International Workshop

A century of Noether's theorem and beyond

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ABSTRACTS

Stäckel transform for Lax equations

Maciej Błaszak

Adam Mickiewicz University, Poznań, Poland

It is well known that multi-parameter Stäckel transform allows for construction of new Liouville integrable systems from old ones. Here we present the action of multi-parameter Stäckel transform at the level of Lax equations. We construct Lax pairs for a wide class of Stäckel systems by applying the multi-parameter Stäckel transform to Lax pairs of suitably chosen systems from the seed class. The resulting Lax pairs of seed systems and transformed systems are parameterized by an arbitrary function of the spectral parameter.

On symmetries of a class of even-order ordinary differential equations

Priscila Leal da Silva

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In this talk we will discuss Lie and Noether symmetries for a class of autonomous even-order ordinary differential equations. We show that for power nonlinearities there is a certain power depending on the order of the equation that makes the Lie symmetries coincide with the Noether symmetries. For this special case, first integrals are established using Noether's theorem and exact solutions are found.

Conservation laws of Camassa–Holm type equations and some consequences

Igor Leite Freire

Universidade Federal do ABC, Brazil & Silesian University in Opava, Czech Republic

In this talk we explore some conservation laws of Camassa–Holm type equations in order to construct and analyze solutions of the equations under study.

Existence of soliton solutions for generalized FitzHugh–Nagumo system

Aleksandra Gawlik

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We sketch the proof of existence of homoclinic orbits corresponding to soliton solutions for a generalized FitzHugh–Nagumo system. The theoretical results are backed by numerical simulations exposing the stability properties of solitary wave solutions (depending on the values of parameters) as well as their inelastic interactions.

Nonlocal symmetries of Gibbons–Tsarev equation

Pavel Holba

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We consider the 3D integrable equation discovered independently by Mikhalev and Pavlov, $u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}$, and its 2D reduction $u_{yy} = (u_y + y)u_{xx} - u_x u_{xy} - 2$, which is equivalent to the Gibbons–Tsarev equation. These equations play an important role in the theory of integrable systems, and the Gibbons–Tsarev equation is also related to the theory of conformal maps. We will present new results on the differential coverings and nonlocal symmetries of the equations in question. This is joint work with I. S. Krasil'shchik, O. I. Morozov, and P. Vojčák; for details please see the paper

P. Holba, I.S. Krasil'shchik, O.I. Morozov and P. Vojčák, 2D reductions of the equation $u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}$ and their nonlocal symmetries, *J. Nonlinear Math. Phys.* **24** (2017), suppl. 1, 36–47.

Noether's theorem and conserved currents in Covariant Classical and Quantum Mechanics

Josef Janyška

Masaryk University, Brno, Czech Republic

Covariant Classical Mechanics and Covariant Quantum Mechanics are geometric approaches to Classical Mechanics and Quantum Mechanics on a curved spacetime fibred over absolute time and equipped with a Riemannian metric on its fibres. We assume a few basic fields from which we can construct, by using covariant (natural) operations, the classical and the quantum Lagrangians. By using the Noether's theorem we can classify conserved currents associated with these Lagrangians.

Explicit triangular decoupling of vector and tensor mode equation on Schwarzschild

Igor Khavkine

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The vector and tensor (aka Lichnerowicz) wave equations have rich geometric properties and are naturally motivated by the study of classical and quantum, respectively, electromagnetic and gravitational perturbations on curved spacetimes. On the Schwarzschild black hole background, both of these equations admit a complete separation of variables. The resulting radial mode equations are complicated coupled systems of ODEs with rational coefficients. The geometric properties of these equations enable us to use an abstract strategy, based on the formal theory of differential equations, to partially decouple these systems into triangular form with rational coefficients. We can prove that this triangular form cannot be fully diagonalized by transformations with rational coefficients. Though the triangular form is no longer self-adjoint, the formal self-adjointness of the original wave equation combines with the reduction to produce a non-trivial recursion operator for the radial mode equation. [arXiv:1711.00585, 1801.02632]

Geometric approach to the multipeakon solutions to the Camassa–Holm equation

Wojciech Kryński

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Multipeakons are special solutions to the Camassa–Holm equation. They can be described by an integrable geodesic flow on a Riemannian manifold. Singular points of the Riemannian metric correspond to collisions of the multipeakons. We consider a bi-Hamiltonian formulation of the system and exploit its first integrals in order to analyse geodesics near singular points of the metric. We present a novel approach to the problem of the dissipative prolongations of multipeakons after the collision time.

