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Cycles in graphs with small density

This talk reports on some work that has been done jointly with A. Gyârfás and M.J. Prömel.

Definition: Let G be a finite graph,

$$\mathcal{L}(G) = \sum \left\{ \frac{1}{\ell} \mid G \text{ contains some cycle of length } \ell \right\}$$

Examples:

$$\mathcal{L}(C_n) = \frac{1}{n}, \quad \mathcal{L}(K_n) = \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$$

$$\mathcal{L}(K_{n,n}) = \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2 \cdot n} .$$

Erdös and Hajnal suggested to consider the function

$$f(\alpha) = \inf \left\{ \mathcal{L}(G) \;\middle|\; \frac{|E(G)|}{|V(G)|} > \alpha \right\},$$

where α is a positive real.

Obviously $$f(1) = 0 .$$

Erdös and Hajnal conjectured that

$$f(\alpha) = (1/2 + o(1)) \cdot \log \alpha$$

as α tends to infinity,

but originally even for large α it was not clear whether $f(\alpha) > 0$ or not.

Theorem: (Gyârfás, Komlós, Szemerédi)

There exists some constant $c > 0$ such that

$f(\alpha) > c \cdot \log \alpha$ for every $\alpha \geq 2$.

This result does not give an answer for small densities α , i.e. for $\alpha \in]1,2[$. Here we can show:

<u>Theorem</u>: For every $\varepsilon > 0$ and every $1 < \alpha < 2$ one has

$$f(\alpha) > \left(\frac{1}{\alpha-1}\right)^{1+\varepsilon} + o(1)$$

as α tends to 1 .

This result can be sharpened somewhat by considering the function $f^*(g,\alpha)$, where

$$f^*(g,\alpha) = \inf \left\{ \mathcal{L}(G) \;\middle|\; \frac{|E(G)|}{|V(G)|} > \alpha \text{ and } \operatorname{girth}(G) > g \right\} ;$$

we can proof the following

<u>Theorem</u>: For every $\varepsilon > 0$ and every $1 < \alpha < 2$ it follows that

$$f^*\left(\left(\frac{1}{\alpha-1}\right)^{1+\varepsilon} + o(1) \;,\; \alpha\right) \geqq \frac{\alpha-1}{17}$$

as α tends to 1 .

<u>Question</u>: For fixed $\alpha > 1$,

is $f^*(g,\alpha)$ bounded (possibly not) as g tends to infinity?

Proof and details will appear elsewhere.

<u>Literatur</u>:

Erdös, P.: "Some recent progress on extremal problems in graph theory", Proc. 6.th S.E. Conference on graph theory, Utilital Math., (1975), 3-14

Gyárfás, A./Komlós J./Szemerédi, E.: "On the sum of reciprocals of cycle lengths in graphs", to appear