Stretched Brownian Motion: convergence of dual optimising sequences *

Walter Schachermayer †1 and Pietro Siorpaes ‡2

1 Department of Mathematics, University of Vienna
2 Department of Mathematics, Imperial College London

August 26, 2025.

We consider an irreducible pair $\mu \leq_{\mathrm{c}} \nu$ of probability measures on $\mathbb{R}^d$ in convex order. In [BBST25], Backhoff, Beiglböck, Schachermayer and Tschiderer have shown that the Stretched Brownian Motion from μ to ν is a Bass martingale, that there exists a dual optimiser $\psi_{\lim}$, and the following somewhat surprising convergence result: by adding affine functions, one can make any dual optimising sequence $(\psi_n)_n$ (satisfying some minor technical conditions) converge pointwise to $\psi_{\lim}$, save possibly on the relative boundary of the convex hull of the support of ν. In the present paper we deal with the more delicate issue of convergence on said boundary, showing in particular that $\psi_{\lim}$ is ν a.s. finite, and $(\psi_n)_n$ converges to $\psi_{\lim}$ in ν-measure.

Keywords: Martingale Optimal transport, Benamou-Brenier, Stretched Brownian motion, Bass martingale.
Mathematics Subject Classification (2010): Primary 60G42, 60G44; Secondary 91G20.

1 Introduction

In mathematical finance one can construct several arbitrage-free models which are compatible with observed market prices of vanilla options on the spot price $S = (S_t)_{t\in[0,T]}$. In practice, only options with some possible maturities $0 = T_0 < T_1 < \ldots < T_n = T$ are

*This research was funded by the Austrian Science Fund (FWF) [Grant DOIs: 10.55776/P35197 and 10.55776/P35519).

† walter.schachermayer@univie.ac.at
‡ p.siorpaes@imperial.ac.uk

traded, whereas to calibrate most models one would need vanilla options prices across the whole continuum of times $t \in [0, T]$. This has traditionally been dealt with via a time-interpolation of the volatility at the unobserved maturities, which can however introduce instabilities.

A possible and quite recent solution is to consider models which are instead calibrated to discrete marginals, such as the local variance Gamma model [CN17], the martingale Schrödinger bridge [HL19], and the Bass local volatility model which, in the continuous-time limit, converges to the well-known Dupire local volatility model [Dup97].

The Bass local volatility model S arises in a very natural way, as S is the continuous process such that $(S_t)_{T_i \leq t \leq T_{i+1}}$ is the martingale diffusion which best interpolates between the (known!) marginals $\mu_i := Law(S_{T_i})$ and $\mu_{i+1} := Law(S_{T_{i+1}})$, in the sense that it is the one which is as close as possible to Brownian Motion [BBHK20]. Since we only need to restrict our attention to the generic interval $[T_i, T_{i+1}]$, we assume w.l.o.g. that $T_i = 0, T_{i+1} = 1$, and write μ, ν for μ_i, μ_{i+1}.

The corresponding optimisation problem, whose solution S has been called Stretched Brownian Motion (SBM), is the Martingale Benamou-Brenier (MBB) problem, which was introduced in [BBHK20] using a probabilistic approach and in [HT19] using PDEs, and had already appeared independently in [Loe18] in the context of market impact in finance. SBM has been studied in [BBST25, BST23, ST24]. If (μ, ν) is irreducible the corresponding SBM S is a Bass martingale [BBST25, Theorem 1.3], and thus admits an explicit construction in terms of a probability α on $\mathbb{R}^d$, and a convex function $\psi : \mathbb{R}^d \to (-\infty, \infty]$. To calibrate the model one needs to compute such α, which is the solution to a fixed-point equation and is the minimiser of the so-called Bass functional, and thus can be computed via a fixed-point iteration scheme and be identified via the gradient descent for the Bass functional, see [CHL21, AMP23, JLO23, QCYF24, BST23, BPS24].

Since the SBM is defined as the solution to a convex optimisation problem, its study is carried out also by considering the corresponding dual optimisation problem. The main result of this paper strengthens the existing results about the convergence of optimising sequences for such dual optimisation problem, whose solution ψ (which exists if (μ, ν) is irreducible) is the convex function mentioned above.

Before stating our main result, we now introduce some notations and definitions. Let $\mathcal{P}(\mathbb{R}^d)$ be the space of Borel probabilities on $\mathbb{R}^d$, $\mathcal{P}_p(\mathbb{R}^d)$ be its subspace of probabilities with finite p^{th} moment, and for $p \geq 1$ let $\mathcal{P}_p^x(\mathbb{R}^d)$ be the subspace of all the $\beta \in \mathcal{P}_p(\mathbb{R}^d)$ whose barycentre $\text{bar}(\beta) := \int y \beta(dy)$ equals $x \in \mathbb{R}^d$. We assume that $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, and denote by $\mathsf{Cpl}(\mu, \nu)$ the set of transports from μ to ν, i.e., the set of probabilities π on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and ν. Each coupling $\pi \in \mathsf{Cpl}(\mu, \nu)$ can be disintegrated with respect to μ, i.e. there exists a (μ a.s. unique) kernel $(\pi_x)_x$ such that $\pi(dx, dy) = \mu(dx)\pi_x(dy)$, called the μ-disintegration of π; clearly $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ implies $\pi_x \in \mathcal{P}_2(\mathbb{R}^d)$ for μ a.e. x. We call π a martingale transport, and $(\pi_x)_x$ a martingale kernel, if $\pi_x \in \mathcal{P}_2^x(\mathbb{R}^d)$ holds $\mu(dx)$-a.e.; we denote with $MT(\mu, \nu)$ the set of all martingale transports in $\mathsf{Cpl}(\mu, \nu)$. It follows from no-arbitrage arguments that μ, ν are in convex order $\mu \leq_c \nu$, or equivalently (by Strassen's theorem) that there exists a martingale transport $\pi \in \mathsf{MT}(\mu, \nu)$.