Multisymplectic structures and Lie systems

Javier de Lucas

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In short, a Lie system is a non-autonomous system of first-order ordinary differential equations whose general solution can be described as a function of a finite family of particular solutions and some constants: the so-called superposition rule. Lie systems cover as particular cases non-autonomous linear systems of first-order ordinary differential equations, matrix Riccati equations, Smorodinsky–Winternitz oscillators, Schwarz equations, and another relevant physical, control theory, and mathematical models. The Lie–Scheffers theorem states that a Lie system amounts to a differential equation describing the integral curves of a non-autonomous vector field taking values in a finite-dimensional Lie algebra of vector fields, a so-called Vessiot–Guldberg Lie algebra.

In this talk, I will introduce the analysis of a new class of Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a multisymplectic structure: the multisymplectic Lie systems. This allows us to show that multisymplectic geometry, which is mainly aimed at the study of partial differential equations and field theories, may be used to provide new approaches to the analysis of systems of ordinary differential equations. More specifically, multisymplectic methods are developed to consider a Lie system as a multisymplectic one. By attaching a multisymplectic Lie system via its multisymplectic structure with a tensor coalgebra, we find geometric and coalgebra methods to derive superposition rules, constants of motion, and invariant tensor fields relative to its evolution. In particular, this extends the so-called coalgebra symmetry method for studying constants of motion of Hamiltonian systems to a much more general realm. Our results are illustrated with examples occurring in physics such as Schwarz equations or diffusion-type equations.

The Kovalevskaya exponents — true and false integrability criteria

Andrzej J. Maciejewski

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S. V. Kovalevskaya introduced an easy method to check whether all solutions of a given system are single-valued as functions of complex time. She noticed that for many integrable system their solutions are single-valued. Thus she formulated an easy-to-check necessary condition for integrability. Later A. M. Lyapunov simplified her considerations. This approach, developed further by P. Painlevé and others, was transformed to the Painlevé analysis which allowed to detect many integrable systems. Somehow unexpectedly H. Yoshida reformulated the Kovalevskaya–Lyapunov method into a purely algebraic algorithm which gives necessary conditions for integrability. This approach got enormous popularity, and many authors attracted by its simplicity formulated their own theorems. Unfortunately many of them, including one of Yoshida, are not correct.

In my talk I will present the Kovalevskaya–Lyapunov method and give examples of easy but false integrability criteria.

Scalar invariants of spactimes with symmetry

Michal Marvan

Silesian University in Opava, Czech Republic

Scalar invariants in general relativity start at the second order. However, a majority of known metrics admit Killing fields and, consequently, also the associated semi-Riemannian submersion. Such a submersion may possess invariants of lower order than two, while the equivalence problems are interconvertible. Dealing with the important case of two commuting Killing fields, we show that generic orthogonally transitive metrics possess four independent first-order scalar invariants, while generic orthogonally intransitive metrics possess six independent first-order scalar invariants and a first-order invariant frame. We also discuss practical challenges associated with classification in invariant terms.

Lax representations and exotic cohomology of symmetry algebras of PDEs

Oleg Morozov

AGH University of Science and Technology, Kraków, Poland

My talk will describe the technique of constructing Lax representations for partial differential equations via non-central extensions of their contact symmetry algebras. The examples will include Lax representations with non-removable parameters, two-component generalizations, and integrable hierarchies associated to some PDEs.

On discrete Fourier analysis for the functions sampled on the weight lattices or model sets of semisimple Lie groups

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A method for finite discrete analysis of almost periodic functions (defined on model sets of semisimple Lie groups) that is entirely based on group-theoretical methods, primarily finite groups and their duals, is developed. Examples of functions based on the standard Fibonacci quasicrystal and two-dimensional quasicrystal are proposed.

Infinite families of orbit functions (functions that are symmetric with respect to the affine Weyl group of a semisimple Lie group) are revisited. It is shown that these functions do satisfy the discrete Fourier analysis conditions on the three-dimensional refined fundamental regions of the respective simple Lie groups.

