T$ associating to each binary relation R its "tolerance (i.e. symmetric and reflexive) closure" $\bar{R}^T=R\cup R^{-1}\cup\Delta_X$ (where, as usual: $R^{-1}=\{(x,y)\mid(y,x)\in R\}$), that is the least tolerance containing R (for most results and constructions see e.g. [Dub], p. 56).

There is an isomorphism (cf. [CNZ], [BC]) between the class of tolerances on X and that of coverings (named also *plots* in [BC] and T-coverings in [CNZ]) $\mathscr{C}$ of X with non-void subsets (again called $\mathscr{C}$-blocks or briefly blocks) satisfying:

(M.1) $(B,A_i\in\mathscr{C},\ B\subseteq\bigcup_i A_i)\Rightarrow(B\supseteq\bigcap_i A_i)$

(M.2) if every two-elements subset of A is contained in some block of $\mathscr{C}$, then A also is a $\mathscr{C}$-block.

The plot corresponding to a tolerance T is exactly the T-plot $\mathscr{M}^M(T)$; the tolerance corresponding to a plot $\mathscr{C}$ is $T_{\mathscr{C}}=\{(x,y)\mid x,y$ belong to some $A\in\mathscr{C}\}$. The respective refinement relations are also preserved (for tolerances, T is finer than T' if $T\subseteq T'$; for coverings, $\mathscr{C}$ is finer than $\mathscr{C}'$ if every $\mathscr{C}$-block is contained in some $\mathscr{C}'$-block).

Thus, by abuse of language, definitions and statements concerning tolerances can be referred to the associated plots, and vice-versa.

Let T be a tolerance; a covering $\mathscr{C}$ of X will be *compatible* with

T if $\mathscr{C} \subseteq \mathscr{M}^S(T)$, i.e. its subsets are T-blocks. A tolerance $\tilde{T}_{\mathscr{C}}$ can be *induced* on a covering $\mathscr{C}$ compatible with T, by defining: $(A,B) \in \tilde{T}_{\mathscr{C}}$ if $A \times B \subseteq T$, for subsets $A,B \in \mathscr{C}$.

Given a covering $\mathscr{C}$ compatible with T, there are the following relationships between T, $\mathscr{M}^S(T)$, $\mathscr{M}^M(T)$ on the one hand, and $\tilde{T}_{\mathscr{C}}$, $\mathscr{M}^S(\tilde{T}_{\mathscr{C}})$, $\mathscr{M}^M(\tilde{T}_{\mathscr{C}})$ on the other hand: if $\mathscr{C}$ has the property (S), then

$$\mathscr{M}^S(T) = \{ \bigcup_{B\in\mathscr{B}} B \mid \mathscr{B} \text{ block of } \mathscr{M}^S(\tilde{T}_{\mathscr{C}}) \}$$

and

$$\mathscr{M}^M(T) = \{ \bigcup_{B\in\mathscr{B}} B \mid \mathscr{B} \text{ block of } \mathscr{M}^S(\tilde{T}_{\mathscr{C}}) \}^M =$$

$$= \{ \bigcup_{B\in\mathscr{B}} B \mid \mathscr{B} \text{ block of } \mathscr{M}^M(\tilde{T}_{\mathscr{C}}) \} .$$

Regarding tolerances and plots from the viewpoint of other structures, we notice that a plot on X is, in particular, a covering with non-void subsets, hence a *hypergraph* on X (see also [Ber]); the converse is not true, because a hypergraph on X need not verify properties (M.1) and (M.2) of plots.

Since a hypergraph $\mathscr{H}$ on X can be viewed also as the set of lines of an incidence structure $(X,\mathscr{H}, \in)$ (see e.g. [Dem]), where the set of points is X and the incidence relation is the inclusion ($\in$), we shall study the case in which the incidence structure is derived from a plot and moreover is a partial plane (here in a slightly broader sense than in [Dem]), i.e. two distinct points are contained in at most one line/block. Some operations and results on plots of partial planes will be deduced in connection with operations and results on their associated tolerances.