By definition the Stretched Brownian Motion M^* between μ and ν is the unique optimiser to the continuous-time optimisation problem

$$\inf_{M_0\sim\mu, M_1\sim v, M_t=M_0+\int_0^t \sigma_s dB_s} \mathbb{E}\left[\int_0^1 |\sigma_t - \mathrm{Id}|_{\mathrm{HS}}^2 \, dt\right], \tag{1}$$

where B is a Brownian motion on $\mathbb{R}^d$ and $|\cdot|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm. It turns out that problem (1) is equivalent to the discrete-time optimisation problem

$$\sup_{\pi\in\mathsf{MT}(\mu,\nu)} \int \mathrm{MCov}(\pi_x, \gamma)\mu(dx), \tag{2}$$

where γ denotes the standard Gaussian law on $\mathbb{R}^d$ and

$$\mathrm{MCov}\left(p_1, p_2\right) := \sup_{q\in\mathsf{Cpl}(p_1,p_2)} \int \langle x_1, x_2\rangle \, q\left(dx_1, dx_2\right), \quad p_1, p_2 \in \mathcal{P}_2(\mathbb{R}^d)$$

is the maximal covariance between p_1 and p_2. Indeed, the unique optimiser π^{SBM} of eq. (2) is closely related to M^*, and so are the corresponding optimal values; so, π^{SBM} is also called the Stretched Brownian Motion between μ and ν. To study problems (1),(2) it is useful to consider the dual optimisation problem

$$D(\mu,\nu) := \inf_{\substack{\mu(\psi<\infty)=1 \\ \psi \text{ convex}}} \mathcal{D}(\psi), \tag{3}$$

where given some $\pi \in \mathsf{MT}(\mu,\nu)$ the functional

$$\mathcal{D}(\psi) := \int \left(\int \psi(y)\pi_x(dy) - \varphi^\psi(x)\right) \mu(dx), \tag{4}$$

is defined for $\psi : \mathbb{R}^d \to (-\infty, \infty]$ convex and μ a.s. finite via the auxiliary function

$$\varphi^\psi(x) := \inf_{p\in\mathcal{P}_2^x(\mathbb{R}^d)} \left(\int \psi dp - \mathrm{MCov}(p,\gamma)\right). \tag{5}$$

One can check that $\mathcal{D}(\psi + a) = \mathcal{D}(\psi)$ if a is affine. By taking $p = \delta_x$ in eq. (5) and applying Jensen's inequality, we get

$$-\infty \le \varphi^\psi(x) \le \psi(x) \in \mathbb{R} \quad \text{and} \quad \mathbb{R} \ni \psi(x) \le \int \psi(y)\pi_x(dy) \le \infty \text{ for } \mu \text{ a.e. } x, \tag{6}$$

which shows that $\int \psi d\pi_x - \phi^\psi(x)$ is well defined and belongs to $[0,\infty]$ for μ a.e. x, and thus $\mathcal{D}(\psi)$ is well defined and $\mathcal{D}(\psi) \in [0,\infty]$, and if $\mathcal{D}(\psi) < \infty$ holds then

$$-\infty < \varphi^\psi(x) \le \int \psi d\pi_x < \infty \text{ for } \mu \text{ a.e. } x. \tag{7}$$

We recall that $D(\mu,\nu)$ does not depend on the choice of $\pi \in \mathsf{MT}(\mu,\nu)$ used in the definition of $\mathcal{D}$ and [BBST25, Theorem 3.3, Lemma 3.7]

$$\sup_{\pi\in\mathsf{MT}(\mu,\nu)} \int \mathrm{MCov}(\pi_x,\gamma)\mu(dx) = \int \mathrm{MCov}(\pi_x^{SBM},\gamma)\mu(dx) = D(\mu,\nu) \in \mathbb{R}. \tag{8}$$

Denote by $C := \overline{\mathrm{co}}\,(\mathrm{spt}\,(\nu))$ the closed convex hull of the support $\mathrm{spt}\,(\nu)$ of ν, and by $I = \mathrm{ri}\,(C)$ its relative interior. We define (μ,ν) to be *irreducible* [BBST25, Def. 1.2] if for any Borel sets A, B such that $\mu(A) > 0, \nu(B) > 0$, there is $\pi \in \mathsf{MT}(\mu,\nu)$ with $\pi[A\times B] > 0$; intuitively, this means that π transports positive mass from A to B. In the rest of this section we assume that (μ,ν) is irreducible. In particular this implies that π^{SBM} is a Bass martingale [BBST25, Theorem 1.3], the dual problem (3) admits a lower semicontinuous solution ψ_{lim} which satisfies $\mu(\mathrm{ri}\,(\psi<\infty)) = 1$ [BBST25, Theorem 7.6], and this unique modulo affine functions [BBST25, Definition 7.14, Lemma 7.19]. Moreover, there exists [BBST25, Theorem 7.8 and Propositions 7.13 and 7.20] a dual optimising sequence $\psi_n \geq 0, n \in \mathbb{N}$ such that $\sup_n \psi_n < +\infty$ on I and which belongs to the space $\mathrm{C_q^{aff}}\left(\mathbb{R}^{\mathrm{d}}\right)$ of continuous test functions which satisfy a convenient quadratic growth condition defined in [BBST25, Eq. (2.1)], and surprisingly for any such $(\psi_n)_n$ there exists affine functions $(a_n)_n$ such that the sequence $(\psi_n + a_n)_n$ (which is also dual optimising) converges pointwise in $I\cup C^c$ to $\psi_{\mathrm{lim}} \geq 0$, and $I \subseteq \{\psi_{\mathrm{lim}} < \infty\} \subseteq C$. The fact that $(\psi_n)_n$ is dual optimising and ψ_{lim} a dual optimiser means that

$$\inf_{\substack{\mu(\psi<\infty)=1\\ \psi \text{ convex}}} \mathcal{D}(\psi) = \lim_n \mathcal{D}(\psi_n) = \mathcal{D}(\psi_{\mathrm{lim}}). \tag{9}$$

We can now state our main result, which refines the above convergence results by considering the behaviour of $(\psi_n)_n$ also on the relative boundary of C, on which ν may very well put strictly positive mass. We denote with $L^0(\nu)$ the space of ν-equivalence classes of *real-valued* functions on $\mathbb{R}^d$, equipped with the convergence in ν-measure. Given $V \subseteq \mathbb{R}^d$, the notation $K \Subset V$ means that K is a compact subset of V.