Bond systems, conservation laws and some other tools to relate discrete integrable systems

Maciej Nieszporski

University of Warsaw, Poland

I will present variety of discrete integrable systems and mutual relations between the systems.

Symmetries of Schrödinger–Pauli equations with position dependent mass

A. G. Nikitin

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Both classical Lie and higher-order symmetries of the position-dependent mass Schrödinger equation with matrix potential are presented. The connection of these symmetries with shape invariance is discussed. The complete sets of solutions for bound states of selected systems are found.

Invariant Nijenhuis operators on homogeneous spaces and integrable geodesic flows

Andriy Panasyuk

University of Warmia and Mazury in Olsztyn, Poland

We study invariant Nijenhuis operators on a homogeneous space G/K of a semisimple Lie group G from the point of view of integrability of a system of differential equations which appears as follows. Such an operator induces an invariant symplectic form ω on the cotangent bundle $\mathrm{T}^*(G/K)$, which is Poisson compatible with the canonical symplectic form $\Omega_{\mathrm{T}^*(G/K)}$. This Poisson pair can be reduced to the space of G-invariant functions on $\mathrm{T}^*(G/K)$ and produces a family of Poisson commuting functions. We give, in Lie algebraic terms, necessary and sufficient conditions of the completeness of this family. As an application we obtain two new classes of metrics on homogeneous spaces with integrable geodesic flow. (A joint work with Konrad Lompert.)

Generalization of second Noether's theorem

Roman O. Popovych

Silesian University in Opava, Czech Republic & University of Vienna, Austria
& Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

We enhance Noether's second theorem and generalize it to systems of differential equations that are not necessarily Euler–Lagrange equations. More specifically, we prove that a system of differential equations is abnormal, i.e., it has an identically vanishing differential consequence, if and only if it possesses a trivial conserved current corresponding to a nontrivial characteristic. Moreover, the above properties are also equivalent to the following: the system admits a family of characteristics that is parameterized by an arbitrary function of all independent variables in a local way and whose Fréchet derivative with respect to the parameter-function does not vanish on solutions of the system for a value of the parameter function. The theorem is illustrated by physically relevant examples.

Integrability analysis of the stretch-twist-fold flow

Maria Przybylska

University of Zielona Góra, Poland

We present the result of the integrability analysis for 7-parameter family of 3D spherically confined steady Stokes flows introduced by Bajer and Moffatt. This flow was constructed to model stretch-twist-fold mechanism of the fast dynamo magnetohydrodynamical model. We analyse integrability of this system using Darboux polynomials. In particular complete classification of degree one Darboux polynomials is presented. On the one hand, for some values of parameters using found Darboux polynomials we construct one first integral. Since the system preserves the phase volume just one first integral is sufficient for its integrability. We also construct for these values of parameters the second first integral using the integrating factor. On the other hand, Darboux polynomials are also useful for proving non-integrability. Namely, they give particular solutions along which one can linearise equations of motion. Then using properties of differential Galois group of linearized equations we prove non-integrability for certain multiparameter families of such systems.

Generalized Darboux transformation and explicit solutions of dynamical systems and nonlocal NLS

Alexander Sakhnovich

University of Vienna, Austria

GBDT version of Darboux transformation was introduced in [1] and actively developed later on. We will discuss its applications to important dynamical systems and to nonlocal NLS (see [2,3] and references therein).

[1] A. L. Sakhnovich, Dressing procedure for solutions of nonlinear equations and the method of operator identities, *Inverse Problems* **10** (1994) 699–710.

[2] A. L. Sakhnovich, Dynamics of electrons and explicit solutions of Dirac–Weyl systems, *J. Phys. A: Math. Theor.* **50** (2017) 115201.

[3] J. Michor and A. L. Sakhnovich, GBDT and algebro-geometric approaches to explicit solutions and wave functions for nonlocal NLS, *J. Phys. A: Math. Theor.*, in press, https://doi.org/10.1088/1751-8121/aaedeb.

Multidimensional Integrability via Geometry

Artur Sergyeyev

Silesian University in Opava, Czech Republic

The search for integrable partial differential systems in four independent variables (4D) is a longstanding problem of mathematical physics. In the present talk we address this problem and show that integrable 4D systems are significantly less exceptional than it appeared before: in addition to a few previously known examples like the (anti)self-dual Yang–Mills equations there is a large entirely new class of integrable 4D systems with Lax pairs of a novel kind related to contact geometry. In particular, we present explicit form for two infinite families of integrable 4D systems from this class along with their Lax pairs.