Furthermore, a tolerance T on a set X determines a *graph* $(X, \mathscr{E}_T)$, where the sets of vertices and edges are X and $\mathscr{E}_T=\{\{x,y\} \mid (x,y) \in T\}$

respectively (here a graph is undirected and simple, without loops and multiple edges). Conversely, a graph $(X, \mathcal{E})$ determines a tolerance on X, i.e. the *adjacency* relation $T_{\mathcal{E}} = \{(x,y) \mid$ the vertices x and y are joined by an edge of $\mathcal{E}\}$. It is easy to check that $(T)_{(\mathcal{E}_T)} = T$ and $(\mathcal{E})_{(T_{\mathcal{E}})} = \mathcal{E}$, which provides a complete identification between tolerances and graphs on the same base-set X.

We recall that a *clique* of a (hyper)graph (cf. [Ber], p. 432) is a set of vertices, of which any two are adjacent; one-vertex sets, hence also isolated vertices (i.e. adjacent to no other one), are trivial cliques. Then a further identification (block ↔ clique) can be made between blocks of a tolerance and cliques of the corresponding graph.

A hypergraph on X can be derived from a graph $(X, \mathcal{E})$ by taking its maximal cliques; this process (via the identifications: plot ↔ tolerance ↔ graph) yields the correspondence plot → hypergraph previously described.

The product between two tolerance T_1, T_2 is the usual composition of binary relations: $(x,y) \in T_1 \circ T_2$ if $(x,z) \in T_1$ and $(z,y) \in T_2$ for some z. The two tolerances *commute* if $T_1 \circ T_2 = T_2 \circ T_1$, i.e. $(x,z) \in T_1$ and $(z,y) \in T_2$ for some z iff $(x,w) \in T_2$ and $(w,y) \in T_1$ for some w. T_1 and T_2 are *independent* if $T_1 \circ T_2 = X^2$, i.e. for every x,y there exists some z such that $(x,z) \in T_1$ and $(z,y) \in T_2$. Clearly independent tolerances commute, but the converse is not always true.

The equivalence relations on X are the transitive tolerances (i.e. $T \circ T \subseteq T$), and form a complete lattice, with partial order: $\subseteq$; absolute maximum (1): X^2; absolute minimum (0): Δ_X; infimum of a family $\{E_i\}_{i \in I}$: $\bigcap_{i \in I} E_i$; supremum of a family $\{E_i\}_{i \in I}$: $\bigvee_{i \in I} E_i = \overline{(\bigcup_{i \in I} E_i)}^{TR}$.

Remark. By $\bar{R}^{TR} = \bigcup_{n \geq 1} R^n$ (with $R^1 = R$ and $R^{n+1} = R^n \circ R$) we denote the transitive closure of a relation R (cf. [Dub], pp. 57-58), i.e. the least transitive relation containing R. Clearly $R \to \bar{R}^{TR}$ is a closure operator, and it holds: $(x,y) \in \bar{R}^{TR}$ iff there is a finite sequence $x_0,\dots,x_n$ with $x=x_0$, $y=x_n$ and $(x_k,x_{k+1}) \in R$. If T is a tolerance, then $\bar{T}^{TR}$ is an equivalence relation (cf. [Dub], p. 58).

For a finite number of equivalence relations, say E and F, the supremum admits a further representation as the transitive closure ot their composition (cf. [Dub], pp. 57-58): $E \vee F = \overline{E \circ F}^{TR}$ (it is easily checked that $E \cup F \subseteq E \circ F \subseteq \overline{E \cup F}^{TR}$, hence $\overline{E \cup F}^{TR} = \overline{E \circ F}^{TR}$).

We recall that, for two equivalence relations E and F, the following statements are equivalent (cf. [DD], pp. 74-75; [Hai], p. 16):

(i) E,F commute

(ii) $E \vee F = E \circ F$

(iii) $E \circ F$ is an equivalence relation.