Theorem 1. *Given an irreducible pair $\mu \leq_c \nu$ in $\mathcal{P}_2(\mathbb{R}^d)$ define $C := \overline{co}\,(spt\,(\nu))$ and $I := ri\,(C)$. Let $(\psi_n)_n$ be a dual optimising sequence and ψ_{lim} a dual optimiser, i.e. $\psi_n, \psi_{lim} : \mathbb{R}^d \to (-\infty,\infty], n\in\mathbb{N}$, are convex and μ a.s. finite and such that eq. (9) holds. Assume w.l.o.g. that $(\psi_n)_n$ are positive and converges pointwise on $I \cup C^c$ to $\psi_{lim} \geq 0$, and ψ_{lim} is lower semicontinuous and satisfies $\mu(ri\,(\psi_{lim} < \infty)) = 1$. Then $\psi_{lim} \in L^0(\nu)$ (i.e. $\psi_{lim} < \infty$, ν a.s.), $(\psi_n)_n$ converges to ψ_{lim} in $L^0(\nu)$, and*

$$\liminf_{n\to\infty} \psi_n(y) \geq \psi_{lim}(y) \quad \text{for all } y \in \mathbb{R}^d. \tag{10}$$

Moreover, if $spt\,(\mu) \Subset I$ then $\psi_{lim} \in L^1(\nu)$ and $(\psi_n)_n$ converges to ψ_{lim} in $L^1(\nu)$.

2 Proof of Theorem 1

In this section we state some auxiliary results and prove theorem 1. To prove the $L^1(\nu)$ convergence in theorem 1 we will need the following approximation lemma, which will allow us to replace a kernel concentrated on $C = \bar{I}$ by a kernel concentrated on some $K \Subset I$; note that we know that $(\psi_n)_n$ is converging uniformly on any $K \Subset I$.

Lemma 2. *Let μ, ν, C, I be as in theorem 1, and $\pi \in \mathsf{MT}(\mu, \nu)$ have μ-disintegration $(\pi_x)_{x \in I}$, so that $\pi_x \in \mathcal{P}_2^x(\mathbb{R}^d)$ and $spt(\pi_x) \subseteq C$ for μ a.e.. Then there exists an increasing sequence of compact convex sets $(K^j)_j \subseteq I$ and a sequence of kernels $((\pi_x^j)_{x \in I})_j$ such that $\cup_j K^j = I$, $K^j \subseteq ri\left(K^{j+1}\right)$, $\pi_x^j \in \mathcal{P}_2^x(\mathbb{R}^d)$ for $j \in \mathbb{N}$ and μ a.e. $x \in I$, and:*

1. *$spt\left(\pi_x^j\right) \subseteq K^j$ for μ a.e. $x \in K^j$, and $\pi_x^j = \delta_x$ for μ a.e. $x \in I \setminus K^j$, for all $j \in \mathbb{N}$,*
2. *$\pi_x^j \leq_{\mathrm{c}} \pi_x^{j+1} \leq_{\mathrm{c}} \pi_x$ for $j \in \mathbb{N}$ and μ a.e. $x \in I$,*
3. *For μ a.e. $x \in I$, as $j \to \infty$ we have $\mathcal{W}_2(\pi_x^j, \pi_x) \to 0$ and*

$$0 \leq MCov\left(\pi_x^j, \gamma\right) \uparrow MCov(\pi_x, \gamma) \leq \int \frac{1}{2}\|y\|^2(\pi_x + \gamma)(dy) \in L^1(\mu(dx)). \tag{11}$$

To prove the convergence in measure in theorem 1 we will use the statement about $L^1(\nu)$ convergence in theorem 1, plus the localisation procedure described in the next lemma, which is of independent interest; note that an analogous statement holds (with analogous proof) if μ^B is replaced by any probability $\mu' \ll \mu$ with bounded density $\frac{d\mu'}{d\mu}$.

Lemma 3. *Given $\mu \leq_{\mathrm{c}} \nu$ in $\mathcal{P}_2(\mathbb{R}^d)$, let π^{SBM} be the Stretched Brownian Motion between μ and ν and $(\pi_x^{SBM})_x$ be its μ-disintegration. Given a Borel set $B \subseteq \mathbb{R}^d$ with $\mu(B) > 0$, define*

$$\mu^B := \frac{\mu(B \cap \cdot)}{\mu(B)}, \quad \nu^B := \int \mu^B(dx)\pi_x^{SBM}, \quad \pi^B(dx, dy) := \mu^B(dx)\pi_x^{SBM}(dy). \tag{12}$$

Then the Stretched Brownian Motion between μ^B and ν^B is π^B. Let $\psi_n, n \in \mathbb{N}$ be convex and μ a.s. finite; if $(\psi_n)_{n \in \mathbb{N}}$ is a dual optimising sequence for (μ, ν) then it is a dual optimising sequence for (μ^B, ν^B). Moreover, if (μ, ν) are irreducible then so are (μ^B, ν^B) (equivalently, if π^{SBM} is a Bass martingale then so is π^B), $\nu_B \sim \nu$ holds (so in particular $\overline{co}(spt(\nu_B)) = \overline{co}(spt(\nu))$), and the dual optimiser ψ_{lim} for (μ, ν) is a dual optimiser for (μ^B, ν^B).