For further details please see A. Sergyeyev, *Lett. Math. Phys.* **108** (2018), no. 2, 359–376.

Lie symmetry approach to study of heat and mass transfer equations

Irina Stepanova

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The talk is devoted to analysis of equations describing thermal diffusion process in a binary liquid mixture via Lie symmetry approach. Thermal diffusion is the component separation in non-uniformly heated fluid. Mathematical model of the process consists of Navier–Stokes equations supplemented by mass and heat transport equations. We will review solutions of the group classification problems for the above system and for simpler systems that do not take into account the influence of convective terms. The group classification is carried out with respect to the coefficients related to physical properties of a mixture.

On a construction of infinite-dimensional Frobenius manifolds

Błażej Szablikowski

Adam Mickiewicz University, Poznań, Poland

I will present some results on the construction of infinite-dimensional Frobenius manifolds, related to dKP and dToda $(2+1)$-dimensional hydrodynamic-type hierarchies, one on a contour in the complex plane, by means of the Cauchy transform, and another by means of a coupled algebra of Laurent series, both within the framework of my previous work: Classical r-matrix like approach to Frobenius manifolds, WDVV equations and flat metrics, *J. Phys. A: Math. Theor.* **48** (2015) 315203.

On a reduction of nonlinear evolution and wave type equations via non-point symmetry method

Ivan Tsyfra

AGH University of Science and Technology, Kraków, Poland

We study the symmetry reduction of partial differential equations by using non-point symmetry operators. We construct an ansatz for dependent variable or its derivatives which reduces a scalar partial differential equation to a system of ordinary differential equations. We use both operators of classical and conditional symmetry. It turns out that combining non-point and conditional symmetry enables us to construct not only solutions but also Bäcklund transformations for the equation under study. Using operators of Lie–Bäcklund symmetry we construct the solutions of nonlinear hyperbolic equations depending on an arbitrary smooth function of one variable.

Symmetries and conservation laws for a generalization of Kawahara equation

Jakub Vašiček

Silesian University in Opava, Czech Republic

In this talk we study formal and generalized symmetries and local conservation laws for a fifth-order nonlinear evolutionary partial differential equation in two independent and one dependent variable which generalizes the Kawahara equation. In particular, we establish nonexistence of formal symmetries of rank greater than 13 and give a complete description of generalized symmetries and local conservation laws for the equation under study.

Soliton-like solutions to a non-integrable hydrodynamic-type model and their spectral stability

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A model of nonlinear elastic medium with internal structure is considered. The medium is assumed to contain cavities, microcracks or inclusions consisting of substances that differ sharply in physical properties from the base material. To describe the wave processes in such a medium, the averaged values of physical fields are used. This leads to nonlinear evolutionary PDEs differing from the classical hydrodynamic-type balance equations. Using some transformations these equations can be presented in the Hamiltonian form.

The purpose of this talk is twofold. In the first part, it is shown that the system under investigation has invariant soliton-like solutions. The second part contains analytical results on studying the spectral stability of the soliton-like solutions.

Structure functions and Spencer cohomology in zero and positive characteristics

Pasha Zusmanovich

University of Ostrava, Czech Republic

Structure functions are obstructions to integrability of G-structures, where G is a Lie group, on a real or complex manifold. These well-known invariants are expressed in terms of Spencer cohomology of the corresponding graded Lie algebra. Many important particular cases of structure functions are known under the names of Riemann tensor, Nijenhuis tensor, etc.

Structure functions can be defined, via Spencer cohomology, also over arbitrary fields, including the case of positive characteristic, though immediate connection with the corresponding geometric situation is then lost. Nevertheless, these invariants are important in a purely algebraic situation, namely, the structure theory of Lie algebras in positive characteristic.

While vanishing of certain important structure functions is a classical result due to Serre and others, the positive characteristic case is, somewhat surprisingly, was not systematically treated in the literature. We present a unifying approach to computation of the relevant Spencer cohomology. Our approach clearly demonstrates the difference between zero and positive characteristics: the (non)vanishing of Spencer cohomology is controlled by (non)vanishing of coinvariants of the derivatives of the corresponding polynomial algebras.