The isomorphism between tolerances and plots yields for equivalence relations the well-known identification with partitions, i.e. plots $\mathscr{P}$ where every element of X is contained precisely in one block or (equivalence) class of $\mathscr{P}$. It is easy to prove that the refiniment relation for partitions $\mathscr{P}$ and $\mathscr{P}'$ can be rephrased as follows: $\mathscr{P}$ is finer than $\mathscr{P}'$ (also: $\mathscr{P}$ consecutive to $\mathscr{P}'$, in [DD], pp. 67-68) iff $\mathscr{P}'$-classes are *saturated* (cf. [Dub], p. 21) by $\mathscr{P}$-classes, i.e. $A \subseteq A'$ if $A \in \mathscr{P}$, $A' \in \mathscr{P}'$ and $A \cap A' \neq \phi$.

Two partitions, $\mathscr{P}$ and $\mathscr{P}'$, are *semi-consecutive* (see [DD], p. 69) if $A \cap A' \neq \phi$ implies $A \subseteq A'$ or $A' \subseteq A$, for any two classes $A \in \mathscr{P}$, $A' \in \mathscr{P}'$.

It is easy to prove that two equivalence relations are independent iff any two classes of their corresponding partitions have non-void intersection (cf. the definition in [NR], p. 21).

A characterization of the commutability of two equivalence relations E, F in terms of independent partitions is the following (see e.g. [Dub], p. 63): E and F commute iff their restrictions to any class A of $E \vee F$ determine independent partitions on A.

OPERATIONS ON TOLERANCES

1. Representation of the intersection

Let $\{T_i\}_{i\in I}$ be a family of tolerances on X, and let $T=\bigcap_{i\in I} T_i$.

Proposition (I). $\mathcal{M}^M(T)=\{\bigcap_{i\in I} A_i \mid A_i \in \mathcal{M}^M(T_i)\}^M$ (cf. [BC]).

Corollary. *If each* $\mathcal{M}^M(T_i)$ *is a partition (resp. a partial plane), then also* $\mathcal{M}^M(T)$ *is.*

2. Representation of the union

Let $\{T_i\}_{i\in I}$ be a family ot tolerances on X, and let $T=\bigcap_{i\in I} T_i$. We give a characterization of T-blocks among the subsets of X.

Proposition (U1). $(\phi\neq)A\in\mathcal{M}^M(T)$ *iff*

(i) $A\times A\subseteq T$

(ii) $(A\subseteq \bigcup_k A_k,\ A_k\in \bigcup_{i\in I}\mathcal{M}^M(T_i))\Rightarrow(A\supseteq\bigcap_k A_k)$.

Then we look for representations on which algorithms to derive the T-plot from T_i-plots can be based. Let $\mathcal{C}=\mathcal{M}^S(T_i)$ or also $\mathcal{C}=\bigcup_{i\in I}\mathcal{M}^S(T_i)$; if we consider the induced tolerance $\tilde{T}_{\mathcal{C}}$, we have the following relationships between T and $\mathcal{C}$:

Proposition (U2). *It holds:*

$$\mathcal{M}^S(T)=\{\bigcup_{B\in\mathcal{B}} B \mid \mathcal{B}\subseteq\mathcal{C};\ B_1,B_2\in\mathcal{B}\Rightarrow B_1\times B_2\subseteq T\}$$

$$\mathcal{M}^M(T) = (\mathcal{M}^S(T))^M = \{ \bigcup_{B \in \mathcal{B}} B \mid \mathcal{B} \subseteq \mathcal{C};\ \mathcal{B} \text{ maximal family such that:}$$

$$B_1, B_2 \in \mathcal{B} \Rightarrow B_1 \times B_2 \subseteq T \}.$$

Now we turn our attention to the union of partitions. Let $\{E_i\}_{i \in I}$ be a family of partitions, and let $E = \bigcup_{i \in I} E_i$.