To combine lemmas 2 and 3 we will need the following result.

Lemma 4. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$, and $\mu(I^c) = 0$ for some Borel set $I \subseteq \mathbb{R}^d$. Let $g, g_n, h, h_n : I \to \mathbb{R}$ be Borel functions, and $(I_j)_{j \in \mathbb{N}}$ be Borel subsets of $\mathbb{R}^d$ such that $\mu(I \setminus \cup_j I_j) = 0$ and $\mu(I_j) > 0$ for all j. Define $\mu_j := \mu(I_j \cap \cdot)/\mu(I_j) \in \mathcal{P}(\mathbb{R}^d)$. Then:*

1. *$g_n \to g$ in $L^0(\mu)$ if and only if $g_n \to g$ in $L^0(\mu_j)$ for all $j \in \mathbb{N}$.*
2. *Given a martingale kernel $(\pi_x)_x$, define*

$$\nu := \int \mu(dx)\pi_x, \quad \nu_j := \int \mu_j(dx)\pi_x \quad \in \mathcal{P}_1(\mathbb{R}^d),$$

then $h_n \to h$ in $L^0(\nu)$ if and only if $h_n \to h$ in $L^0(\nu_j)$ for all $j \in \mathbb{N}$.

The proof of lemmas 2 to 4 are postponed to section 3.

Proof of theorem 1. We can assume w.l.o.g. that the affine hull of the support of ν equals $\mathbb{R}^d$ [BBST25, Assumption 3.1 and text that follows], and so I is open. Recall that if $\mathcal{D}(\psi) < \infty$ then eq. (7) holds, and so $\varphi^\psi, \int \psi d\pi. \in L^0(\mu)$; for this reason we assume that $\mathcal{D}(\psi_n) < \infty$ for all n, which we can do w.l.o.g. since $\mathcal{D}(\psi_n) \to \mathcal{D}(\psi_{\lim}) < \infty$.

We now prove eq. (10). Assume by contradiction that eq. (10) fails at some point y; clearly $y \in C \setminus I$, since by assumption $(\psi_n(y))_n \to \psi_{\lim}(y)$ for all $y \in I \cup C^c = (C \setminus I)^c$. Choose a $x_0 \in I$; by restricting all functions to the segment from y to x_0 (i.e. by replacing $\mathbb{R}^d \ni z \mapsto f(z)$ with $[0,1] \ni t \mapsto f(tx_0 + (1-t)y))$ we can assume w.l.o.g. that $d = 1, y = 0, x_0 = 1, I = (0,a)$ for some $a > 1$. We focus on the case $\psi_{\lim}(0) < \infty$, and leave the case $\psi_{\lim}(0) = \infty$ to the reader. As we assumed that eq. (10) fails at y, by passing to a subsequence (without relabelling) we get that $\epsilon := \psi_{\lim}(0) - \lim_{n\to\infty} \psi_n(0) > 0$ and so for all big enough $n \in \mathbb{N}$

$$\psi_n(0) \le \psi_{\lim}(0) - \frac{3\epsilon}{4}. \tag{13}$$

Since $\psi_{\lim}$ is lower semicontinuous, we have $\psi_{\lim}(x) \ge \psi_{\lim}(0) - \frac{\epsilon}{4}$ for all $x \in (0,\delta)$ for some $\delta \in (0,1)$. Thus for all $x \in (0,\delta)$ for all big enough $n \ge N(x) \in \mathbb{N}$ we have

$$\psi_n(x) \ge \psi_{\lim}(0) - \frac{\epsilon}{2}. \tag{14}$$

If follows from eqs. (13) and (14) that the slope $\frac{\psi_n(x)-\psi_n(0)}{x-0}$ of ψ_n between 0 and x is bounded below by $\frac{\epsilon}{4x}$. Since ψ_n is convex, such slope is bounded above by the left-derivative $\psi_n'(x-)$ of ψ_n at x, and so $\frac{\epsilon}{4x} \le \psi_n'(x-)$ for all $n \ge N(x)$. From this and eq. (14), using the convexity of ψ_n we conclude that

$$\liminf_n \psi_n(1) \ge \liminf_n \psi_n(x) + \psi_n'(x-)(1-x) \ge \left(\psi_{\lim}(0) - \frac{\epsilon}{2}\right) + \frac{\epsilon}{4x}(1-x).$$

Taking $\lim_{x\downarrow 0}$ gives $\liminf_n \psi_n(1) = \infty$, contradicting $\psi_n(1) \to \psi_{\lim}(1) \in \mathbb{R}$. Thus, eq. (10) holds.

We now prove that $\psi_{lim} \in L^1(\nu)$ if $spt(\mu) \Subset I$. Let π^{SBM} be the Stretched Brownian Motion π^{SBM} from μ to ν and $(\pi_x^{SBM})_x$ be its disintegration with respect to μ. By [BBST25, Lemma 7.9] and Fatou's lemma

$$0 \le A := \int\int (\psi_{\lim}(y) - \psi_{\lim}(x))\, \pi_x^{SBM}(dy)\mu(dx) < \infty,$$

i.e.

$$\int\int \psi_{\lim}(y)\pi_x^{SBM}(dy)\mu(dx) \le A + \int\int (\psi_{\lim}(x))\, \pi_x^{SBM}(dy)\mu(dx),$$

or equivalently $\int \psi_{\lim}\, d\nu \le A + \int \psi_{\lim}\, d\mu$. So, from the fact that $\psi_{\lim}$ is continuous and finite on I, and thus bounded on the compact set $\operatorname{spt}(\mu) \subseteq I$, we conclude $\psi_{\lim} \in L^1(\nu)$.

