SYMMETRIZED BIDETERMINANTS ON SKEW SHAPES

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Carter and Lusztig (1) in their study of the modular representations of the general linear group introduced two modules, which are called Weyl modules of the first and second kind. Weyl modules of the first kind can be constructed following Schur's thesis of 1901 (ref.2), where the irreducible representations of GL(n,**C**) were first explicitely given.

In 1979, Clausen (3) succeeded in constructing Weyl modules of the second kind by the technique of symmetrized bideterminants on Young tableaux. The theories of Carter and Lusztig and Clausen are the background of the present work.

Akin, Buchsbaum and Weyman (ref.4) introduced Schur and Coschur functors, motivated by the program of extending to commutative rings the construction over the complex numbers given by Schur (5) and Weyl (6), respectively. Indeed, Schur and Coschur modules are isomorphic over the rationals, but far from isomorphic over the integers.

In fact, it has been noted that Schur and Coschur functors are closely related to Weyl modules of the first and second kind. Led by the analogy between Clausen's work and the work of Akin, Buchsbaum and Weyman, we introduce the notions of skew determinants and symmetrized skew determinants, thereby succeeding in describing Schur and Coschur modules on skew Young shapes by a technique that appears to be simple and manageable.

Schur modules are spanned by skew determinants, and Coschur modules are spanned by symmetrized skew determinants.

Viewed as functors from the cathegory of commutative rings to the cathegory of modules, Schur(λ/μ) and Coschur(λ/μ) are universally free functors. These and other cathegorical properties become trivial in the present approach.

Actually, Schur modules are obtained as the images of the Capelli operators, as introduced by Désarménien, Kung and Rota (ref.7). In this way, Coschur modules turn out to be the images under the adjoints of Capelli operators, relative to a pair of symmetric bilinear forms.

We single out a submodule M of the domain of the adjoint of the Capelli operator which has the following properties: (a) the image of M under the adjoint of the Capelli operator is the Coschur module; (b) the pairing between M and the Schur module is triangular. This yields a complete symmetry between Schur and Coschur modules; as a byproduct, we obtain the standard basis theorem for Coschur modules from the analogous theorem for Schur modules.

Finally, we obtain an explicit formula for the computation of coefficients in the expression of a skew determinant as a linear combination of standard skew determinants. These coefficients are obtained by inverting a triangular matrix all of whose diagonal entries equal one. By inverting the transpose of the same matrix, we obtain the coefficients in the expression of a symmetrized skew determinant as linear combination of standard symmetrized skew determinants.

As an application of the present theory, we derive two standard basis theorems over the rationals, which imply the direct sum decompositions of tensors space into irreducible symmetry classes in the sense of Schur and Weyl, respectively. Proposition 7.1 below can also be used to derive the Gordan-Capelli formula of classical invariant theory in the canonical form first given by Wallace (8).

References

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