Proposition (UP1). *The following is equivalent:*

(i) E is a partition

(ii) $E = \bigvee_{i \in I} E_i$

(iii) $(x,z) \in E_{i_1}$, $(z,y) \in E_{i_2} \Rightarrow \exists\, i_3 : (x,y) \in E_{i_3}$.

Corollary. *If* E *is a partition, then* E*-classes are satureted by* E_i*-classes. If each union* $E_{i_1} \cup E_{i_2}$ *is a partition, then also* E *is.*

Let E_1, E_2 be two partitions, and let $E = E_1 \cup E_2$.

Proposition (UP2). *The following is equivalent:*

(i) E is a partition

(ii) $E = E_1 \circ E_2$

(iii) $(x,z) \in E_1$, $(z,y) \in E_2 \Rightarrow (x,y) \in E_1$ or $(x,y) \in E_2$

(iv) $E = E_1 \vee E_2$

(v) E_1, E_2 are semi-consecutive

(vi) $\mathcal{M}^M(E) = (\mathcal{M}^M(E_1) \cup \mathcal{M}^M(E_2))^M$.

Corollary. *If* E *is a partition, then* E_1, E_2 *commute.*

We now introduce some new concepts.

Let $\mathcal{C}$ and $\mathcal{C}'$ be two plots; then $\mathcal{C}'$-blocks will be *2-saturated* by $\mathcal{C}$-blocks, if a block $A' \in \mathcal{C}'$ "absorbs" a whole $\mathcal{C}$-block $A \in \mathcal{C}$ whenever A' contains two distinct points of A (i.e. $A \subseteq A'$ if $|A \cap A'| \geq 2$, for every $A \in \mathcal{C}$, $A' \in \mathcal{C}'$).

Let $(X, \mathcal{E})$ be a graph. A *quadrangle* is a subgraph generated by four vertices and having at least four (of the six possible) edges; it is *quasi-complete* if its edges are either four or six.

Lemma. *A tolerance is associated to a partial plane iff the quadrangles of its graph are quasi-complete. Furthermore: let $\mathcal{C}$ and $\mathcal{C}'$ be the plots of two partial planes; then $\mathcal{C}$ is finer than $\mathcal{C}'$ iff $\mathcal{C}'$-blocks are 2-saturated by $\mathcal{C}$-blocks.*

Let $(X, \mathcal{E}_T)$ be the graph associated to the union $T=T_1 \cup T_2$ of two tolerances T_1, T_2. Each of the k edges of a quadrangle (hence $4 \leq k \leq 6$) may then have a "colour" of type T_1 or T_2; let m be the maximum number of its edges having the same colour (T_1 or T_2). Clearly $2 \leq m \leq 6$ and $k/2 \leq m$, while $m=2$ implies $k=4$; the quadrangle will be *bicolouring-complete* if $m \geq 3$ implies $k=6$.

Now we study the union of partial planes.

Let T_1, T_2 be two partial planes, and let $T=T_1 \cup T_2$.

Proposition (UPP1). T *is a partial plane iff the quadrangles of its graph are bicolouring-complete.*

Some consequences can be drawn for the unions $T= \bigcup_{i \in I} T_i$ of a family of partial planes $\{T_i\}_{i \in I}$.

Proposition (UPP2). *If each union* $T_{i_1} \cup T_{i_2}$ *is a partial plane, then also*

T *is.*

Proposition (UPP3). *If* T *is a partial plane, then* T*-blocks are 2-saturated by* T_i*-blocks.*

3. Representation of the composition

Let T_1, T_2 be two tolerances on X, and let $T = T_1 \circ T_2$.

Proposition (P1). T *is a tolerance iff* T_1, T_2 *commute.*

Corollary. *If the plots of* T_1 , T_2 *are independent, then* T *is a tolerance.*

Proposition (P2). *If* T_1, T_2 *commute, then also their restrictions* $T_1|_A$, $T_2|_A$ *commute, for any equivalence class* A *of* $\overline{T}^{TR}$.

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