We now prove that $(\psi_n)_n \to \psi_{lim}$ in $L^1(\nu)$ if $spt(\mu) \Subset I$. Since $\psi_{\mathrm{lim}}, \psi_n \geq 0$, we get

$$0 \leq (\psi_n - \psi_{\mathrm{lim}})^- \leq \psi_{\mathrm{lim}},$$

and as eq. (10) is equivalent to $(\psi_n - \psi_{\mathrm{lim}})^-(y) \to 0$ for all $y \in \mathbb{R}^d$, since $\psi_{\mathrm{lim}} \in L^1(\nu)$ by dominated convergence we conclude that $(\psi_n - \psi_{\mathrm{lim}})^- \to 0$ in $L^1(\nu)$. Thus, to prove $(\psi_n)_n \to \psi_{\mathrm{lim}}$ in $L^1(\nu)$ it suffices to show that $\int \psi_n d\nu \to \int \psi_{\mathrm{lim}}\, d\nu$. Since eq. (10) and Fatou's lemma imply that $\liminf_n \int \psi_n d\nu \geq \int \psi_{\mathrm{lim}} d\nu$, it suffices to show that

$$\limsup_n \int \psi_n d\nu \leq \int \psi_{\mathrm{lim}} d\nu. \tag{15}$$

Since we assumed that $(\psi_n)_n$ is a dual optimising sequence, i.e.

$$\int \psi_n d\nu - \int \varphi^{\psi_n} d\mu = \mathcal{D}(\psi_n) \to \mathcal{D}(\psi_{\mathrm{lim}}) = \int \psi_{\mathrm{lim}} d\nu - \int \varphi^{\psi_{\mathrm{lim}}} d\mu, \tag{16}$$

to prove eq. (15) it remains to show that

$$\limsup_n \int \varphi^{\psi_n} d\mu \leq \int \varphi^{\psi_{\mathrm{lim}}} d\mu. \tag{17}$$

For $\pi_x := \pi_x^{SBM}$, let K^j, π_x^j be as in lemma 2. Since $\pi_x^j \in \mathcal{P}_2^x(\mathbb{R}^d)$, the definition of φ^ψ gives

$$\varphi^{\psi_n} \leq \int \psi_n d\pi_\cdot^j - \mathrm{MCov}(\pi_\cdot^j, \gamma) \quad \mu \text{ a.e.}. \tag{18}$$

Since $(\psi_n)_n$ are convex and $\psi_n \to \psi < \infty$ on I, the convergence $\psi_n \to \psi$ is uniform on compacts [HUL01, Theorem 3.1.4]. It follows from item 1 of lemma 2 that the support of $\nu^j := \int \mu(dx)\pi_x^j$ satisfies $\mathrm{spt}\,(\nu^j) \subseteq K^j \cup \mathrm{spt}\,(\mu)$, and thus it is compact. Thus we get

$$\lim_n \int \left(\int \psi_n d\pi_\cdot^j \right) d\mu = \lim_n \int \psi_n d\nu^j = \int \psi d\nu^j = \int \left(\int \psi d\pi_\cdot^j \right) d\mu. \tag{19}$$

Since $\pi_x^j \leq_{\mathrm{c}} \pi_x$ gives $\int \left(\int \psi d\pi_\cdot^j \right) d\mu \leq \int \left(\int \psi d\pi_\cdot \right) d\mu$, integrating eq. (18), taking $\limsup_n$ and using eq. (19) we get that

$$\limsup_n \int \varphi^{\psi_n} d\mu \leq \int \left(\int \psi d\pi_\cdot - \mathrm{MCov}(\pi_\cdot^j, \gamma) \right) d\mu. \tag{20}$$

By dominated convergence it follows from eq. (11) that $\mathrm{MCov}\left(\pi_\cdot^j, \gamma\right) \to \mathrm{MCov}\,(\pi_\cdot, \gamma)$ in $L^1(\mu)$, and so taking $\lim_j$ of eq. (20) we conclude that eq. (17) holds, since π_x is the solution of the minimisation problem (5) for μ a.e. $x \in I$ when $\psi = \psi_{\mathrm{lim}}$ [BBST25, Section 5, sentence after eq. (5.2)]. We have thus proved that $(\psi_n)_n \to \psi_{\mathrm{lim}} \in L^1(\nu)$ if $\mathrm{spt}\,(\mu)$ is compact.

We now prove that $\psi_{lim} \in L^0(\nu)$ and $(\psi_n)_n \to \psi_{\lim}$ in $L^0(\nu)$. For $\pi_x := \pi_x^{SBM}$, let K^j be as in lemma 2, and define

$$\mu_j := \mu(\cdot|K^j) := \frac{\mu(K^j \cap \cdot)}{\mu(K^j)}, \quad \nu_j := \int \mu_j(dx)\pi_x^{SBM}.$$

For any $j \in \mathbb{N}$, by lemma 3 $\psi_{\lim}$ is a dual optimiser and $(\psi_n)_n$ is a dual optimising sequence also for (μ_j, ν_j). Moreover, (μ_j, ν_j) satisfy $\operatorname{spt}(\mu_j) = K^j \Subset I = \operatorname{ri}(\overline{\operatorname{co}}(\operatorname{spt}(\nu_j)))$ and $\nu_j \sim \nu$. Thus from the previously proved statements we get that $\psi_{\lim} \in L^1(\nu_j)$, and so $\psi_{\lim}$ is finite ν_j a.s. and so ν a.s. (i.e. $\psi_{\lim} \in L^0(\nu)$), and $\psi_n \to \psi_{\lim}$ in $L^1(\nu_j)$, and so $\psi_n \to \psi_{\lim}$ in $L^0(\nu_j)$, and so by lemma 4 $\psi_n \to \psi_{\lim}$ in $L^0(\nu)$. □

3 Proofs of lemmas

In this section we first present the proof of lemma 4, and then prove lemmas 2 and 3 with the aid the additional lemmas 5 and 6.

Proof of lemma 4. Item 1 is well known. Let us prove item 2. Let $\theta \in \mathcal{P}(\mathbb{R}^d)$ and $(\pi_x)_x$ be a martingale kernel, and define $\beta := \int_{\mathbb{R}^d} \theta(dx)\pi_x$. As $(h_n)_n \to h$ in $L^0(\theta)$ iff $(1_{\{|h_n-h|>\epsilon\}})_n \to 0$ in $L^1(\theta)$ for all $\epsilon > 0$, we get that $(h_n)_n \to h$ in $L^0(\beta)$ iff, for all $\epsilon > 0$, the sequence

$$v_n^\epsilon := \int_{\mathbb{R}^d} \pi_\cdot(dy) 1_{\{|h_n(y)-h(y)|>\epsilon\}}, \quad n \in \mathbb{N},$$

converges to 0 in $L^1(\theta)$, or equivalently (since $|v_n^\epsilon| \le 1$) in $L^0(\theta)$. Thus item 2 follows from item 1 applying this fact to $\theta = \mu$ and then to $\theta = \mu_j$ with $g_n := 1_{\{|h_n-h|>\epsilon\}}, g = 0$. □

Lemma 5. *If $\alpha, \beta, \zeta \in \mathcal{P}_1(\mathbb{R}^d)$, $\pi^1 \in \mathsf{MT}(\alpha, \beta)$, $\pi^2 \in \mathsf{Cpl}(\alpha, \zeta)$, then there exist random variables A, B, Z such that $(A, B) \sim \pi^1, (A, Z) \sim \pi^2$ and $\mathbb{E}[B|A, Z] = A$.*

Proof. One can take (A, B, Z) to be any random vector whose law π is given by

$$\pi(da, db, dz) := \alpha(da)\pi_a^1(db)\pi_a^2(dz), \qquad \text{where } \pi^i(da, dx) = \alpha(da)\pi_a^i(dx),$$

i.e. the kernel $(\pi_a^i)_a$ denotes the disintegration of π^i with respect to α for $i = 1, 2$. □

Lemma 6. *If $\alpha, \beta, \zeta \in \mathcal{P}_2(\mathbb{R}^d)$ and $\alpha \le_c \beta$ then*

$$\langle bar(\alpha), bar(\zeta)\rangle \le MCov(\alpha, \zeta) \le MCov(\beta, \zeta) \le \int \frac{1}{2}\|x\|^2(\beta + \zeta)(dx) < \infty.$$

Proof. Fix any $\pi^2 \in \mathsf{Cpl}(\alpha, \zeta)$. By Strassen's theorem $\exists \pi^1 \in \mathsf{MT}(\alpha, \beta)$, so applying lemma 5 yields (A, B, Z) such that $A \sim \alpha, B \sim \beta, Z \sim \zeta$ and

$$\mathbb{E}\langle Z, B\rangle = \mathbb{E}\left(\mathbb{E}[\langle Z, B\rangle|A, Z]\right) = \mathbb{E}\langle Z, \mathbb{E}[B|A, Z]\rangle = \mathbb{E}\langle Z, A\rangle = \textstyle\int \pi^2(da, dz)\langle a, z\rangle,$$

which, since $\pi^2 \in \mathsf{Cpl}(\alpha,\zeta)$ was arbitrary, shows that $\alpha \leq_c \beta$ implies $\operatorname{MCov}(\alpha,\zeta) \leq \operatorname{MCov}(\beta,\zeta) = \operatorname{MCov}(\zeta,\beta)$, which applied to $\delta_{\operatorname{bar}(\alpha)} \leq_c \alpha$ and then to $\delta_{\operatorname{bar}(\zeta)} \leq_c \zeta$ gives

$$\langle \operatorname{bar}(\alpha), \operatorname{bar}(\zeta)\rangle = \operatorname{MCov}\left(\delta_{\operatorname{bar}(\alpha)}, \delta_{\operatorname{bar}(\zeta)}\right) \leq \operatorname{MCov}\left(\alpha, \delta_{\operatorname{bar}(\zeta)}\right) \leq \operatorname{MCov}(\alpha,\zeta).$$

Finally $\operatorname{MCov}(\beta,\zeta) \leq \frac{1}{2}\int \|x\|^2(\beta+\zeta)(dx)$ follows from $2\langle x,y\rangle \leq \|x\|^2 + \|y\|^2$. □

Proof of lemma 2. As usual, by restricting to the affine hull of C, we may assume w.l.o.g. (see [BBST25, Assumption 3.1 and text that follows]) that C has dimension d, i.e. that I is open. Let K^j denote the intersection of the closed ball of radius j with the set of $x \in C$ whose distance $\operatorname{dist}(x, I^c) := \inf\{\|x-b\| : b \in I^c\}$ from I^c is at least $1/j$, i.e.

$$K^j := \left\{ z \in C : \operatorname{dist}(z, I^c) \geq \tfrac{1}{j}, \quad \|z\| \leq j \right\}. \tag{21}$$

Then $(K^j)_j \subseteq I$ is an increasing sequence of compact convex sets such that $\cup_j K^j = I$ and $K^j \subseteq \operatorname{int}\left(K^{j+1}\right)$. For $x \in I$ let $(M^x_t)_{t\in[0,1]}$ be the Stretched Brownian motion between δ_x and π_x, and define the stopping times

$$\tau^j_x := \inf\left\{t \in [0,1] : M^x_t \notin K^j\right\} \wedge 1, \quad j \in \mathbb{N}.$$

Note that, for $x \notin K^j$, we have $\tau^j_x = 0$. Since the limit τ_x of the increasing sequence $(\tau^j_x)_j$ equals 1 a.s. for all $x \in I$ [BBST25, Corollary 6.8] and M^x is continuous, we get that $M^x_{\tau^j} \to M^x_1$ a.s., and thus also in L^2 since Doob's L^2-inequality and $M^x_1 \sim \pi_x \in \mathcal{P}_2(\mathbb{R}^d)$ imply $\sup_t \|M^x_t\| \in L^2$. Thus the law π^j_x of $M^x_{\tau^j}$ converges weakly to the law π_x of M^x_1, and its second moment is finite and also converges, i.e. $\mathcal{M}_2(\pi^j_x) \to \mathcal{M}_2(\pi_x)$. It follows [Vil03, Theorem 7.12] that $\mathcal{W}_2(\pi^j_x, \pi_x) \to 0$ as $j \to \infty$, and so the identity[1]

$$\mathcal{M}_2(p) - 2\operatorname{MCov}(p,q) + \mathcal{M}_2(q) = \mathcal{W}_2(p,q), \quad p,q \in \mathcal{P}_2(\mathbb{R}^d)$$

implies $\operatorname{MCov}\left(\pi^j_x, \gamma\right) \to \operatorname{MCov}(\pi_x,\gamma)$ as $j \to \infty$. Since M^x is a martingale we get $\pi^j_x \leq_c \pi^{j+1}_x \leq_c \pi_x$ for all j, so Lemma 6 implies

$$0 \leq \operatorname{MCov}\left(\pi^j_x, \gamma\right) \leq \operatorname{MCov}\left(\pi^{j+1}_x, \gamma\right) \leq \operatorname{MCov}(\pi_x, \gamma) < \infty. \tag{22}$$

Clearly $M^x_{\tau^j}$ has values in K^j for $x \in K^j$, and $M^x_{\tau^j} = x$ for $x \in I \setminus K^j$, and so π^j_x is supported in K^j (resp. $\{x\}$) for $x \in K^j$ (resp. $x \in I \setminus K^j$). Finally, lemma 6 gives

$$\operatorname{MCov}(\pi_x, \gamma) \leq \int \frac{1}{2}\|y\|^2(\pi_x + \gamma)(dy) =: g(x),$$

and since $\nu = \int \mu(dx)\pi_x \in \mathcal{P}_2(\mathbb{R}^d) \ni \gamma$ it follows that $g \in L^1(\mu)$. □

[1] This identity follows integrating the formula $\|x\|^2 - 2\langle x,y\rangle + \|y\|^2 = \|x-y\|^2$ w.r.t. $r(dx,dy)$ and taking the infimum over $r \in \mathsf{Cpl}(p,q)$.

Lemma 7. *Given $\mu \leq_c \nu$ in $\mathcal{P}_2(\mathbb{R}^d)$, let π^{SBM} be the Stretched Brownian Motion between μ and ν and $(\pi_x^{SBM})_x$ be its μ-disintegration. For any $\psi : \mathbb{R}^d \to (-\infty, \infty]$ convex and μ a.s. finite define*

$$L(\psi)(x) := \int \psi(y)\pi_x^{SBM}(dy) - \varphi^{\psi}(x), \quad x \in \mathbb{R}^d.$$

Then

$$L(\psi) \geq MCov(\pi_\cdot^{SBM}, \gamma) \geq 0 \quad \mu \text{ a.e.} \tag{23}$$

for any such ψ, and $(\psi_n)_n$ is dual optimising (i.e. $\psi_n : \mathbb{R}^d \to (-\infty, \infty]$ is convex and μ a.s. finite for any $n \in \mathbb{N}$, and satisfies eq. (9)) if and only if

$$\|L(\psi_n) - MCov(\pi_\cdot^{SBM}, \gamma)\|_{L^1(\mu)} \to 0 \quad \text{as} \quad n \to \infty. \tag{24}$$

Proof. The first inequality in eq. (23) follows from the definition of φ^{ψ} (eq. (5)), the second from lemma 6. By eq. (8) $(\psi_n)_n$ is dual optimising if and only if

$$\mathcal{D}(\psi_n) = \int L(\psi_n) d\mu \to \int \mathrm{MCov}(\pi_\cdot^{SBM}, \gamma) d\mu < \infty$$

which by eq. (23) holds if and only if eq. (24) holds. □

Proof of lemma 3. Trivially $\pi^B \in \mathsf{MT}(\mu^B, \nu^B)$. Choose any $\pi \in \mathsf{MT}(\mu^B, \nu^B)$, and let $(\pi_x)_x$ be its μ-disintegration. Define

$$\tilde{\pi}_x := \begin{cases} \pi_x & \text{for } x \in B, \\ \pi_x^{SBM} & \text{for } x \in \mathbb{R}^d \setminus B. \end{cases}, \quad \tilde{\pi}(dx, dy) := \mu(dx)\tilde{\pi}_x(dy).$$

Notice that $\tilde{\pi}(\mathbb{R}^d \times \cdot) = \nu$, and so $\tilde{\pi} \in \mathsf{MT}(\mu, \nu)$. The inequality

$$\int_C \mu(dx)\, \mathrm{MCov}(\pi_x^{SBM}, \gamma) \geq \int_C \mu(dx)\, \mathrm{MCov}(\tilde{\pi}_x, \gamma) \tag{25}$$

holds when $C = B$: indeed it holds when $C = \mathbb{R}^d$ (since $\tilde{\pi} \in \mathsf{MT}(\mu, \nu)$, this follows from eq. (8)), and it holds with equality when $C = \mathbb{R}^d \setminus B$ (by definition of $\tilde{\pi}_x$). Evaluating eq. (25) with $C = B$ and dividing by $\mu(B)$ we get that

$$\int_{\mathbb{R}^d} \mu^B(dx)\, \mathrm{MCov}(\pi_x^{SBM}, \gamma) \geq \int_{\mathbb{R}^d} \mu^B(dx)\, \mathrm{MCov}(\pi_x, \gamma)$$

and given that $\pi \in \mathsf{MT}(\mu^B, \nu^B)$ was arbitrary, it follows that π^B is the Stretched Brownian Motion between μ^B and ν^B: indeed by [BBST25, Theorem 3.3] the unique maximiser of $\pi \mapsto \int \mathrm{MCov}(\pi_x, \gamma)\mu'(dx)$ over $\pi \in \mathsf{MT}(\mu', \nu')$ is the Stretched Brownian Motion between μ' and ν', so the thesis follows taking $\mu' = \mu^B, \nu' = \nu^B$.

If eq. (24) holds then it holds with μ replaced by μ^B, and since the μ-disintegration $(\pi_x^B)_x$ of the Stretched Brownian Motion π^B between μ^B and ν^B equals the μ-disintegration

$(\pi_x^{SBM})_x$ of the Stretched Brownian Motion between μ and ν, if $(\psi_n)_n$ is a dual optimising sequence for (μ,ν) then by lemma 7 it is also is a dual optimising sequence for (μ^B,ν^B). In particular, if a dual optimiser ψ_{lim} for (μ,ν) exists, then it is also a dual optimiser for (μ^B,ν^B), since ψ is a dual optimiser iff $\psi_n=\psi, n\in\mathbb{N}$ is a dual optimising sequence (here an alternative proof: use $(\pi_x^B)_x=(\pi_x^{SBM})_x$ and apply [BBST25, Definition 7.14, Lemma 7.19]).

By [BBST25, Theorems 1.4 and Remark D.3] two probabilities are irreducible iff there exists a Bass martingale connecting them, and by [BBST25, Theorem 1.3] the Stretched Brownian Motion between irreducible measures is a Bass martingale, so the statements about irreducibility and Bass martingales are equivalent.

To show that they hold, assume that (μ,ν) is irreducible, so $\pi_x^{SBM}\sim\nu$ for μ a.e. x by [BBST25, Corollary 7.7]. Since, for any Borel $A\subseteq\mathbb{R}^d$, $\nu_B(A)=0$ holds iff $\pi_x^{SBM}(A)=0$ for μ_B a.e. x, we conclude that $\nu_B(A)=0$ iff $\nu(A)=0$, i.e. $\nu_B\sim\nu$; thus $C_B:=\overline{\text{co}}\,(\text{spt}\,(\nu_B))$ equals $C:=\overline{\text{co}}\,(\text{spt}\,(\nu))$, and [BBST25, Theorem D.1] implies that (μ_B,ν_B) is irreducible. Finally, recall that if (μ,ν) is irreducible, the dual optimiser ψ_{lim} exists [BBST25, Theorem 7.6]. □

References

[AMP23] Beatrice Acciaio, Antonio Marini, and Gudmund Pammer. Calibration of the Bass local volatility model. *arXiv preprint arXiv:2311.14567*, 2023.

[BBHK20] Julio Backhoff-Veraguas, Mathias Beiglböck, Martin Huesmann, and Sigrid Källblad. Martingale Benamou-Brenier- a probabilistic perspective. *The Annals of Probability*, 48(5):2258–2289, 2020.

[BBST25] Julio Backhoff-Veraguas, Mathias Beiglböck, Walter Schachermayer, and Bertram Tschiderer. Existence of Bass martingales and the martingale Benamou-Brenier problem in $\mathbb{R}^d$. *arXiv preprint arXiv:2306.11019*, 2025.

[BPS24] Julio Backhoff-Veraguas, Gudmund Pammer, and Walter Schachermayer. The gradient flow of the bass functional in martingale optimal transport. *arXiv preprint arXiv:2407.18781*, 2024.

[BST23] Julio Backhoff-Veraguas, Walter Schachermayer, and Bertram Tschiderer. The Bass functional of martingale transport. *arXiv preprint arXiv:2309.11181*, 2023.

[CHL21] Antoine Conze and Pierre Henry-Labordère. Bass construction with multi-marginals: lightspeed computation in a new local volatility model. *Available at SSRN 3853085*, 2021.

[CN17] Peter Carr and Sergey Nadtochiy. Local variance gamma and explicit calibration to option prices. *Mathematical Finance*, 27(1):151–193, 2017.

[Dup97] Bruno Dupire. Pricing and hedging with smiles. In *Mathematics of derivative securities (Cambridge, 1995)*, volume 15 of *Publ. Newton Inst.*, pages 103–111. Cambridge Univ. Press, Cambridge, 1997.

[HL19] Pierre Henry-Labordère. From (martingale) Schrodinger bridges to a new class of stochastic volatility models. *arXiv preprint arXiv:1904.04554*, 2019.

[HT19] Martin Huesmann and Dario Trevisan. A Benamou-Brenier formulation of martingale optimal transport. *Bernoulli*, 25(4A):2729 – 2757, November 2019.

[HUL01] Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal. *Fundamentals of convex analysis*. Grundlehren Text Editions. Springer-Verlag, Berlin, 2001.

[JLO23] Benjamin Joseph, Grégoire Loeper, and Jan Obłój. The measure preserving martingale Sinkhorn algorithm. *arXiv preprint arXiv:2310.13797*, 2023.

[Loe18] Grégoire Loeper. Option pricing with linear market impact and nonlinear Black-Scholes equations. *The Annals of Applied Probability*, 28(5):2664–2726, 2018.

[QCYF24] Hao Qin, Charlie Che, Ruozhong Yang, and Liming Feng. Robust and fast Bass local volatility. *arXiv preprint arXiv:2411.04321*, 2024.

[ST24] Walter Schachermayer and Bertram Tschiderer. The decomposition of stretched Brownian motion into Bass martingales. *Preprint*, 2024.

[Vil03] Cédric Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.